Exercise 9 Let $A \to B \to C$ and $A \to D$ be ring homomorphisms. Show the following.

1) If $A \to B$ and $B \to C$ are unramified, so is $A \to C$
2) a) —
   b) — $\Big\}$ online
   c) If $A \to B$ is unramified, then so is $D \to D \otimes_A B$

3. Etale morphisms

Def A morphism of finite presentation of rings is *étale* if it is flat and unramified.

Exercise 10 Let $A \to B \to C$ and $A \to D$ be ring homomorphisms. Show the following.

1) If $A \to B$ and $B \to C$ are étale, so is $A \to C$
2) If $A \to B$ is étale, then so is $D \to D \otimes_A B$

Example. Suppose $Y \to X$ is a morphism of smooth affine $\mathbb{C}$-varieties, say $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$. Then $A \to B$ is étale if and only if $Y(\mathbb{C}) \to X(\mathbb{C})$ is a local homeomorphism of topological spaces.
($Y(\mathbb{C}) \subset \underset{\mathbb{R}^{2n}}{\underset{\shortparallel}{\mathbb{C}^n}}$, $X(\mathbb{C}) \subset \underset{\mathbb{R}^{2m}}{\underset{\shortparallel}{\mathbb{C}^m}}$, are given the "classical" topology)

Topology I

These families are called **coverings** (被覆)

Definition A **(Grothendieck) topology** on a category $\mathcal{C}$ is the data of: for every $U \in \mathcal{C}$, a collection of families of morphisms $\{U_i \to U\}_{i \in I}$. This data is required to satisfy:

1) $\{U \xrightarrow{\mathrm{id}} U\}$ is a covering, for every object U.
2) If $\{U_i \to U\}_{i \in I}$ is a covering of U, and $V \to U$ is any morphism, then each fibre product $U_i \times_U V$ exists and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.
3) If $\{U_i \to U\}_{i \in I}$ is a covering of U, and for each $i \in I$, we have a covering $\{U_{ij} \to U_i\}_{j \in J_i}$ of U_i then $\{U_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering of U.

A category equipped with a Grothendieck topology is called a **site**.

Exercise 1 Suppose X is a topological space. Define $Op(X)$ to be the category whose objects are open sets of X, and morphisms are inclusions $\left(\hom(U, V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V \\ U \hookrightarrow V & \text{if } U \subseteq V \end{cases} \right)$

For $U \in Op(X)$, define the coverings of U to be the families $\{U_i \to U\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$. Show that this defines a Grothendieck topology. (Note: In $Op(X)$ we have $V \times_U W = V \cap W$).

Exercise 2 Let X be a topological space. Define $LH(X)$ to be the category whose objects are local homeomorphisms $f: Y \to X$. (i.e., $\forall\, y \in Y$, $\exists$ open neighbourhood $y \in V \subset Y$ s.t. f induces a homeomorphism $V \to f(V)$)

Morphisms are commutative triangles
$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}.$$

Ex. 2. cont. Show $LH(X)$ has fibre products.
For $Y \in LH(X)$, define the coverings of Y to be the families $\{f_i : Y_i \to Y\}_{i \in I}$ such that $\bigcup_{i \in I} f_i(Y_i) = Y$. Show that this defines a Grothendieck topology on $LH(X)$.

Definition A morphism of schemes $f : Y \to X$ is **étale** if it is locally of finite presentation and for all $y \in Y$, $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is étale.

Remark $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale iff $A \to B$ is étale

Exercise 3 Let X be a scheme. Let $Et(X)$ denote the category whose objects are are étale morphisms $Y \to X$ and morphisms are commutative triangles
$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}.$$
Show $Et(X)$ has fibre products.
Define covering families to be those families $\{Y_i \xrightarrow{f_i} Y\}_{i \in I}$ such that $\bigcup_{i \in I} f_i(Y_i) = Y$. Show that this defines a Grothendieck topology.

Definition A **presheaf** F on a category $\mathcal{C}$ is just a functor $\mathcal{C}^{op} \to Set$.

$\forall X \in \mathcal{C}, \mapsto F(X) \in Set$ (or Ab)
$\forall f : Y \to X, \mapsto F(Y) \xleftarrow{F(f)} F(X)$ (homomorphism)
$F(id) = id$
$F(f \circ g) = F(g) \circ F(f)$

A **morphism** of presheaves $F \to G$ is a natural transformation of functors.

