Topology II

§1 The pro-étale site

Recall: A morphism $f: Y \to X$ of finite presentation of schemes is étale if it is flat and $Y \to Y \times_X Y$ is flat.

Def A map $f: Y \to X$ of schemes is weakly étale if it is flat and the diagonal $Y \to Y \times_X Y$ is flat.

Def $X_{proét}$ = category of weakly étale X-schemes.

Example

1) pro-étale morphisms are weakly étale
(e.g., if (B_λ) is a filtered system of étale A-algebras then
$$\varprojlim \operatorname{Spec} B_\lambda = \operatorname{Spec} \operatorname{colim}_{\to} B_\lambda \to \operatorname{Spec} A$$
is weakly étale)

2) Given a scheme X, and a profinite set $S = \varprojlim_{i \in I} S_i$, the morphism
$$X \otimes S := \varprojlim_{i \in I} \Big(\coprod_{s \in S_i} X \Big) \to X$$
is pro-étale. This defines a functor
$$\mathrm{ProFinSet} \times X_{proét} \to X_{proét}$$
$$(S, Y) \longmapsto Y \otimes S$$

$X =$ ⋮

k^{sep}/k $\qquad \operatorname{Spec}(k^{sep} \otimes_k k^{sep}) \leftarrow \operatorname{Spec}(k^{sep})$

$\qquad \operatorname{Gal}(k^{sep}/k) \leftrightarrow \{id\}$

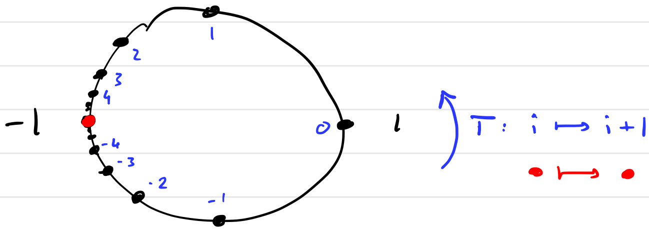
$X \otimes S$  X

S

Cantor set

3) [BS 4.1.12]

$$S = \left\{ e^{\pi i (1 - \frac{1}{2^n})} : n \in \mathbb{Z}, n \geq 0 \right\} \cup \left\{ e^{\pi i (2^n - 1)} : n \in \mathbb{Z}, n \leq 0 \right\} \cup \{-1\} \subseteq \mathbb{C}$$

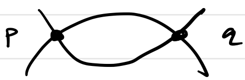


Opens : 1) $\{\bullet\}$ for $\bullet \neq \bullet$

2) $\{\}$

3) cofinite sets containing $\bullet$

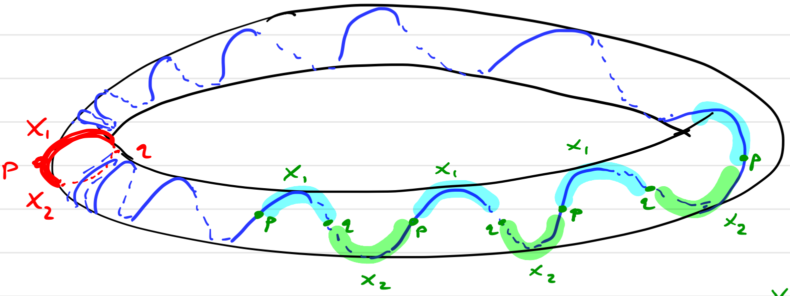
Choose smooth curves $X_1, X_2 \subset \mathbb{A}^2_{\mathbb{C}}$ meeting transversally at two points

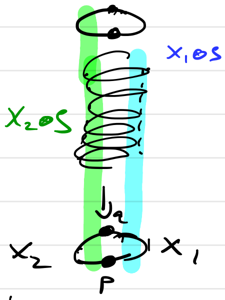


$X = X_1 \cup X_2$

$X_1 \cap X_2 = \{p, q\}$

Define $Y \to X$ to be $X_1 \otimes S$ glued to $X_2 \otimes S$ using Id at p and T at q.





$$\downarrow$$

$$X$$

Over $X \setminus \{p\}$ and $X \setminus \{q\}$, $Y \to X$ is pro-étale, so $Y \to X$ is locally weakly étale, so $Y \to X$ is weakly étale. On the other hand one can show that $Y \to X$ is not pro-étale using the fact that the section  has no (nontrivial) open-closed neighbourhoods (see [BS, 4.1.12]).

