Functoriality II.

Etale vs. Pro Etale

Today: ① $\mathrm{Shv}(X_{et}) \xrightarrow{f.f.} \mathrm{Shv}(X_{proet})$

Image $= \{ F \mid F(\varprojlim U_\lambda) = \varinjlim F(U_\lambda) \}$

② $D^+(X_{et}) \xrightarrow{f.f.} D^+(X_{proet})$

Image $= \{ K \mid \underline{H}^n K \in \mathrm{Shv}(X_{et}) \ \forall n \}$

③ $\hat{D}(X_{et}) \cong \{ K \in D(X_{proet}) \mid H^n(K) \in \mathrm{Shv}(X_{et}) \ \forall n \}$

④ $D^+_{Ek}(X_{et}, \mathbb{Z}_\ell) \cong \left\{ K \in D^+(X_{proet}, \mathbb{Z}_\ell) : \begin{array}{l} H^n(K/\ell) \in \mathrm{Shv}_{et} \\ K \simeq R\varprojlim(K \overset{L}{\otimes} \mathbb{Z}/\ell^n) \end{array} \right\}$

⑤ $H^i_{cont.}(X_{et}, (\mathbb{Z}/\ell^n)_\bullet) \cong H^i(X_{proet}, \mathbb{Z}_\ell)$

Jannsen's Continuous Cohomology

§1 Étale to Proétale

Every étale morphism is weakly étale, so $\exists$ canonical

$$\nu : X_{\acute{e}t} \to X_{pro\acute{e}t}$$

ν sends coverings to coverings, so there is an induced adjunction

$$\nu^* : Shv(X_{\acute{e}t}) \rightleftarrows Shv(X_{pro\acute{e}t}) : \nu_*$$

$$F|_{X_{\acute{e}t}} \longleftarrow F$$

ν^* sends $F \in Shv(X_{\acute{e}t})$ to the <u>sheafification</u> of the presheaf

$$\underset{\in X_{pro\acute{e}t}}{U} \longmapsto \operatorname{colim}_{U \to \underbrace{V}_{\in X_{\acute{e}t}} \to X} F(V) \quad =: (\nu^p F)(U) \quad (*)$$

Exercise 1 Show that ν^* is exact.

Recall, last time we defined

$$X^{aff}_{pro\acute{e}t} \subseteq X_{pro\acute{e}t}$$

as the full subcategory of proétale X-schemes that can be written as

$$U = \varprojlim \operatorname{Spec}(A_\lambda)$$

for some filtered system of affine étale X-schemes $\operatorname{Spec}(A_\lambda)$.

$\boxed{\text{NB.}}$ $$Shv(X_{proét}) \simeq Shv(X^{aff}_{proét})$$

NB. Every morphism in $X^{aff}_{proét}$ is pro étale

Remark $$Shv(X_{zar}) \cong Shv(X^{aff}_{zar})$$

affine opens

Exercise 2 (Advanced)

Show that a presheaf on $X^{aff}_{proét}$ is a sheaf if and only if

1) $\forall$ surjection $V \to U$,
$$F(U) \to F(V) \rightrightarrows F(V \times_U V) \quad \text{is exact.}$$

2) $F|_{op(Y)}$ is a Zariski sheaf $\forall\, Y \in X^{aff}_{proét}$

Lemma 1. For $F \in Shv(X_{ét})$ and $U \in X^{aff}_{proét}$ with presentation $U = \varprojlim U_\lambda$, we have

$$(\nu^* F)(U) \boxed{=} \varinjlim F(U_\lambda). \overset{def}{=} \nu^p F(U)$$

to prove

That is, $(*)$ is already a sheaf on $X^{aff}_{proét}$.

Proof To show $(*)$ is a sheaf, suffices to check 1) and 2). I'll just do (1).

First, suppose $V \to U$ is a surjective étale morphism of finite presentation. Then $\exists$ a surjective étale morphism of finite presentation $V_\lambda \to U_\lambda$ for some λ such that $V = U \times_{U_\lambda} V_\lambda$.

