(Pro)Etale Cohomology
Lecture 12. Topology II

In this talk we define the pro-étale site of a scheme, giving a number of examples of pro-étale schemes. We discuss the case of a field in detail, and in particular, mention the equivalence of categories

$$G\text{-ProFinSet} \cong \mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}.$$

We see that in general the pro-étale topos is locally weakly contractible, and therefore is replete, and left complete. Finally, we observe that the pro-étale topos gives a good setting to study the cohomology of compactly generated topological abelian groups. In particular, for a large class of "nice" topological groups, continuous cohomology agrees with the pro-étale cohomology

$$H^n_{cont}(G, M) \cong H^n_{\mathsf{pro\acute{e}t}}(G\text{-ProFinSet}, F_M).$$

1 The pro-étale site

Recall that a morphism of schemes $f : Y \to X$ of finite presentation is *étale*, if it is flat and the diagonal $Y \to Y \times_X Y$ is flat.[1]

Definition 1 (Def.4.1.1)**.** *A map $f : Y \to X$ of schemes is* weakly étale *if it is flat, and the diagonal $\Delta : Y \to Y \times_X Y$ is flat. The category of weakly étale X-schemes is denoted $X_{\mathsf{pro\acute{e}t}}$.*

Example 2.

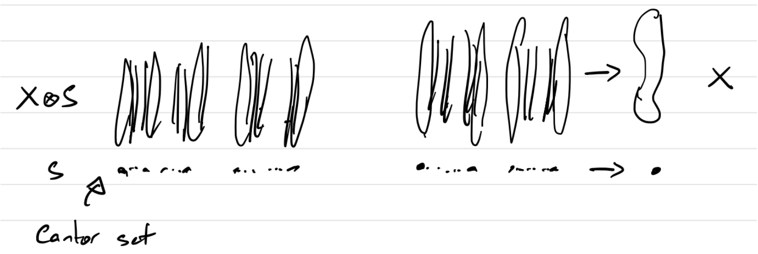
1. Suppose $A \to B$ is an ind-étale morphism of rings (so $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is pro-étale). Then we saw previously that $A \to B$ is a weakly étale morphism of rings [Prop.2.3.3], so $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a weakly étale morphism of schemes.

2. Bhatt, Scholze choose to work with weakly étale maps instead of pro-étale morphisms of schemes in general because pro-étale morphisms of schemes are not so well-behaved. Specifically, there exist morphisms which are locally pro-étale but not globally pro-étale, cf. [Exa.4.1.12] reproduced below.

[1] The diagonal being flat is one of a number of equivalent definitions for a finite presentation morphism to be unramified, [Stacks project, 02GE, 01KJ, 025G]

3. [Exa.4.1.9] Given a scheme X, and a profinite set $S = \varprojlim S_i$, the morphism $X \otimes S := \varprojlim_{i \in I}(\sqcup_{s \in S_i} X) \to X$ is pro-étale. This defines a functor

$$\mathsf{ProFinSet} \times X_{\mathsf{pro\acute{e}t}} \to X_{\mathsf{pro\acute{e}t}}; \qquad (S, Y) \mapsto S \otimes Y$$

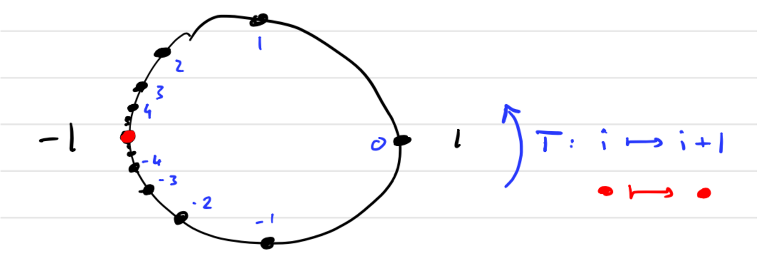


4. [Exa.4.1.12] Here we give an example of a morphism of schemes which is locally pro-étale but not globally pro-étale. Consider the one point compactification of $\mathbb{Z}$ with the discrete topology. Concretely, this profinite set can be realised as the image of the map