4) If k is a field, then a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(k)$ is weakly étale if and only if $k \to R$ is ind-étale.

5) For any scheme X, point $x \in X$, geometric point $\bar{x} \to X$

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \to X, \qquad \mathrm{Spec}(\mathcal{O}^h_{X,x}) \to X, \qquad \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \to X$$

are pro-étale.

$$\mathrm{Spec}(\mathcal{O}^h_{X,x}) = \lim_{\substack{x \to U \to X \\ \text{étale}}} U, \qquad \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) = \lim_{\substack{\bar{x} \to U \to X \\ \text{étale}}} U$$

Exercise 1 Show that if $f: Y \to X$ is weakly étale then $X' \times_X Y \to X'$ is weakly étale for any $X' \to X$.

Exercise 2 Show that if $Y \to X$ and $W \to Y$ are weakly étale then $W \to X$ is weakly étale.

Show that if $Y, Y' \rightrightarrows X$ are weakly étale then any morphism $Y' \to Y$ s.t.
$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$
commutes is weakly étale

Exercise 3 Use Ex. 1 and Ex. 2 to show that $X_{\text{proét}}$ has finite limits.

Definition The pro étale topology on $X_{\text{proét}}$ is generated by the Zariski topology and finite jointly surjective families of affine schemes.

That is, $\{Y_i \to Y\}_{i \in I}$ in $X_{\text{proét}}$ is a covering if $\forall$ open affine $\mathcal{U} \subseteq Y$, $\exists$ finite subset $J \subseteq I$ and open affines $V_j \subseteq Y_j$ s.t. $\coprod_J V_j \to \mathcal{U}$ is surjective.

Example

$$\{ \operatorname{Spec}(\mathbb{Z}^{sh}_{(p_1)}), \ldots, \operatorname{Spec}(\mathbb{Z}^{sh}_{(p_n)}), \operatorname{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \ldots, \tfrac{1}{p_n}]) \}$$

is a proétale covering of $\operatorname{Spec} \mathbb{Z}$.

$\{ \operatorname{Spec}(\mathbb{Z}^{sh}_{(p)}) \}_{p \text{ prime}}$ is <u>not</u>.

§2. Pro étale site of a field.

Example Suppose k is separably closed.

Let R be an ind-étale algebra.
$$k \to R = \varinjlim R_\lambda$$

So $R_\lambda \cong \prod_{i \in I_\lambda} k = \text{hom}(I_\lambda, k)$ $\quad I_\lambda$ finite.

$R_\lambda \to R_{\lambda'}$ are induced (and determined) by maps of sets $I_{\lambda'} \to I_\lambda$.

$$\Rightarrow \quad (\text{Spec } R)_{top} = \varprojlim_\lambda I_\lambda =: I.$$

$\varprojlim (\text{Spec } A_\lambda)_{top}$
$= (\varprojlim \text{Spec } A_\lambda)_{top}$
$= (\text{Spec } \varinjlim A_\lambda)_{top}$

One can show that for any profinite set $S = \varprojlim S_\lambda$ and discrete topological space X, we have $\text{hom}_{cont.}(\varprojlim S_\lambda, X) = \varinjlim \text{hom}(S_\lambda, X)$

Hence,
$$\Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) = \text{hom}_{cont.}(I, k) \leftarrow \text{discrete topology}$$

In general, $\Gamma(U, \mathcal{O}_{\text{Spec}(R)}) = \text{hom}_{cont.}(U, k)$

for any open $U \subset I$. Can also show $\mathcal{O}_{\text{Spec} R, x} \cong k$.