Setting $V_m := U_m \times_{U_\lambda} V_\lambda$ for $m \geq \lambda$,

$$(\nu^p F)(U) \to (\nu^p F)(V) \rightrightarrows (\nu^p F)(V \times_U V) \qquad \bullet$$

$$\begin{array}{ccccc} \wr\wr & & \wr\wr & & \wr\wr \end{array} \quad \to \text{def. of } \nu^p F$$

$$\operatorname*{colim}_{m \geq \lambda} F(\underset{\in X_{et}}{\boxed{U_m}}) \to \operatorname{colim} F(\underset{\in X_{et}}{\boxed{V_m}}) \rightrightarrows \operatorname{colim} F(\underset{\in X_{et}}{\boxed{V_m \times_{U_m} V_m}}) \qquad \bullet$$

Since filtered colimits preserve finite limits, exactness of $\bullet$ follows from exactness of $\bullet$.

Now let $V \to U$ be a surj. pro étale morphism in $X^{\text{aff}}_{\text{proet}}$. Choose a presentation $V = \varprojlim V_\lambda \to U$.

$$(\nu^p F)(U) \to (\nu^p F)(V) \rightrightarrows (\nu^p F)(V \times_U V) \qquad \bullet$$

$$\begin{array}{ccccc} \| & & \wr\wr & & \wr\wr \end{array}$$

$$(\nu^p F)(U) \to \operatorname{colim}(\nu^p F)(V_\lambda) \rightrightarrows \operatorname{colim}(\nu^p F)(V_\lambda \times_{U_\lambda} V_\lambda) \qquad \bullet$$

$\bullet$ is a filtered colimit of sequences of the form $\bullet$ so $\bullet$ is exact, so $\bullet$ is exact.

Zariski case is similar. $\square$

Aside:

$$\begin{tikzcd}
PSh(X^{\text{aff}}_{et}) \arrow[r, shift left, "\nu^p"] \arrow[d, shift right, "a"'] & PSh(X^{\text{aff}}_{\text{proet}}) \arrow[l, shift left, "\nu_p"] \arrow[d, shift right, "a"'] \\
Shv(X^{\text{aff}}_{et}) \arrow[r, shift left, "\nu^*"] \arrow[u, shift right, "i"'] & Shv(X^{\text{aff}}_{\text{proet}}) \arrow[l, shift left, "\nu_*"] \arrow[u, shift right, "i"']
\end{tikzcd}$$

$i \circ \nu_* = \nu_p \circ i$

$a \nu^p i =: \nu^*$

Lemma If we use affines, then $\nu^p F \in Shv(X^{\text{aff}}_{\text{proet}})$

$X_{et} \subset X_{\text{proet}}$ is fully faithful so $\nu_p \nu^p F = F$

Exercise 3 Prove that filtered colimits preserve exact sequences.

Example 2 If k^{sep}/k is not finite, then $\hom(-, \mathrm{Spec}(k^{sep}))$ is in $\mathrm{Shv}(k_{pro\acute{e}t})$ but not in $\mathrm{Shv}(k_{\acute{e}t})$.

Lemma 3

$$\nu^* : \mathrm{Shv}(X_{\acute{e}t}) \to \mathrm{Shv}(X_{pro\acute{e}t})$$

is fully faithful. Its essential image are sheaves satisfying:

$$(\mathrm{Cl}_{\cdot}) \qquad F(U) = \varinjlim F(U_\lambda) \text{ for all } U = \varprojlim U_\lambda \in X^{aff}_{pro\acute{e}t}$$

(Follows from Lemma 1, see notes).

Definition Sheaves in the image of $\mathrm{Shv}(X_{\acute{e}t}) \subseteq \mathrm{Shv}(X_{pro\acute{e}t})$ are called <u>classical</u>.

