(Pro)Etale Cohomology
Lecture 13. Functoriality II

In this talk we compare the pro-étale site with the étale site. First we will see that $\mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{proét}})$ is fully faithful, and a sheaf is in the image if and only if it "commutes" with filtered limits.

$$\mathrm{Image}\Big(\mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{proét}})\Big) = \Big\{F : F(\varprojlim_{\lambda\in\Lambda} U_\lambda) = \varinjlim_{\lambda\in\Lambda} F(U_\lambda)\Big\}.$$

Similarly, a complex is in the image of $D^+(X_{\mathsf{et}}) \to D^+(X_{\mathsf{proét}})$ if and only if its cohomology is in the image of $\mathsf{Shv}_{\mathsf{et}}(X)$.

$$\mathrm{Image}\Big(D^+(X_{\mathsf{et}}) \to D^+(X_{\mathsf{proét}})\Big) = \Big\{K : H^nK \in \mathsf{Shv}(X_{\mathsf{et}})\ \forall\ n\Big\}.$$

Then we see how the pro-étale site offers a technically simpler way to left complete the étale site. There is a canonical identification of $\widehat{D}(X_{\mathsf{et}})$ with the subcategory of $D(X_{\mathsf{proét}})$ of objects whose cohomology lies in the image of X_{et}.

$$\widehat{D}(X_{\mathsf{et}}) \cong \Big\{K \in D(X_{\mathsf{proét}}) : H^nK \in \mathsf{Shv}(X_{\mathsf{et}})\ \forall\ n\Big\}.$$

We also show how the pro-étale site can be used to recover the classical derived category of l-adic sheaves,

$$D^+_{Ek}(X_{\mathsf{et}}, \mathbb{Z}_\ell) \cong \left\{K \in D^+(X_{\mathsf{proét}}, \mathbb{Z}_\ell) : \begin{array}{c} H^n(K/\ell) \in \mathsf{Shv}(X_{\mathsf{et}})\ \forall\ n, \text{ and} \\ K \cong R\varprojlim_n (K \overset{L}{\otimes} \mathbb{Z}/\ell^n) \end{array}\right\}.$$

and Jannsen's continuous cohomology,

$$H^i_{cont}(X_{\mathsf{et}}, (\mathbb{Z}/\ell^n)_\bullet) \cong H^i(X_{\mathsf{proét}}, \mathbb{Z}_\ell).$$

1 From étale to pro-étale

Since every étale morphism is weakly étale, for any scheme X we have a canonical fully faithful functor

$$\nu : X_{\mathsf{et}} \to X_{\mathsf{pro\acute{e}t}}.$$

Remark 1. If X is the spectrum of a separably closed field, then (when restricted to affines), ν is canonically identified with the inclusion of the category of finite sets into the category of profinite sets.

As explained in "Lecture 5: Functoriality I", such a functor leads to an adjunction

$$\nu^p : \mathsf{PreShv}(X_{\mathsf{et}}) \leftrightarrows \mathsf{PreShv}(X_{\mathsf{pro\acute{e}t}}) : \nu_p$$

where $(\nu_p F)(Y)) = F(\nu(Y))$ and ν^p can be calculated as

$$(\nu^p F)(U) = \varinjlim_{U \to V \to X} F(V),$$

for $U \in X_{\mathsf{pro\acute{e}t}}$ where the colimit is indexed by factorisations $U \to V \to X$ such that $V \in X_{\mathsf{et}}$.

Definition 2. *In the last lecture we defined*

$$X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}} \subseteq X_{\mathsf{pro\acute{e}t}}$$

as the full subcategory of weakly étale X-schemes that can be written as $\mathrm{Spec}(A) = \varprojlim \mathrm{Spec}(A_\lambda)$ *for some filtered system of affine étale X-schemes* $\mathrm{Spec}(A_\lambda)$.

Remark 3. We note for later use that every morphism in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ is also pro-étale [BS, Lem.4.2.2]. In other words, for every $V \to U$ in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ we have $V \in U^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$.