$\forall X, \quad F(X) \to G(X)$
$\forall f : Y \to X,$
$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G(Y) \arrow[d] \\ F(X) \arrow[r] & G(X) \end{tikzcd}$$

Example Let X be a topological space, $\mathcal{C} = Op(X)$, define
$$F(U) := \{\text{continuous functions } U \to \mathbb{R}\}.$$
Then F is a presheaf. $\left(V \hookrightarrow U \,, \quad \begin{array}{c} F(U) \to F(V) \\ (f: U \to \mathbb{R}) \mapsto (V \to U \to \mathbb{R}) \end{array} \right)$

Definition If $\mathcal{C}$ is equipped with a Grothendieck topology, then a presheaf F is called a **sheaf** if for any object U and any covering $\{U_i \to U\}_{i\in I}$ we have

$$F(U) = eq\left(\prod_{i\in I} F(U_i) \overset{proj_1}{\underset{proj_2}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \times_U U_j) \right) \qquad (*)$$

A **morphism** of sheaves is a morphism of presheaves.
A sheaf on $Et(X)$ for some scheme X is called an **étale sheaf**.

Remark If A is a ring, write $Et(A) := Et(Spec(A))$ and if $A \to B$ is an étale algebra, F is a presheaf on $Et(A)$, write $F(B) := F(Spec(B))$

Remark $\forall$ scheme X $\exists$ étale covering $\{U_i \to X\}_{i\in I}$ s.t. U_i are affine. Cf. Exercise 9

Example If A is a ring, the following are étale sheaves on $Et(A)$.

1) $\mathcal{O} : B \longmapsto (B, +) \in$ Abelian group
2) $\mathcal{O}^* : B \longmapsto (B^*, *) \in$ Abelian group
3) $\mu_n : B \longmapsto \{ b \in B \mid b^n = 1 \}$
4) $\Omega : B \longmapsto \Omega_B$

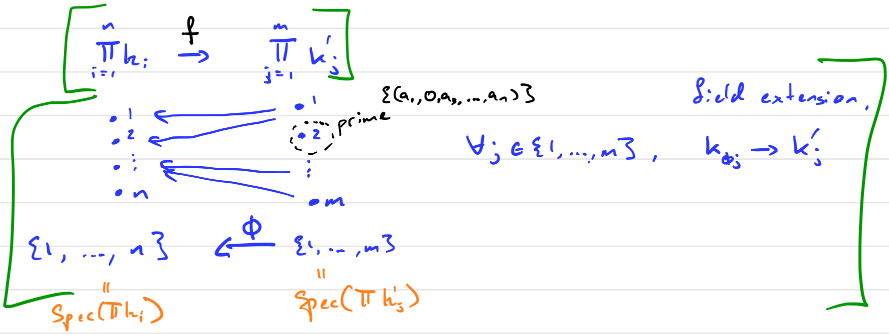
Remark If a presheaf takes values in the category of abelian groups then the sheaf condition $(*)$ is equivalent to asking that the sequence

$$0 \to F(U) \to \prod_{i\in I} F(U_i) \xrightarrow{proj_1 - proj_2} \prod_{i,j\in I} F(U_i \times_U U_j)$$

be exact.

Remark Let k be a field. A k-algebra $k \to A$ is an étale morphism if and only if $A \cong \prod_{i=1}^n k_i$ for some finite separable field extensions k_i/k.

Note: $\mathrm{Spec}\left(\prod_{i=1}^n k_i\right) = \coprod_{i=1}^n \mathrm{Spec}(k_i)$



$Et(k)$ encodes all finite separable field extensions of k + all embeddings.

Exercise 5 Let $\mathrm{Spec}(L) \to \mathrm{Spec}(L')$ be a morphism in $Et(k)$ such that L'/L is a Galois field extension with Galois group $G := \mathrm{Gal}(L'/L)$. Recall that there is a canonical isomorphism

$$L' \otimes_L L' \xrightarrow{\cong} \prod_G L'$$

where the two morphisms $L' \rightrightarrows L' \otimes_L L'$; $a \mapsto 1 \otimes a$, $a \mapsto a \otimes 1$ correspond to $L' \to \prod_G L'$; $a \mapsto (a, a, \dots, a)$, $a \mapsto (a^{g_1}, a^{g_2}, \dots, a^{g_n})$ $(G = \{g_1, \dots, g_n\})$. Show that if F is an étale sheaf on $Et(k)$, then

1) $F\left(\prod_G L'\right) \cong \prod_G F(L')$, and

2) $F(L') \cong F(L')^G$

Ex. 5. cont.