$$\mathbb{Z} \amalg \{\infty\} \to \mathbb{C}$$

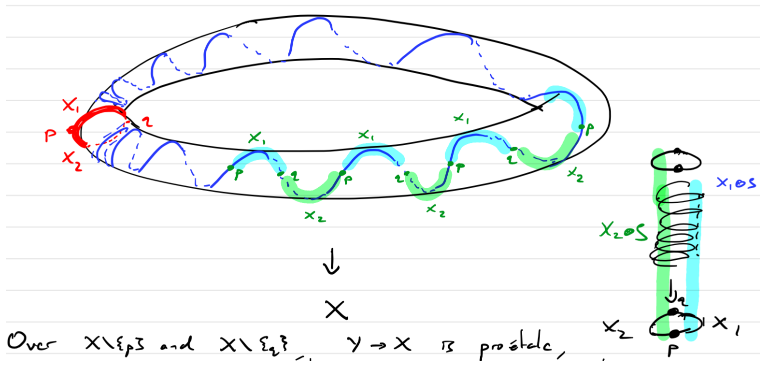
$$n \mapsto \begin{cases} e^{\pi i (1 - \frac{1}{2^n})} & n \geq 0, n \neq \infty \\ e^{\pi i (2^n - 1)} & n \leq 0, n \neq \infty \\ -1 & n = \infty \end{cases}$$

with the induced topology from $\mathbb{C}$ (with the usual analytic topology). The profinite set S is equipped with the translation function $T : n \mapsto n + 1$ (the point $n = \infty$ is sent to itself).



Now let $X_1, X_2 \subseteq \mathbb{A}^2_{\mathbb{C}}$ be two smooth curves meeting transversally at points p and q, and $X = X_1 \cup X_2$.

Then consider the X-scheme Y which is $S \otimes X_1$ glued to $S \otimes X_2$ using the identity at p and the translation function T at q.



Then $Y \to X$ is locally pro-étale. Indeed, away from p (or q), it is just $S \otimes (X \setminus \{p\}) \to (X \setminus \{p\})$. However, one can show that $Y \to X$ is not globally pro-étale, because the section $X \to Y$ corresponding to the point $n = \infty$ does not have any non-trivial open-closed neighbourhoods. Indeed, if $Y = \varprojlim Y_i$ for some étale $Y_i \to X$, then the section $X \to Y$ decomposes each Y_i as $Y_i = X \amalg Y_i'$ (any section of an étale morphism is the inclusion of a direct summand). So the image of $X \to Y = \varprojlim (X \amalg Y_i')$ should be an intersection of subsets $\pi_i^{-1} X \subseteq Y$ (here $\pi_i : Y \to Y_i = X \amalg Y_i'$) which are both open and closed. But this is not the case.

5. [Exa.4.1.4] If k is a field, then a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(k)$ is weakly étale if and only if $k \to R$ is ind-étale.[2]

6. [Exa.4.1.5] For any scheme X, point $x \in X$, and geometric point $\overline{x} \to X$, the morphisms

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \to X, \qquad \mathrm{Spec}(\mathcal{O}_{X,x}^h) \to X, \qquad \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}^{sh}) \to X$$

are all weakly étale.

Recall that flat morphisms are preserved by base change and composition.

Exercise 1 ([Lem.4.1.6])**.** *Weakly étale morphisms are preserved by base change.* Show that if $f : Y \to X$ is weakly étale then $X' \times_X Y \to X'$ is weakly étale for any morphism $X' \to X$.

[2] By [BS, Thm.2.3.4], for every weakly étale morphism $A \to B$ there is a faithfully flat ind-étale morphism $B \to C$ such that $A \to C$ is ind-étale. In particular, B is a sub-A-algebra of an ind-étale A-algebra. But for fields k, every sub-k-algebra B of an ind-étale k-algebra C is again an ind-étale k-algebra: Indeed, write $C = \varinjlim_{i \in I} C_i$ where the C_i are étale k-algebras, and recall that this means that each C_i is a finite product of finite separable k-field extensions. Replacing each C_i with its image in C, we can assume all morphisms $C_i \to C$ are injective. Then taking $B_i = C_i \cap B$, we produce a system $(B_i)_{i \in I}$ such that each B_i is an étale k-algebra and $B = \varinjlim_{i \in I} B_i$.

Exercise 2 ([Lem.4.1.6], [Lem.4.1.7])**.** *Weakly étale morphisms are preserved by composition, and all morphisms in $X_{\text{proét}}$ are weakly étale.* Let $g : W \to Y$ and $f : Y \to X$, $f' : Y' \to X$ be weakly étale morphisms, and $h : Y' \to Y$ any X-morphism.