Lemma 4. *Let X be a scheme. A presheaf F is in the image of the composition*

$$\mathsf{PreShv}(X_{\mathsf{et}}) \xrightarrow{\nu^p} \mathsf{PreShv}(X_{\mathsf{pro\acute{e}t}}) \xrightarrow{(-)|_{X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}}} \mathsf{PreShv}(X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}})$$

if and only if

$$F(\varprojlim U_\lambda) = \varinjlim F(U_\lambda)$$

for every filtered system $(U_\lambda)_{\lambda \in \Lambda}$ of affine étale X-schemes U_λ.

Proof. ($\Rightarrow$) By [EGA IV-3, Prop.8.13.1][1], for any morphism $V \to X$ locally of finite presentation, we have $\hom_X(\varprojlim U_\lambda, V) = \varinjlim \hom_X(U_\lambda, V)$, so the system $(U_\lambda)_{\lambda \in \Lambda}$ is cofinal in the system of all factorisations $\varprojlim U_\lambda \to V \to X$ through $V \in X_{\mathsf{et}}$.

($\Leftarrow$) Given a presheaf F commuting with filtered limits as in the statement, define $G \in \mathsf{PreShv}(X_{\mathsf{et}})$ by $G(Y) = \varprojlim_{Y_\mu \to Y} F(Y_i)$ where the limit is over all affine étale Y-schemes Y_μ.[2] If Y is affine, then it is initial in the system of $(Y_\mu \to Y)$, so $G(Y) = F(Y)$. Hence, inserting the definitions, we find that $\nu^p(G)|_{X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}} = F$. □

[1] Or [Stacks project, 01ZC].

[2] That is, we *right* Kan extend from $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ to $X_{\mathsf{pro\acute{e}t}}$ then apply ν_p.

Since ν sends étale covering families to proétale coverings families (cf. Exercise 9 in "Lecture 5: Functoriality I"), there adjunction (ν^p, ν_p) induces an adjunction

$$\nu^* : \mathsf{Shv}(X_{\mathsf{et}}) \rightleftarrows \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}) : \nu_*$$

such that $\nu_* = \nu_p$ and $\nu^* = a \circ \nu^p$ where $a : \mathsf{PreShv}(X_{\mathsf{pro\acute{e}t}}) \to \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ is the sheafification functor.

Exercise 1. Show that ν^* is exact. That is, show that it preserves finite colimits and finite limits of sheaves. Hint: Look at the proof of Lemma 10 in "Lecture 5. Functoriality I".

Exercise 2 (Advanced)**.** Following the strategy of Exercise 9 from "Lecture 3. Topology I" from last quarter show that a presheaf on $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ is a sheaf if and only if:

1. For any surjection $V \to U$ in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$, the sequence $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.
2. $F|_{Op(Y)}$ is a Zariski sheaf for each $Y \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$.

Lemma 5 ([Lem.5.1.1])**.** *For $F \in \mathsf{Shv}(X_{\mathsf{et}})$ and $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ with presentation $U = \varprojlim_\lambda U_\lambda$, we have $(\nu^* F)(U) = \varinjlim_\lambda F(U_\lambda)$. In other words, $\nu^p F$ already satisfies the sheaf condition on $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ before sheafification.*

Proof. We want to show that the *presheaf* $\nu^p F$ on $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ is a sheaf. It suffices to check the two conditions in Exercise 2. We check the first one, since the second one is similar. First suppose that $V \to U$ is a surjective *étale* (i.e., not a general pro-étale) morphism of finite presentation in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$, and let $U = \varprojlim U_\lambda$ be a presentation for U. Then for some λ, there exists a surjective étale morphism $V_\lambda \to U_\lambda$ such that $V = U \times_{U_\lambda} V_\lambda$,[3] [Stacks project, 01ZM, 07RP, 081D]. Then the sheaf condition for $V \to U$ is the filtered colimit of the sheaf conditions for $V_\mu := U_\mu \times_{U_\lambda} V_\lambda \to U_\mu$ for $\mu \geq \lambda$

$$\begin{array}{ccccc}
\nu^p F(U) & \longrightarrow & \nu^p F(V) & \rightrightarrows & \nu^p F(V \times_U V) \\
\| & & \| & & \| \\
\varinjlim_{\mu \geq \lambda} F(U_\mu) & \longrightarrow & \varinjlim_{\mu \geq \lambda} F(V_\mu) & \rightrightarrows & \varinjlim_j F(V_\mu \times_{U_\mu} V_\mu)
\end{array}$$

