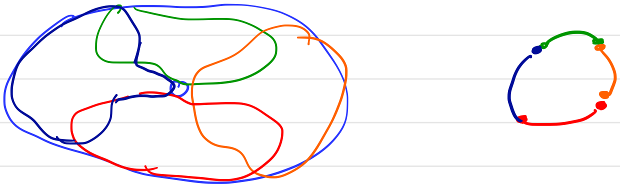
Commutative algebra I

We want a cohomology theory. We saw that over $\mathbb{C}$, as an analytic space, the set $\mathbb{C}\setminus\{0\} \sim S^1$

$$H^i_{sing}(\mathbb{C}\setminus\{0\}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$



← analytic picture.

Problem: algebraic open subsets are two big.

Solution: use local homeomorphisms instead.

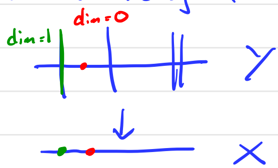
↖ today: algebraic notion of local homeomorphism

Example:

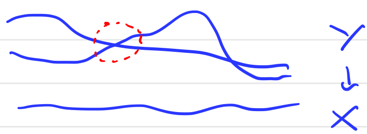
$$\mathbb{C}\setminus\{0\} \xrightarrow{z \mapsto z^4} \mathbb{C}\setminus\{0\}$$

Two obstacles to a morphism of varieties $Y \to X$ being a "local homeomorphism"

1) dimension of fibres can vary (not flat)



2) can have "branch points" (ramified)



1) Flatness

Definition 1 Let A be a ring. An A-module M is ==flat== if for every monomorphism of A-modules $N \subseteq N'$, the morphism $M \otimes_A N \to M \otimes_A N'$ is also a monomorphism. That is, $M \otimes_A -$ preserves monomorphisms. An A-algebra is flat if it is flat as an A-module.

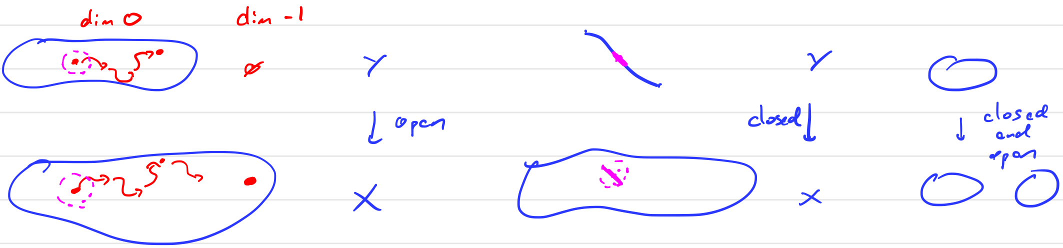
Exercise 1. Show that if k is a field, every k-module (and therefore k-algebra) is flat.

Exercise 2. Show that if A is a ring and $S \subseteq A$ is a multiplicatively closed subset, then $A \to A[S^{-1}]$ is flat.

Example In fact an A-module M is flat if and only if for every prime $\mathfrak{p} \subset A$ the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is flat.

Exercise 3 Let A be a ring, and $I \subseteq A$ an ideal

1) Show that if A/I is a flat A-algebra then $I = I^2$
2) (Advanced) Show that if I is finitely generated and $I = I^2$ then A/I is a flat A-algebra.
3) (Advanced) Give an example of an ideal I of a ring A such that $I = I^2$ but A/I is not a flat A-algebra.



Exercise 4

1) if $A \to B$ and $B \to C$ are flat, show $A \to C$ is flat
2) if $A \to B$ is flat and $A \to D$ is any ring homomorphism, show $D \to B \otimes_A D$ is flat.

Definition Let A be a ring. An A-module M is faithfully flat if it is flat and given any morphism $N \to N'$ of A-modules such that $M \otimes_A N \to M \otimes_A N'$ is a monomorphism, then $N \to N'$ is a monomorphism.

[$N \to N'$ monic $\Rightarrow$ $M \otimes_A N \to M \otimes_A N'$ monic] flatness
$M \otimes_A N \to M \otimes_A N'$ monic $\Rightarrow$ $N \to N'$ monic] faithfully flatness

An A-algebra B is faithfully flat if its faithfully flat when considered as an A-module.

Example One can show that $A \to B$ is faithfully flat if and only if $A \to B$ is flat and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.
In particular, if k is a field, every k-algebra is faithfully flat.
If A is a ring and $f_1, \dots, f_n \in A$ generate the unit ideal then $\{\mathrm{Spec}\, A_{f_i} \to \mathrm{Spec}\, A\}_{i=1}^{n}$ is an open covering so $A \to \prod_{i=1}^{n} A_{f_i}$ is faithfully flat.

Note: 1) For A-modules, $M_1 \times \dots \times M_n = M_1 \oplus \dots \oplus M_n$
2) $N \otimes_A (M_1 \oplus \dots \oplus M_n) = (N \otimes_A M_1) \oplus \dots \oplus (N \otimes_A M_n)$
3) If $M_1, \dots, M_n$ are flat then $M_1 \oplus \dots \oplus M_n$ is flat
$\phi : N \to N'$ is monic then $\bigoplus_{i=1}^{n} (\phi \otimes_A M_i) = \phi \otimes_A \left(\bigoplus_{i=1}^{n} M_i\right)$ is monic

[Aside: filtered colimit of flat modules is flat
Conversely (Lazard) every flat module is a filtered colimit of finite free modules]

<u>Exercise 5</u> Let M be an A-module, and $A \to B$ a faithfully flat morphism. Show that if $M \otimes_A B$ is a flat B-module then M is a flat A-module.