1. Use the fact that $W \cong (Y \times_X W) \times_{(Y \times_X Y)} Y$ to show that $W \to Y \times_X W$ is flat.

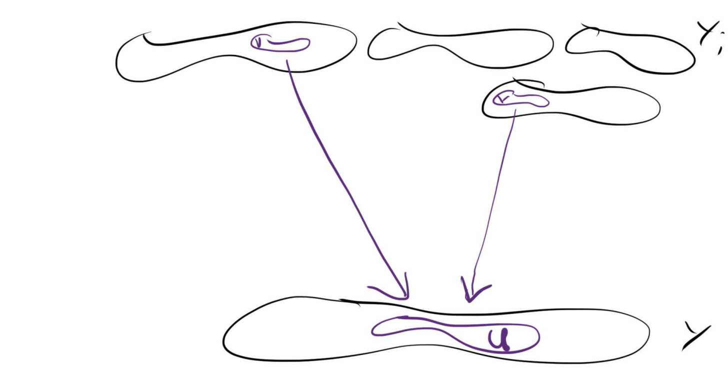
2. Use part (1), the fact that $Y \times_X W \times_X W \cong (Y \times_X W) \times_Y (Y \times_X W)$, and a clever factorisation of $W \to W \times_X W$ to show that $f \circ g : W \to X$ is weakly étale. (Alternatively, use the isomorphism $W \times_Y W \cong (W \times_X W) \times_{(Y \times_X Y)} Y$).

3. As in part (1), show that $Y' \to Y \times_X Y'$ is flat.

4. Use part (3) and the fact that $Y' \cong (Y' \times_Y Y') \times_{(Y' \times_X Y')} (Y')$ to show that $Y' \to Y$ is weakly étale.

Exercise 3 ([Lem.4.1.8])**.** Use Exercise 1 and Exercise 2 to show that $X_{\text{proét}}$ has fibre products. Deduce that $X_{\text{proét}}$ has all finite limits. (Recall, that an exercise earlier in the course was to show that a category has all finite limits if and only if it has fibre products and a terminal object).

Definition 3. *A family $\{Y_i \to Y\}_{i \in I}$ in $X_{\text{proét}}$ is a covering, if for every open affine $U \subseteq Y$, there is a finite subset $J \subseteq I$ and open affines $V_j \subseteq Y_j$ for each $j \in J$ such that $\amalg_{j \in J} V_j \to U$ is surjective.*



The finiteness in the above definition is important, and affects the topology:

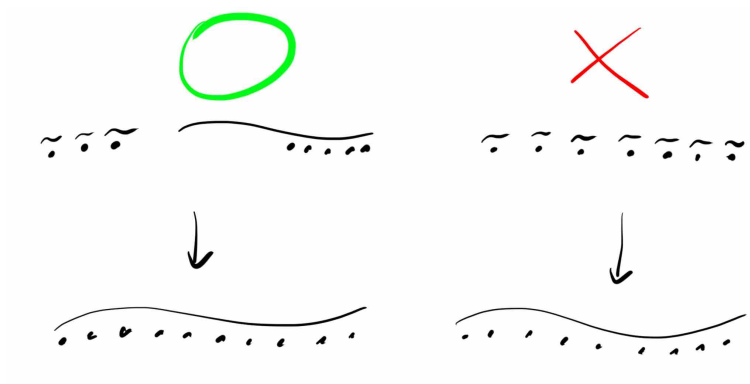
Example 4 ([Exa.4.1.13])**.** Consider $\operatorname{Spec}(\mathbb{Z})$. If $p_1, \ldots, p_n$ are finitely many primes, then

$$\left\{ \operatorname{Spec}(\mathbb{Z}^{sh}_{(p_1)}), \ldots, \operatorname{Spec}(\mathbb{Z}^{sh}_{(p_n)}), \operatorname{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \ldots, \tfrac{1}{p_n}]) \right\}$$

is a weakly étale cover. However,

$$\left\{ \operatorname{Spec}(\mathbb{Z}^{sh}_{(p)}) : p \text{ is prime} \right\}$$

is *not* a weakly étale cover.



2 The pro-étale site of a field

Example 5. In this example we study affine pro-étale schemes over a separably closed field k in great detail, giving a concrete description of them as locally ringed spaces. Explicitly, we will show that a locally ringed space $(X, \mathcal{O}_X)$ is the spectrum of a weakly étale k-algebra if and only if X is profinite, and $\mathcal{O}_X(U) = \hom_{cont.}(U, k)$.