Note: $\forall\, g \in G$, $\quad \mathrm{Spec}(L') \xrightarrow{g} \mathrm{Spec}(L')$

$\mapsto\ g^*: F(L') \to F(L')$

$$F(L')^G = \{ s \in F(L') \mid g^* s = s \ \forall\ g \in G \}.$$

Deduce that if $F_1 \to F_2$ is a morphism of étale sheaves such that $F_1(L) \cong F_2(L)$ for every Galois extension L/k, then $F_1 \cong F_2$. ($F_1(B) \cong F_2(B)\ \forall\ B \in Et(k)$)

Theorem Suppose k is a field, k^{sep}/k is a separable closure, and $G = \mathrm{Gal}(k^{sep}/k)$. Then there is an equivalence of categories

$$\left\{ \begin{array}{c} \text{discrete} \\ G\text{-sets} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{étale sheaves} \\ \text{on } Et(k) \end{array} \right\}$$

set S, $\quad G \times S \to S$

$(g, s) \longmapsto s^g \quad$ s.t. $s^{(gh)} = (s^g)^h$

$\exists\ N \trianglelefteq G$ s.t. G/N is finite and $s^n = s \ \forall \ \begin{array}{l} s \in S \\ n \in N \end{array}$.

Next week: Sketch of proof.

$$Y = \begin{matrix} \circ \\ \bullet \end{matrix}$$

$$\downarrow$$

$$X = \{0, \ldots, \tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, 1\} \subset \mathbb{R}$$

$X = \operatorname{Spec} A$ $\qquad A = \operatorname{colim}\left(k \to k^2 \to k^3 \to \ldots \to k^n \to \ldots\right)$

$$(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, a_n)$$

(Next quarter: $k \to A$ is pro-étale)

$$A = \{(a_1, a_2, \ldots) \in k^{\mathbb{N}} \mid \exists N \text{ s.t. } a_i = a_{i+1} \ \forall i > N\}$$

$$\bigoplus_{\mathbb{N}} k \subset A \subset \prod_{\mathbb{N}} k$$

$$\cap$$

$$\mathfrak{p}$$

$$\uparrow$$

not finitely generated

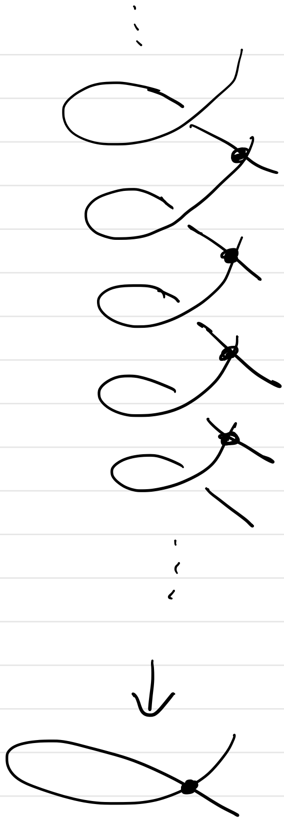
Claim: $A_{\mathfrak{p}} \cong k$

so $A/\mathfrak{p} \leftarrow A$ is not finite presentation.

$$\mathrm{Shv}_{et}(\text{etale } X\text{-schemes}) \xrightarrow{\cong} \mathrm{Shv}_{et}(\text{etale affine } X\text{-schemes})$$

$$\mathrm{Shv}_{et}(\mathrm{Et}(k)) \xrightarrow{\cong} \mathrm{Shv}_{et}(\text{Galois } k\text{-algebras})$$

Y ← not affine (not quasi-compact)



$\downarrow$ étale

X ← affine

$\mathrm{Aut}(Y/X) \cong \mathbb{Z}$

$$\pi_1^{et}(X) = \varprojlim_{\substack{Y \to X \\ \text{finite étale}}} \mathrm{Aut}(Y)$$

$$\pi_1^{et}(\alpha) = \hat{\mathbb{Z}}$$