Conversely, if $(X, \mathcal{O}_X)$ is a locally ringed space s.t. $X = \varprojlim X_\lambda$ is profinite, $\mathcal{O}_X(U) = \text{hom}_{cont.}(U, k)$ then $(X, \mathcal{O}_X) \cong \text{Spec } \varinjlim \prod_{X_\lambda} k$. that is, X is the affine scheme associated to an ind-étale k-algebra.

<u>Proposition</u> Suppose k is a separably closed field and $X \in \mathrm{Spec}(k)_{\mathrm{proet}}$. The following are equivalent.

1) X is affine
2) X is the spectrum of an ind. étale k-algebra
3) X is qcqs
4) $X = \mathrm{Spec}(k) \otimes S$ for some $S \in \mathrm{ProFinSet}$.

(Proof in notes)

Write $\mathrm{Spec}(k)^{\mathrm{aff}}_{\mathrm{proet}}$ for the full subcategory of $\mathrm{Spec}(k)_{\mathrm{proet}}$ of those objects satisfying the above conditions.

<u>Cor</u> If $k = k^{\mathrm{sep}}$,

$$\mathrm{ProFinSet} \cong \mathrm{Spec}(k)^{\mathrm{aff}}_{\mathrm{proet}}$$

$$S \longmapsto \mathrm{Spec}(k) \otimes S$$

Under this identification, coverings of $\mathrm{Spec}(k) \otimes S$ are jointly surjective families of profinite sets $\{S_i \to S\}_{i \in I}$ which admit a jointly surjective finite sub family $\{S_{i_j} \to S\}_{j=1}^{n}$.

<u>Example</u> S infinite profinite set, $\{\{s\} \to S\}_{s \in S}$ is <u>not</u> a covering.

Proposition Let k be any field. Choose k^{sep}/k, let $G := \mathrm{Gal}(k^{sep}/k)$, there is an equivalence of categories between profinite sets equipped with a continuous G-action and the affine objects in $\mathrm{Spec}(k)_{pro\acute{e}t}$.

$$G\text{-}\mathrm{ProFinSet} \xleftarrow{\;\cong\;} \mathrm{Spec}(k)^{aff}_{pro\acute{e}t}$$

Under this identification, coverings are families $\{S_i \to S\}_{i \in I}$ which admit a jointly surjective finite subfamily $\{S_{i_j} \to S\}_{j=1}^{n}$.

Sketch of proof

In one direction use $\mathrm{Spec}(k)^{aff}_{pro\acute{e}t} \xrightarrow{k^{sep} \otimes_k -} \mathrm{Spec}(k^{sep})^{aff}_{pro\acute{e}t}$

and $\mathrm{Spec}(k^{sep})^{aff}_{pro\acute{e}t} \cong \mathrm{ProFinSet}$.

G-action comes from G acting on k^{sep}.

In the other direction,

for $S \in G\text{-}\mathrm{ProFinSet}$ $\qquad \mathrm{Spec}\left(\mathrm{hom}_{cont}(S, k^{sep})^{G}\right)$ $\square$

Special case $S = G/\mathrm{Stab}(L) \cong \mathrm{Gal}(L/k)$

$\prod_S k^{sep} = \mathrm{hom}_{cont}(S, k^{sep}) \cong L$ $\qquad g \cdot f := g(f(g^{-1} -))$

§3 The pro étale topos

Rmk $\mathrm{Shv}(\mathrm{Spec}(k)_{pro\acute{e}t}) \cong \mathrm{Shv}(\mathrm{Spec}(k)^{aff}_{pro\acute{e}t})$

If $k = k^{sep}$, $\quad \mathrm{Shv}(\mathrm{Spec}(k)_{pro\acute{e}t}) \cong \mathrm{Shv}(\mathrm{ProFinSet})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Condensed sets

Prop. For any scheme X, the topos $Shv(X_{pro\acute{e}t})$ is locally weakly contractible. therefore $D(X_{pro\acute{e}t})$ is left complete.

Lemma Suppose X is a scheme, T a topological space (e.g. $\mathbb{Z}_\ell$) then the presheaf

$$F_T : X_{pro\acute{e}t}^{op} \to Set \quad ; \quad U \mapsto Map_{cont.}(U, T)$$

is a sheaf.