Here the vertical equalities are Lemma 4. Since F is an étale sheaf, the lower row is a filtered colimit of exact sequences. Filtered colimits preserve finite limits, so the lower row is exact, and therefore the upper row is exact.

[3] Note, any morphism between affines that is locally of finite presentation, is in fact, of finite presentation.

Now let $V = \varprojlim V_\lambda \to U$ be a presentation for a general surjective morphism in $X^{\mathsf{aff}}_{\mathsf{proét}}$. Then, again, the sheaf condition for $V \to U$ is the filtered colimit of the sheaf conditions for the $V_\lambda \to U$.

$$\begin{array}{ccccc}
\nu^p F(U) & \longrightarrow & \nu^p F(V) & \rightrightarrows & \nu^p F(V \times_U V) \\
\| & & \| & & \| \\
\nu^p F(U) & \longrightarrow & \varinjlim_\lambda \nu^p F(V_\lambda) & \rightrightarrows & \varinjlim_\lambda \nu^p F(V_\lambda \times_U V_\lambda)
\end{array}$$

Again, the vertical equalities are Lemma 4. We have just shown that the lower line is a filtered colimit of exact sequences (because each $V_\lambda \to U$ is surjective étale of finite presentation), so it follows that the upper line is exact.

For the Zariski case, since affine schemes are quasicompact, and basic opens $\mathrm{Spec}(A[a^{-1}]) \subseteq \mathrm{Spec}(A)$ form a base for the Zariski topology, it suffices to check the sheaf condition for coverings of the form $\{\mathrm{Spec}(A[a_i^{-1}]) \to \mathrm{Spec}(A)\}_{i=1}^n$.[4] If $A = \varinjlim A_\lambda$ is a presentation for the ind-étale algebra A, then we descend the covering $\{\mathrm{Spec}(A[a_i^{-1}]) \to \mathrm{Spec}(A)\}_{i=1}^n$ to some A_λ as in the previous case, and argue as in the previous case. $\square$

Exercise 3. Prove the claim that filtered colimits preserve exact sequences. That is, suppose that Λ is a filtered category, and $A, B, C : \Lambda \to Ab$ are functors from Λ to the category of abelian groups, and $A \to B \to C$ are natural transformations such that for each $\lambda \in \Lambda$, the sequence

$$0 \to A_\lambda \to B_\lambda \to C_\lambda \to 0$$

is exact. Then show that

$$0 \to \varinjlim_{\lambda} A_\lambda \to \varinjlim_{\lambda} B_\lambda \to \varinjlim_{\lambda} C_\lambda \to 0$$

is an exact sequence.

Example 6. Suppose k is a field with separable closure k^{sep} such that k^{sep}/k is not a finite extension. Then consider the sheaf $F(-) = \hom(-, \mathrm{Spec}(k^{sep}))$ on the category $\mathrm{Spec}(k)_{\mathsf{proét}}$. For any $\mathrm{Spec}(A) \in \mathrm{Spec}(k)_{\mathsf{proét}}$ we have $F(\mathrm{Spec}(A)) = \varnothing$. However, $\mathrm{Spec}(k^{sep}) \in \mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{proét}}$ and we have $F(\mathrm{Spec}(k^{sep})) \neq \varnothing = \varinjlim_{k \subseteq L \subseteq k^{sep}} F(\mathrm{Spec}(L))$ where the limit is over finite subextensions of k^{sep}/k. So F is not in the image of ν^*.

Lemma 7 ([Lem.5.1.2])**.** *The functor*

$$\nu^* : \mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{proét}})$$

is fully faithful. Its essential image consists of those sheaves F which satisfy:

[4] This reduction can be proven in a similar way to Exercise 9 from "Lecture 3. Topology I".

