Etale cohomology

1. Counting points with zeta functions

Question 1. Let X be a smooth projective variety over $\mathbb{F}_q$. How many elements are in the set

$$X(\mathbb{F}_{q^n}) = \hom_{\mathrm{Spec}(\mathbb{F}_q)}(\mathrm{Spec}(\mathbb{F}_{q^n}), X) \, ?$$

Equivalently,

Question 2. If $f_1, \ldots, f_e \in \mathbb{F}_q[t_0, \ldots, t_d]$ are the homogeneous polynomials defining X, how many solutions of $f_1, \ldots, f_e$ have in $\mathbb{F}_{q^n}$ for each n.

In order to work with $X(\mathbb{F}_{q^n})$ all at once, we introduce the zeta function.

residue field at x

$\deg(x) := [k(x) : \mathbb{F}_q]$

$$Z(X, t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n!} \right) \overset{(*)}{=} \prod_{x \in |X|} \left(1 - t^{\deg(x)}\right)^{-1}$$

closed points

Exercise 1 Applying log and using the identity $\log(1-T)^{-1} = \sum_{n=1}^{\infty} T^n/n$, prove $(*)$.

Remark Above definition works for any $\mathbb{F}_q$-variety X.

Remark $Y_0 \subset Y_1 \subset \ldots \subset Y_n = X$ closed subvarieties then we have

$$Z(X, t) = \prod_i Z(Y_i \setminus Y_{i-1}, t)$$

Question: Calculate $Z(X,t)$.

Example $|\mathbb{A}^d(\mathbb{F}_{q^n})| = |\mathbb{F}_{q^n}^d| = q^{nd}$

$$Z(\mathbb{A}^d,t) = \exp \sum_{n=1}^{\infty} (q^d t)^n/n = \exp\left(-\log(1-q^d t)\right) = \frac{1}{(1-q^d t)}$$

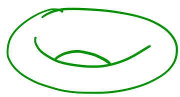
Example $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \dots \subset \mathbb{P}^n$, $\mathbb{A}^i \cong \mathbb{P}^i \setminus \mathbb{P}^{i-1}$

$$Z(\mathbb{P}^d,t) = \frac{1}{(1-t)(1-qt)\dots(1-q^d t)}$$

Example X = elliptic curve $/\mathbb{F}_q$

Using the action $\phi_\ell : T_\ell E \to T_\ell E$ of the Frobenius ϕ on the Tate module $T_\ell E = \varprojlim \big(\underbrace{\ker(E(\overline{\mathbb{F}}_q) \xrightarrow{\ell^n} E(\overline{\mathbb{F}}_q))}_{\in\, \mathbb{Z}/\ell^n\text{-mod}} \big) \in \mathbb{Z}_\ell\text{-mod}$

(lift E to $E/\mathbb{Z}_{(p)}$ $\rightsquigarrow$ $E(\mathbb{C}) \in$ complex analytic space $\cong \mathbb{C}/\Gamma$ $\quad \Gamma \cong \mathbb{Z} + \tau\mathbb{Z}$)



can calculate

$$|E(\mathbb{F}_{q^n})| = \deg(1-\phi^n) = \det(1-\phi_\ell^n)$$
$$= 1 - \alpha^n - \beta^n + q^n$$

for some $\alpha, \beta \in \mathbb{C}$ which are complex conjugate with absolute value $\sqrt{q}$.

Use the log argument we used in the $\mathbb{A}^d$ case, we get

$$Z(E,t) = \frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-qt)}$$

<u>Example</u> If X is a curve, the zeta function can be written in terms of divisors, and from there, in terms of linear systems of divisors of line bundles. Then using Riemann-Roch theorem for curves, one can calculate

$$Z(X,t) = \frac{f(t)}{(1-t)(1-qt)}$$

where $f(t) \in \mathbb{Z}[t]$ has degree $2g$ for $g =$ genus.

<u>Example</u> $X =$ smooth hypersurface.

$$Z(X,t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^{r-1}t)} \underbrace{\prod_{\alpha}}_{\substack{\uparrow \\ \text{finite} \\ \text{product}}} \Big(1 - \underbrace{C(\alpha)}_{\in \mathbb{C}}\, t^{\overbrace{\mu(\alpha)}^{\in \mathbb{N}}}\Big)^{\frac{(-1)^{r}}{\mu(\alpha)} \in \mathbb{N}}$$

$$|C(\alpha)| = q^{\frac{(r-1)\mu(\alpha)}{2}}$$

<u>Theorem</u> (Weil conjectures). Suppose X is a connected smooth projective variety of dimension n over $\mathbb{F}_q$. Then $Z(X,t)$ satisfies:

1) (Rationality) $Z(X,t)$ is a rational function of t.
(i.e., $Z(X,t) \in \mathbb{Z}(t) \cap \mathbb{Z}[[t]] \subset \mathbb{Z}((t))$)

2) (Functional equation) There is an integer e such that

$$Z(X, q^{-n}t^{-1}) = \pm q^{en/2} t^{e} Z(X,t)$$

3) (Riemann Hypothesis)

$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

where $P_i(t) \in \mathbb{Z}[t]$ have roots have absolute value $q^{-i/2}$.
Moreover, $P_0(t) = 1-t$ $\quad P_{2n}(t) = 1 - q^n t$.

e.g. $K = \mathbb{Q}$

4) (Betti numbers) Suppose $\exists$ a finite extension $K/\mathbb{Q}$, and homogeneous polynomials $f_1, \dots, f_c \in \mathcal{O}_K[t_0, \dots, t_d]$ where

e.g. $\mathcal{O}_K = \mathbb{Z}$

$\mathcal{O}_K$ = ring of integers of K, such that X is defined by the f_i mod $\mathfrak{p}$, for some prime $\mathfrak{p} \subset \mathcal{O}_K$ (so $\mathcal{O}_K/\mathfrak{p} = \mathbb{F}_q$).

e.g. $q = p$

Suppose further that the complex projective variety $X(\mathbb{C})$ defined by the f_i ($\mathcal{O}_K \subset K \hookrightarrow \mathbb{C}$) is smooth. Then

$$\deg P_i(t) = \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q})$$

where $X(\mathbb{C}) \subseteq \mathbb{P}^d_{\mathbb{C}}$ is given the complex analytic topology.

$\left(H^i(X(\mathbb{C}), F) = H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \right)$

for any char. zero field F

e.g. $\mathbb{Q}_\ell$.

<u>Exercise 2</u> Show that if s is a zero or pole of $Z(X, q^{-s})$ then the real part of s is $j/2$ for some $j \in \mathbb{Z}$.

<u>Remark</u> 1) complex conjugation acts on $H^i_{sing}(X(\mathbb{C}), \mathbb{Q})$.

2) $H^i_{dR}(X(\mathbb{C})) \underset{\text{canonical}}{\cong} H^i_{sing}(X(\mathbb{C}), \mathbb{C}) \rightsquigarrow H^i = \bigoplus_{p+q=i} H^{p,q}$ — Hodge theory

3) Frobenius acts on $H^i_{et}(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell)$ has deep relationship to the Hodge decomposition.

2. Counting points with cohomology

Why should cohomology help calculate $Z(X, t)$?

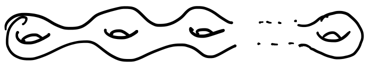
Consider an n-dimensional compact real manifold M.
Its cohomology groups are a sequence $H^0(M, \mathbb{Q}), H^1(M, \mathbb{Q}), H^2(M, \mathbb{Q}) \dots$
of $\mathbb{Q}$-vector spaces. The dimension of the ith space
is roughly how many "i-dimensional holes" M has in some sense.

<u>Example</u> $M = S^m = \{ (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \mid x_0^2 + \dots + x_m^2 = 1 \}$

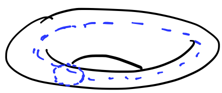
$$H^n(M, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } n = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{C}^m \setminus \{0\} \cong \mathbb{R}^{2m} \setminus \{0\} \twoheadrightarrow S^{2m-1}$$

$$H^n(\mathbb{C}^m \setminus \{0\}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2m-1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Example</u> $M =$  sphere with m-handles

$$H^n(M, \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0 \\ \mathbb{Q}^{2m} & n = 1 \\ \mathbb{Q} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$



Example $M = \mathbb{P}^m(\mathbb{C})$ ← considered as a real manifold

$$H^n(M, \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2, \ldots, 2m \\ 0 & \text{otherwise} \end{cases}$$

In general, when M is a connected, compact, real manifold,

1) (Finiteness) $\dim_{\mathbb{Q}} H^i(M, \mathbb{Q}) < \infty \quad \forall i$.