Proposition 5 For any $K \in D^+(X_{\acute{e}t})$, the map $K \to R\nu_* \nu^* K$ is a weak equivalence. Moreover, for $U = \varprojlim U_\lambda \in X^{aff}_{pro\acute{e}t}$ then

$$R\Gamma_{pro\acute{e}t}(U, \nu^* K) = \varinjlim R\Gamma_{\acute{e}t}(U_\lambda, K).$$

Proof See notes. Reduce (eventually) to showing that $\check{H}^p(U, \nu^* \underset{\substack{\uparrow \\ \text{injective}}}{I}) = 0$ for $p > 0$

Then note that

$$\check{H}^p_{\text{proét}}(\underbrace{U}_{\varprojlim U_\lambda}, \nu^* \mathcal{I}) = \varinjlim \underbrace{\check{H}^p(\underbrace{V_\lambda / U_\lambda}_{\text{étale coverings}}, \mathcal{I})}_{= H^p(\mathcal{I}(V_\lambda) \to \mathcal{I}(V_\lambda \times_{U_\lambda} V_\lambda) \to \cdots)} . \qquad \square$$

$$\operatorname{colim}_{\substack{\longrightarrow \\ V/U \\ \text{proétale}}} \check{H}^p(V/U, \nu^* \mathcal{I})$$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V_\lambda \arrow[r] \arrow[dl] & V_{\lambda m} \arrow[d] \\
U \arrow[rr] & & U_m
\end{tikzcd}$$

Corollary 6 For any scheme X

$$\nu^+ : D^+(X_{\text{ét}}) \to D^+(X_{\text{proét}})$$

is fully faithful, its essential image are those K s.t. $\underline{H}^n K$ are classical $\forall\ n \in \mathbb{Z}$.

§2 Left completion

Recall $\hat{D}(X_{\text{ét}}) \subset D(X_{\text{ét}}^{\mathbb{N}})$ is the subcategory of those $(\cdots \to K_2 \to K_1 \to K_0)$ s.t.

1) $K_n \in D^{\geq -n}(X_{\text{ét}})$ (i.e., $\underline{H}^i K_n = 0$ $i < -n$)
2) $\tau^{\geq -n} K_{n+1} \to K_n$ is a weak equivalence. (i.e., $\underline{H}^i K_{n+1} \simeq \underline{H}^i K_n$ $i \geq -n$)

$\hat{D}(X_{proet})$ is defined similarly, but

$Shv(X_{proet})$ is replete, so $D(X_{proet})$ is left complete.

That is, $\tau : D(X_{proet}) \rightleftarrows \hat{D}(X_{proet}) : R\lim$

$$K \longmapsto (\dots \to \tau^{\geq -2}K \to \tau^{\geq -1}K \to \tau^{\geq 0}K)$$

$$R\lim K_n \longleftarrow (\dots \to K_2 \to K_1 \to K_0)$$

are inverse equivalences of categories.

$$\begin{array}{ccc} D(X_{et}) & \xrightarrow{\nu^*} & D(X_{proet}) \\ \tau \downarrow & & \tau \downarrow \wr\!\wr \ \uparrow R\lim \\ \hat{D}(X_{et}) & \xrightarrow{\nu^*} & \hat{D}(X_{proet}) \end{array}$$

leads to an adjunction

$$\underbrace{(R\lim)}_{\cong \tau^{-1}} \circ \nu^* : \hat{D}(X_{et}) \rightleftarrows D(X_{proet}) : R\nu_* \circ \underbrace{\tau}_{\simeq (R\lim)^{-1}}$$

Def Let $D_{cc}(X_{proet}) \subseteq D(X_{proet})$ be the full subcategory of K s.t. $H^n K$ is classical $\forall n$.

Proposition There is an induced adjunction

$$\nu^* : D(X_{et}) \rightleftarrows D_{cc}(X_{proet}) : R\nu_*$$

isomorphic to the adjunction

$$\tau : D(X_{et}) \rightleftarrows \hat{D}(X_{et}) : R\lim$$

In particular,

$$\hat{D}(X_{et}) \cong D_{cc}(X_{proet}) \overset{ff}{\subseteq} D(X_{proet})$$

Proof Mostly formal, main ingredient is Corollary 6 $\square$

§3 Ekedahl's ℓ-adic sheaves

— see notes —

§4 Jannsen's continuous cohomology.