(Cla) $F(U) = \varinjlim_\lambda F(U_i)$ *for any* $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ *with presentation* $U = \varprojlim_\lambda U_\lambda$.

Proof. A left adjoint is fully faithful if and only if the unit $\mathrm{id} \to \nu_*\nu^*$ is an isomorphism.[5] Isomorphisms of sheaves can be detected locally, cf. Exercise 4 below, and in X_{et} every scheme is locally affine. For any affine étale $U \to X$, the constant diagram (U) is a presentation for U. So then by Lemma 5 above we have $F(U) \cong \nu_*\nu^*F(U)$ for any $F \in \mathsf{Shv}(X_{\mathsf{et}})$.

For the second part, suppose $G \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ satisfies the conditions of the lemma. To show that G is in the image of ν^*, we will show that $\nu^*\nu_*G \to G$ is an isomorphism. Since every weakly étale X-scheme can be covered by affine proétale X-schemes [BS, Thm.2.3.4] ($\leftarrow$ this is a difficult theorem), it suffices to show that $\nu^*\nu_*G(U) \to G(U)$ is an isomorphism for every $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$, cf. Exercise 4 below. But this follows from Lemma 5 and the hypothesis. ☐

Exercise 4. Prove the claim in the above proof that a morphism of sheaves $\phi : F \to G$ on a site (C, τ) is an isomorphism if and only if for every $X \in C$, there is a τ-covering family $\{U_i \to X\}_{i \in I}$ such that $F(U_i) \to G(U_i)$ is an isomorphism for all i.

Hint: The hypothesis is for every $X \in C$, in particular, for any cover $\{U_i \to X\}$ with ϕ an isomorphism on each U_i, there are also covers $\{W_{ijk} \to U_i \times_X U_j\}_{k \in K_{ij}}$ with ϕ an isomorphism on each W_{ijk}.

Definition 8. *Sheaves in the image of* $\mathsf{Shv}(X_{\mathsf{et}}) \subseteq \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$*, that is sheaves satisfying the condition (Cla) in Lemma 7 called* classical.

The recognition of classical sheaves can be used to show that

$$D^+(X_{\mathsf{et}}) \to D^+(X_{\mathsf{pro\acute{e}t}})$$

is also fully faithful.

Proposition 9 ([Cor.5.1.6])**.** *For any* $K \in D^+(X_{\mathsf{et}})$*, the map* $K \to R\nu_*\nu^*K$ *is an equivalence. Moreover, if* $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ *has presentation* $U = \varprojlim_\lambda U_\lambda$ *then* $R\Gamma_{\mathsf{pro\acute{e}t}}(U, \nu^*K) = \varinjlim_\lambda R\Gamma_{\mathsf{et}}(U_\lambda, K)$.

Proof. (Probably omitted from lecture). The first part follows from the second part. Indeed, to prove the first part, it suffices to show that the morphism of presheaves $R\Gamma_{\mathsf{et}}(U, K) \to R\Gamma_{\mathsf{et}}(U, R\nu_*\nu^*K)$ is an isomorphism for every affine étale X-scheme U, cf.Ex.4. But if the second part is true, then for every such U, we have $R\Gamma_{\mathsf{et}}(U, K) = R\Gamma_{\mathsf{pro\acute{e}t}}(U, \nu^*K) = R\Gamma_{\mathsf{et}}(U, R\nu_*\nu^*K)$.[6]

For the second part, it suffices to consider the case that K is concentrated in degree zero. Indeed, if its true for K concentrated in degree zero, then its true for K concentrated in any one degree. Then by the truncation triangles $\tau^{\leq n-1}K \to \tau^{\leq n}K \to \underline{H}^nK[n] \to \tau^{\leq n-1}K[1]$ it is true for any bounded complex.

[5] This is because the composition $\hom(X, Y) \to \hom(LX, LY) \cong \hom(X, RLY)$ induced by the unit $Y \to RLY$.