   If $M = X(\mathbb{C})$ for some complex algebraic variety X then $H^i(M, \mathbb{Q}) = 0$ for $i > 2 \dim_{\mathbb{C}} X(\mathbb{C})$

2) (Functoriality) For any continuous map $\phi: M \to N$ there are induced morphisms $\phi^i: H^i(N, \mathbb{Q}) \to H^i(M, \mathbb{Q})$ (s.t. $(\varphi \circ \phi)^i = \phi^i \circ \psi^i$)

3) (Poincare Duality) $\exists$ $H^{\dim M}(M, \mathbb{Q}) \cong \mathbb{Q}$ ← canonical and a natural perfect pairing

$$H^i(M, \mathbb{Q}) \times H^{\dim M - i}(M, \mathbb{Q}) \to H^{\dim M}(M, \mathbb{Q})$$

(i.e., $H^i \cong \left(H^{\dim M - i}\right)^\vee$ canonically)

4) (Lefschetz Trace Formula) Suppose $\phi: M \to M$ is a continuous map with only simple isolated fixed points. ($\{m \in M \mid \phi m = m\}$ is finite)

   Then

$$\#\{\text{fixed points}\} = \sum_{i=0}^{\dim M} (-1)^i \operatorname{tr}\left(\phi^i : H^i(M, \mathbb{Q}) \to H^i(M, \mathbb{Q})\right)$$

Now. Suppose we had H^i defined for algebraic varieties over finite fields.

$$X(\mathbb{F}_{q^m}) = \{\text{fixed points of } \mathrm{Frob}^m : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q)\}$$

We could hope that a version of (Lefschetz Trace Formula) gives

$$|X(\mathbb{F}_{q^m})| = \sum_{i=0}^{2\dim X} (-1)^i \, tr\left((\Phi^i : H^i \to H^i)^m\right)$$

where Φ = Frobenius. Then

$$Z(X,t) = \exp \sum_{n=1}^{\infty} \left(\sum_{i=0}^{\dim X} (-1)^i tr(\Phi^i)^n \right) \frac{t^n}{n}$$

$$= \prod_{i=1}^{2\dim X} \left(\exp \sum_{i=1}^{\infty} tr(\Phi^i)^n \frac{t^n}{n} \right)^{(-1)^i}$$

(use $\det(1-A)^{-1} = \exp \sum_{n=1}^{\infty} tr A^n / n$)

$$= \prod_{i=0}^{2\dim X} \det(1-\Phi^i \cdot t)^{(-1)^{i+1}}$$

So we get (Rationality) in the Weil conjectures.

An appropriate version of (Poincaré Duality) would give (Functional Equation), and if the new coh. groups are compatible with usual coh. groups for varieties over $\mathbb{C}$, we get (Betti numbers). Finally, this description suggests the polynomials in (Riemann Hypothesis) are

$$P_i(t) = \det(1-\Phi^i \cdot t).$$

If so, then the second part is reformulated as: eigenvalues of $\Phi^i : H^i \to H^i$ have absolute value $q^{-i/2}$.

$H^i_{et} : (\text{schemes})^{op} \to F\text{-vector spaces}$ char. zero field F

For X smooth projective / $\mathbb{C}$, we want

$$H^i_{et}(X, F) \cong H^i_{sing}(X(\mathbb{C}), F).$$

$$\begin{aligned} \overline{\mathbb{F}}_q &\to \overline{\mathbb{F}}_q \\ x &\mapsto x^q \end{aligned}$$ field automorphism

$$\mathbb{F}_q = \{ x \in \overline{\mathbb{F}}_q \mid x^q = x \}$$

(affine case)

$$X(\overline{\mathbb{F}}_q) = \left\{ (x_1, \dots, x_d) \in \overline{\mathbb{F}}_q^d \;\middle|\; \begin{array}{c} f_1(x) = 0 \\ \vdots \\ f_c(x) = 0 \end{array} \right\}$$

$$\begin{aligned} X(\overline{\mathbb{F}}_q) &\xrightarrow{\text{Frob}} X(\overline{\mathbb{F}}_q) \\ (x_1, \dots, x_d) &\mapsto (x_1^q, \dots, x_d^q) \end{aligned}$$

Schemes / $\mathbb{F}_q$

affine schemes: $\text{Frob} : \text{Spec}(A) \to \text{Spec}(A)$

$$\begin{aligned} A &\leftarrow A \\ a^q &\leftarrow\!\shortmid a \\ \mathfrak{p} &\mapsto \text{Frob}^{-1}(\mathfrak{p}) \\ &= \{ a \mid a^q \in \mathfrak{p} \} \\ &= \mathfrak{p} \end{aligned}$$

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{F}_{q^n}) \arrow[r] \arrow[dr, dotted] & \mathbb{P}^d_{\mathbb{F}_q} \\ & X \arrow[u, hook] \end{tikzcd} \qquad \longleftrightarrow \qquad (a_0 : \dots : a_d) \Big/ \mathbb{F}_{q^n}^*$$

$$X = \mathrm{Proj}\left(\frac{\mathbb{F}_q[T_0, \dots, T_d]}{\langle f_1, \dots, f_c \rangle} \right)$$

factors $\iff$ $f_i(a) = 0 \quad \forall \; i$