Consider

$$\gamma : \mathrm{Shv}(X_{et})^{\mathbb{N}} \to \mathrm{Ab}$$

$$\underset{\in \, \mathrm{Shv}(X_{et})^{\mathbb{N}}}{(\cdots \to F_2 \to F_1 \to F_0)} \longmapsto \varprojlim_n F_n(X)$$

Ims

$$R\gamma : D(\mathrm{Shv}(X_{et})^{\mathbb{N}}) \to D(\mathrm{Ab})$$

__Def__ The continuous cohomology of (F_n) is

$$H^i_{cont}(X, (F_n)) := H^i R\gamma((F_n)).$$

__Proposition__ If $F_{n+1} \to F_n$ are surjective, then

$$H^i_{cont}(X, (F_n)) \cong H^i_{proet}(X, \varprojlim \nu^* F_n)$$

In particular,

$$H^i_{cont}(X, (\mathbb{Z}/\ell^n)) \cong H^i_{proet}(X, \mathbb{Z}_\ell)$$

$\gamma = \varprojlim \Gamma(-,-) = \Gamma(-, \varprojlim -)$

<u>Proof</u>

$$R\gamma((F_n)) \cong R\Gamma_{et}(X, R\varprojlim F_n)$$

$$\cong R\varprojlim R\Gamma_{et}(X, F_n)$$

$$\overset{\text{Prop. 5}}{\cong} R\varprojlim R\Gamma_{proet}(X, \nu^* F_n)$$

$$\cong R\Gamma_{proet}(X, R\varprojlim \nu^* F_n)$$

$$\cong R\Gamma_{proet}(X, \varprojlim \nu^* F_n)$$

$\square$

<u>Exercise 7</u> show that $R\varprojlim F_n = \varprojlim F_n$ if the $F_{n+1} \to F_n$ are surjective.

<u>Proposition 5</u> For any $K \in D^+(X_{et})$, the map $K \to R\nu_* \nu^* K$ is a weak equivalence. Moreover, for $U = \varprojlim U_\lambda \in X_{proet}^{aff}$ then

$$R\Gamma_{proet}(U, \nu^* K) = \varinjlim R\Gamma_{et}(U_\lambda, K).$$

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

$$D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{B}) \xrightarrow{RG} D^+(\mathcal{C})$$

$$R(G \circ F) \cong RG \circ RF$$
$$\Updownarrow$$
F sends injectives to acyclics

$$\begin{tikzcd}
\mathrm{Shv}(X_{et})^{\mathbb{N}} \arrow[r, "\lim"] \arrow[d, "{\Gamma(X,-)}"'] \arrow[dr, "\gamma"] & \mathrm{Shv}(X_{et}) \arrow[d, "{\Gamma(X,-)}"] \\
Ab^{\mathbb{N}} \arrow[r, "\lim"] & Ab
\end{tikzcd}$$

$$H^1_{cont} = 0 \overset{?}{\Rightarrow} \{H^1_{et}(F_n)\} = 0$$

$\in Ab$ $\qquad\qquad$ $\in \mathrm{Pro}(Ab)$

$$\begin{tikzcd}
D\left(\mathrm{Shv}(X_{et})^{\mathbb{N}}\right) \arrow[d, "{R\Gamma(X,-)}"'] & \\
D(Ab^{\mathbb{N}}) \arrow[r, "R\lim"] & D(Ab
\end{tikzcd}$$

$$R^0 F = F$$

$$\begin{array}{ccc}
H^1(\Gamma(X, I^\cdot)) & & \\
\| & & \\
H^1(X,-) & = & R^1\Gamma(X,-)
\end{array}$$

$$\mathrm{Shv}(X_{et})^{\mathbb{N}} \to Ab^{\mathbb{N}}$$