[6] The second equality comes from the fact that $\Gamma_{\mathsf{et}}(-, \nu_*-) = \Gamma_{\mathsf{pro\acute{e}t}}(-, -)$, and $R(F \circ G) = RF \circ RG$ for composable left exact functors.

Finally, since $K \cong \varinjlim_n \tau^{\leq n} K$, if it's true for bounded complexes, it's true for bounded below complexes.

So now we are trying to prove that for any $F \in \mathsf{Shv}(X_{\mathsf{et}}, \mathrm{Ab})$ we have $H^n_{\mathsf{pro\acute{e}t}}(U, \nu^* F) = \varinjlim_\lambda H^n_{\mathsf{et}}(U_\lambda, F)$ when $U = \varprojlim U_\lambda$ with $U_\lambda \in X_{\mathsf{et}}$ affine. When $n = 0$ this is just Lemma 5. Now choose a short exact sequence $0 \to F \to I \to G \to 0$ in $\mathsf{Shv}(X_{\mathsf{et}})$ with I injective, and use induction on n. By the morphism of long exact sequences[7]

$$\begin{array}{ccccccccc} \dots \to & H^n_{\mathsf{et}}(U_\lambda, I) & \to & H^n_{\mathsf{et}}(U_\lambda, G) & \to & H^{n+1}_{\mathsf{et}}(U_\lambda, F) & \to & H^{n+1}_{\mathsf{et}}(U_\lambda, I) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & H^n_{\mathsf{pro\acute{e}t}}(U, \nu^* I) & \to & H^n_{\mathsf{pro\acute{e}t}}(U, \nu^* G) & \to & H^{n+1}_{\mathsf{pro\acute{e}t}}(U, \nu^* F) & \to & H^{n+1}_{\mathsf{pro\acute{e}t}}(U, \nu^* I) & \to \dots \end{array}$$

and the fact that since I is injective $H^n_{\mathsf{et}}(U_\lambda, I) = 0$ for all $n > 0$,[8] it suffices to show that $H^n_{\mathsf{pro\acute{e}t}}(U, \nu^* I) = 0$ for $n > 0$.

To show $H^n_{\mathsf{pro\acute{e}t}}(U, \nu^* I) = 0$ for $n > 0$, by the Čech-to-sheaf cohomology spectral sequence [Milne, Prop.III.2.3]

$$\check{H}^p(U, \underline{H}^q_{\mathsf{pro\acute{e}t}} \nu^* I) \Rightarrow H^{p+q}_{\mathsf{pro\acute{e}t}}(U, \nu^* I)$$

and induction on n, it suffices to show that the Čech cohomology $\check{H}^p(U, \nu^* I)$ vanishes.[9]

Similar to what happened in the proof of Lemma 5, to calculate this Čech cohomology, it suffices to take the colimit over coverings of U the form $V := U \times_{U_\lambda} V_\lambda$ for $\lambda \in \Lambda$ and étale coverings $V_\lambda \to U_\lambda$ in $X^{\mathsf{aff}}_{\mathsf{et}}$. By Lemma 5, for such a covering we have

$$\check{H}^n(V/U, \nu^* I) = \varinjlim_{\mu \geq \lambda} \check{H}^n(V_\mu / U_\mu, I)$$

where $V_\mu := U_\mu \times_{U_\lambda} V_\lambda$. The right hand side vanishes for $n > 0$ because I is injective in $\mathsf{Shv}(X_{\mathsf{et}})$. □

Corollary 10 ([BS, Prop.5.2.6(1),(3)]). *Let X be a scheme. Then the functor*

$$\nu^* : D^+(X_{\mathsf{et}}) \to D^+(X_{\mathsf{pro\acute{e}t}})$$

is fully faithful, and its essential image consists of those complexes K whose cohomology sheaves are classical.

Proof. Fully faithfulness follows from Prop.9, since a left adjoint L is fully faithful if and only if the unit $\mathrm{id} \to RL$ is a natural isomorphism.

[7] Note ν^* is exact, so $0 \to \nu^* F \to \nu^* I \to \nu^* G \to 0$ is again a short exact sequence.

[8] One quick way to see this is to note that I is its own injective resolution.