<u>Exercise 6</u> Let M be an A-module.

1) Show that M is faithfully flat if and only if for every A-module N such that $M \otimes_A N \cong 0$, we have $N \cong 0$

2) Show that if M is faithfully flat then given any $N \to N'$ such that $M \otimes_A N \to M \otimes_A N'$ is a surjection, the morphism $N \to N'$ is a surjection.

3) Deduce that if M is faithfully flat, then a sequence of A-modules is exact if it is exact after applying $M \otimes_A -$.

[A sequence $\cdots \xrightarrow{d_{i-1}} N_{i-1} \xrightarrow{d_i} N_i \xrightarrow{d_{i+1}} N_{i+1} \xrightarrow{d_{i+2}} \cdots$ is <u>exact at i</u> if $\ker(d_{i+1}) = \operatorname{im}(d_i)$. Its <u>exact</u> if its exact at all i.]

<u>Theorem</u> Suppose that $f: A \to B$ is a faithfully flat ring morphism. Then

$$0 \to A \xrightarrow{f} B \xrightarrow{d_1} B \otimes_A B \xrightarrow{d_2} B \otimes_A B \otimes_A B \xrightarrow{d_3} \cdots$$

is an exact sequence of A-modules. Here,

$$d = \sum (-1)^i e_i$$

$$e_i : b_0 \otimes \cdots \otimes b_{r-1} \longmapsto b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}$$

<u>Exercise 7</u> Show $d_1 \circ f = 0$ and $d_{n+1} \circ d_n = 0$ $\forall n \geq 1$.

Proof. First suppose there exists a ring homomorphism $r: B \to A$ such that $r \circ f = \mathrm{Id}_A$ $\left(r(f(a)) = a \right)$.
Next define

$$s : b_0 \otimes b_1 \otimes \dots \otimes b_{r+1} \mapsto r(b_0)\, b_1 \otimes \dots \otimes b_{r+1}$$

and check that $s \circ d_1 + d \circ s = \mathrm{Id}$ and $s \circ d_{n+1} + d_n \circ s = \mathrm{Id}$.
[In other words, we have constructed a chain complex homotopy between Id and 0. Therefore the cohomology groups vanish]

$a \in \ker(d)$ then $a = sda + dsa = 0 + dsa$ so $a \in \operatorname{im} d$.

Now consider some A-algebra A', let $B' = A' \otimes_A B$, $f' = A' \otimes_A f$.
Since

$$A' \otimes_A (B \otimes_A \dots \otimes_A B) \cong (A' \otimes_A B) \otimes_{A'} \dots \otimes_{A'} (A' \otimes_A B)$$

applying $A' \otimes_A -$ to the sequence for f produces the sequence for f'. By Exer. 6, if we can find some faithfully flat $A \to A'$ such that f' has a retraction r, then the theorem is proven. Take $A' = B$ with retraction $B \otimes_A B \to B$; $b_1 \otimes b_2 \mapsto b_1 b_2$ finishes the proof $\square$.

$$\begin{array}{ccc} B \otimes_A B & \to & B \\ \| & & \| \\ B' & \to & A' \end{array}$$

Remark The above theorem will be used to show that G, G^*, M_n, GL_n, Ω', ... are étale sheaves.

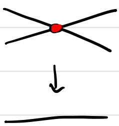
[**Corollary.** τ-cohomology of quasicoherent sheaves on an affine scheme vanishes for any τ coarser than fpqc-topology (e.g. Zariski, étale)]

2. Unramified morphisms

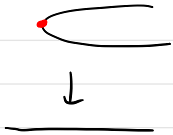
Recall that by definition, the residue field $k(\mathfrak{p})$ of a ring A at a prime $\mathfrak{p} \subseteq A$ is $k(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$

Definition A morphism of rings $\phi : A \to B$ is unramified at a prime $\mathfrak{q} \subseteq B$ if $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$ where $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ and $\mathfrak{q}B_{\mathfrak{q}} = \phi(\mathfrak{p})B_{\mathfrak{q}}$. It is unramified if it is of finite presentation and unramified at all $\mathfrak{q}$.

Example



$$B = \mathbb{C}[x,y]/(x+y)(x-y)$$
$$\uparrow$$
$$A = \mathbb{C}[x]$$



$$B = \mathbb{C}[t] \qquad t^2$$
$$\uparrow \qquad \uparrow$$
$$A = \mathbb{C}[t] \qquad t$$

are unramified everywhere except at the origin •

Suppose $k = \mathbb{F}_p(t)$. The morphism $k[x] \to \dfrac{k[x,y]}{y^p - xy - t}$ is unramified everywhere except $\mathfrak{q} = (x, y^p - t)$.

Remark A finite presentation morphism is unramified if and only if the diagonal $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_A B)$ is an open immersion if and only if $\Omega_{B/A} = 0$.

Next week: étale = flat + unramified

then

Topology II