[9] N.B. We automatically have $\check{H}^0(U, \underline{H}^q_{\mathsf{pro\acute{e}t}} \nu^* I) = 0$ for $q > 0$ and $\check{H}^0(U, \underline{H}^0_{\mathsf{pro\acute{e}t}} \nu^* I) = \underline{H}^0_{\mathsf{pro\acute{e}t}}(U, \nu^* I)$.

For the essential image, we use an argument that appeared in the proof of Prop.9. Certainly, by definition, if a complex $K \in D^+(X_{\mathsf{proét}})$ has only one nonzero cohomology sheaf, and that cohomology sheaf is classical, i.e., in the image of $\nu^* : \mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{proét}})$, then K is in the image of ν^*. By the truncation triangles $\tau^{\leq n-1}K \to \tau^{\leq n}K \to \underline{H}^n K[n] \to \tau^{\leq n-1}K[1]$ and induction on the number of non-zero cohomology sheaves, it is true for any bounded complex. Finally, since $K \cong \varinjlim_n \tau^{\leq n}K$, if it's true for bounded complexes, it's true for bounded below complexes. ☐

2 Left completion via the pro-étale site

Recall that the left completion $\widehat{D}(X_{\mathsf{et}})$ of $D(X_{\mathsf{et}})$ is the subcategory of $D(X_{\mathsf{et}}^{\mathbb{N}})$ consisting of those sequence of chain complexes $(\cdots \to K_2 \to K_1 \to K_0)$ in $Ch(\mathsf{Shv}(X_{\mathsf{et}})^{\mathbb{N}})$ such that

1. $\underline{H}^i K_n = 0$ for $i < -n$,
2. $\underline{H}^i K_{n+1} = \underline{H}^i K_n$ for $i \geq -n$.

Here $\underline{H}^i K$ is the ith cohomology sheaf of K.

Proposition 11 ([Prop.5.3.2])**.** *Let X be a scheme. The functor*

$$\widehat{D}(X_{\mathsf{et}}) \to D(X_{\mathsf{proét}})$$

$$(\cdots \to K_2 \to K_1 \to K_0) \mapsto R\varprojlim \nu^* K_n$$

is fully faithful. It's essential image is the full subcategory of those $K \in D(X_{\mathsf{proét}})$ such that each cohomology sheaf $\underline{H}^i K$ is classical.

Notes about the proof. Most of the proof is formal. The main non-formal ingredients are Cor.10, the fact shown in Exercise 1 that $\nu^* : \mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{proét}})$ is exact, and of course the equivalence $D(X_{\mathsf{proét}}) \cong \widehat{D}(X_{\mathsf{proét}})$ coming from the fact that $\mathsf{Shv}(X_{\mathsf{proét}})$ is replete. ☐

3 Ekedahl's l-adic sheaves via the pro-étale site

Suppose l is a prime, and X is a $\mathbb{Z}[1/l]$-scheme. The l-adic cohomology is classically defined as

$$H^i_{\mathsf{et}}(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\mathsf{et}}(X, \mathbb{Z}/\ell^n).$$

On the other hand, it is useful to have a description of cohomology in terms of derived categories. We have

$$\hom_{D(X_{\mathsf{et}}, \mathbb{Z}/\ell^n)}(\mathbb{Z}/\ell^n, \mathbb{Z}/\ell^n[i]) = H^i_{\mathsf{et}}(X; \mathbb{Z}/\ell^n)$$

but to extend this to l-adic cohomology, we would need to consider something like

$$\varprojlim_n D(X_{\mathsf{et}}, \mathbb{Z}/\ell^n)$$

but categories are only well-defined up to equivalence, so limits of categories are technically complicated to define.

Exercise 5. In this exercise we show that naïve inverse limits of categories are not well-defined up to equivalence of categories. Let $\cdots \to C_2 \to C_1 \to C_0$ be a system of functors of small categories. Define $\varprojlim C_n$ to be the category with set of objects

$$Ob_{\varprojlim C_n} = \varprojlim Ob_{C_n}.$$

Given objects $x = (\dots, x_2, x_1, x_0)$ and $y = (\dots, y_2, y_1, y_0)$ in $\varprojlim C_n$ define

$$\hom_{\varprojlim C_n}(x, y) = \varprojlim \hom_{C_n}(x_n, y_n).$$

1. For an abelian group A, let BA be the category of one object, $*$, and $\hom_{BA}(*, *) = A$ with composition in BA given by addition in A. Note that any group homomorphism $A \to A'$ induces a functor $BA \to BA'$. Show that
$$\varprojlim_n B(\mathbb{Z}/\ell^n) = B\mathbb{Z}_\ell.$$

2. Now define C_n to be the category whose objects are $Ob\ C_n = \{i \in \mathbb{Z} : i \geq n\}$, morphisms are $\hom_{C_n}(i, j) = \mathbb{Z}/\ell^n$ for every i, j, and composition is given by addition in $\mathbb{Z}/\ell^n$. Note that there are canonical functors $C_{n+1} \to C_n$ induced by the group homomorphisms $\mathbb{Z}/\ell^{n+1} \to \mathbb{Z}/\ell^n$ and the inclusions $Ob\ C_{n+1} \subset Ob\ C_n$. Show that
$$\varprojlim_n C_n = \varnothing.$$

3. Show that for every n, the canonical functor $C_n \to B\mathbb{Z}/\ell^n$ is fully faithful, and essentially surjective. That is, it is an equivalence of categories. Deduce that $\varprojlim$, as defined above, does not preserve equivalences of categories.

There is a notion of 2-limit of categories defined by keeping track of isomorphisms, which does preserve equivalences.

One could also use ∞-categories which not only invisibly keep track of chain homotopies, but homotopies between homotopies, and homotopies between homotopies between homotopies, etc. However, since there are now infinitely many compatibility conditions, ∞-categories are not well-suited to concrete calculations.

The following is a more concrete way of dealing with this problem, more suited to calculations that might arise in Galois cohomology.

Definition 12 ([Def.5.5.2]). *Define $D^+_{Ek}(X_{\mathsf{et}}, \mathbb{Z}_\ell)$ as the full subcategory of $D^+(X^{\mathbb{N}}_{\mathsf{et}}, \mathbb{Z}_\ell)$ consisting of those sequences $(\cdots \to M_2 \to M_1 \to M_0)$ of complexes such that each M_n is a complex of sheaves of $\mathbb{Z}/\ell^n$-modules, and the induced maps*[10]

$$M_n \overset{L}{\otimes}_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^{n-1} \to M_{n-1}$$

are quasi-isomorphisms for all n.

The category $D^+_{Ek}(X_{\mathsf{et}}, \mathbb{Z}_\ell)$ (and its unbounded version) is what was used classically to access l-adic cohomology in a derived category setting.

Recall from Exercise 15(3) from "Lecture 11. Homological Algebra II", that a complex K is *derived complete* if and only if

$$K \cong R\varprojlim(K \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n).$$

Proposition 13 (BS, Prop.5.5.4). *There is a fully faithful embedding*

$$(R\varprojlim) \circ \nu^* : D^+_{Ek}(X_{\mathsf{et}}, \mathbb{Z}_\ell) \subseteq D^+(X_{\mathsf{pro\acute{e}t}}, \mathbb{Z}_\ell).$$

The essential image consists of those bounded below complexes K such that

1. *K is derived complete.*
2. *the cohomology sheaves of $K \overset{L}{\otimes}_{\mathbb{Z}_\ell} (\mathbb{Z}/\ell)$ are classical.*

Proof. Again, most of the proof is formal. The main non-formal ingredient is Cor.10. □

Remark 14. If there is an integer N such that for all affine $Y \in X_{\mathsf{et}}$ and sheaves of κ-vector spaces F we have $H^n(Y, F) = 0$ for $n > N$, then the above proposition is true for unbounded complexes too.

Remark 15. Notice that $D^+_{Ek}(X_{\mathsf{et}}, \mathbb{Z}_\ell)$ is defined by adding structure to $D(X_{\mathsf{et}}, \mathbb{Z}_\ell)$, whereas $D^+_{Ek}(X_{\mathsf{pro\acute{e}t}}, \mathbb{Z}_\ell)$ is defined via properties of objects in $D^+(X_{\mathsf{pro\acute{e}t}}, \mathbb{Z}_\ell)$. So one would expect that the latter is easier to work with.

4 Jannsen's continuous cohomology via the proétale site

Given a tower $(\cdots \to F_2 \to F_1 \to F_0) \in \mathsf{Shv}(X_{\mathsf{et}})^{\mathbb{N}}$ of étale sheaves of abelian groups, we can consider the functor

$$\gamma : \mathsf{Shv}(X_{\mathsf{et}})^{\mathbb{N}} \to \mathrm{Ab}; \qquad (F_n) \mapsto \varprojlim F_n(X).$$

[10] Since $\mathbb{Z}/\ell^{n-1} \cong l\mathbb{Z}/\ell^n$ and $0 \to l\mathbb{Z}/\ell^n \to \mathbb{Z}/\ell^n \xrightarrow{l} \mathbb{Z}/\ell^n \to 0$ is a short exact sequence of $\mathbb{Z}_\ell$-modules the functor $- \overset{L}{\otimes}_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^{n-1}$ can be calculated as $\mathrm{Cone}(- \xrightarrow{\ell} -)[-1]$ for chain complexes of sheaves of $\mathbb{Z}/\ell^n$-modules. Similarly, $- \overset{L}{\otimes}_{\mathbb{Z}_\ell} (\mathbb{Z}/\ell)$ can be calculated by $\mathrm{Cone}(- \xrightarrow{\ell} -)$ for complexes of sheaves of $\mathbb{Z}_\ell$-modules.

We can left derive this to obtain a functor

$$R\gamma : D(X_{\mathsf{et}}^{\mathbb{N}}) \to D(\mathrm{Ab})$$

Definition 16. *The* continuous étale cohomology *of a tower* $(F_n) \in \mathsf{Shv}(X_{\mathsf{et}}, \mathrm{Ab})$ *of étale sheaves of abelian groups is the cohomology*

$$H^i_{cont}(X, (F_n)) = H^i R\gamma((F_n))).$$

Proposition 17 ([BS, Prop.5.6.2])**.** *Let* (F_n) *be a tower of étale sheaves of abelian groups on* X_{et} *with surjective transition maps. Then*

$$H^i_{cont}(X_{\mathsf{et}}, (F_n)) \cong H^i(X_{\mathsf{proét}}, \varprojlim \nu^* F_n).$$

In particular, in the case $F_n = \mathbb{Z}/\ell^n$ *we have*

$$H^i_{cont}(X_{\mathsf{et}}, (\mathbb{Z}/\ell^n)) \cong H^i(X_{\mathsf{proét}}, \mathbb{Z}_\ell).$$

Proof. Again, most of the proof is formal. The main non-formal ingredient is Prop.9. The hypothesis that the $F_{n+1} \to F_n$ are surjective implies that $\varprojlim F_n \cong R\varprojlim F_n$, cf.Exercise 6 below.

$$\begin{aligned} R\gamma((F_n)) &\cong R\Gamma_{\mathsf{et}}(X, R\varprojlim F_n) \\ &\cong R\varprojlim R\Gamma_{\mathsf{et}}(X, F_n) \\ &\overset{\text{Prop.9}}{\cong} R\varprojlim R\Gamma_{\mathsf{proét}}(X, \nu^* F_n) \\ &\cong R\Gamma_{\mathsf{proét}}(X, R\varprojlim \nu^* F_n) \\ &\overset{\text{Ex.6}}{\cong} R\Gamma_{\mathsf{proét}}(X, \varprojlim \nu^* F_n) \end{aligned}$$

□

Exercise 6. Using Exercise 7 of "Lecture 11. Homological Algebra II", show that if (F_n) is a tower of étale sheaves of abelian groups on X_{et} with surjective transition maps, then $\varprojlim F_n \cong R\varprojlim F_n$.