Suppose that k is a separably closed field, and R is an ind-étale algebra

$$k \to R = \varinjlim R_\lambda.$$

So each R_λ is a *finite* product $R_\lambda = \prod_{i \in I_\lambda} k$. Moreover, every morphism $\prod_{i \in I_\lambda} k = R_\lambda \to R'_\lambda = \prod_{j \in I'_\lambda} k$ is induced by morphisms of sets $\phi_{\lambda,\lambda'} : I'_\lambda \to I_\lambda$. Since the underlying topological space of Spec of a filtered colimit of rings is the inverse limit of the underlying topological spaces, [EGAIV, §8], we see that the underlying topological space of $\mathrm{Spec}(R)$ is the profinite set $I = \varprojlim I_\lambda$.

$$\mathrm{Spec}(R)_{top} = I.$$

For any profinite set $\varprojlim S_\lambda$, and any discrete set X, one can see that we have[3] $\hom_{cont.}(\varprojlim S_\lambda, X) = \varinjlim \hom(S_\lambda, X)$. Hence,

$$\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = \hom_{cont.}(I, k)$$

is the set of continuous morphisms where k is given the discrete topology. Moreover, for any open subset of the form $U_{\lambda,i} = \phi_\lambda^{-1}(i)$ where $i \in I_\lambda$ and $\phi_\lambda : I \to I_\lambda$ is the canonical projection, $U_{\lambda,i}$ is again ind-étale, and so $\Gamma(U_{\lambda,i}, \mathcal{O}_{\mathrm{Spec}(R)}) =$

[3] If $f : \varprojlim S_\lambda \to X$ is a continuous morphism, then for every point $x \in X$, $f^{-1}x$ is open, and by definition of the limit topology, admits a covering of the form $\mathscr{U}_x = \{U_{\lambda,s} : \lambda \in \Lambda_x, s \in S_\lambda\}$ where $U_{\lambda,s} = \phi^{-1}(s)$ is the preimage of s under the canonical projection $\phi : S \to S_\lambda$. Since S is profinite, it is quasicompact, so the coverling family $\cup_{x \in X} \mathscr{U}_x$ admits a finite subcovering $\{V_{\lambda_i,s} : 1 \le i \le n, s \in S_i\}$. Then by construction any λ with $\lambda \le \lambda_1, \dots, \lambda_n$, has the property that f is constant on the fibres of $S \to S_\lambda$. Hence, f factors as $S \to S_\lambda \to X$.

$\hom_{cont.}(U_{\lambda,i}, k)$. Finally, if $U \subseteq I$ is any open subset, and $\{U_i \subset U\}$ an open covering, we have

$$\Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)}) = \mathrm{Eq}\Big(\prod_{i\in I} \Gamma(U_i, \mathcal{O}_{\mathrm{Spec}(R)})) \rightrightarrows \prod_{i,j\in I} \Gamma(U_i \cap U_j, \mathcal{O}_{\mathrm{Spec}(R)}))\Big)$$

By definition, every open of I is covered by opens of the form $U_{\lambda,i}$ so we deduce that in general,

$$\Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)}) = \hom_{cont.}(U, k).$$

Given any point $x \in I$, and any open $x \in U$, there is a smaller open containing x of the form $U_{\lambda,i}$. For any continuous function $f : U \to k$, there is a refinement $x \in U_{\lambda,i} \subseteq U$. But $U_{\lambda,i}$ is profinite, and therefore quasicompact, so the image $f(U_{\lambda,i})$ is finite, so there is a further refinement $x \in V \subseteq U_{\lambda,i}$ such that $f : V \to k$ is constant. It follows that all local rings of $\mathrm{Spec}(R)$ are isomorphic to k.

$$\mathcal{O}_{\mathrm{Spec}(R),x} \cong k.$$

Conversely, if $(X, \mathcal{O}_X)$ is a locally ringed space such that $X = \varprojlim X_\lambda$ is profinite, and $\mathcal{O}_X(U) = \hom_{cont.}(U, k)$ for some separable closed field k, then $(X, \mathcal{O}_X) \cong \mathrm{Spec}(\varinjlim \prod_{X_\lambda} k)$, i.e., $(X, \mathcal{O}_X)$ is the affine scheme associated to an ind-étale k-algebra.

Proposition 6. *Suppose k is a separable closed field and $X \in \mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$. The following are equivalent.*

1. *X is affine.*
2. *X is the spectrum of an ind-étale algebra.*
3. *X is qcqs.*
4. *X is of the form $\mathrm{Spec}(k) \otimes S$ for a profinite set S.*

Proof. ($1 \iff 2$) We have seen, Exa.2(5) that the affine schemes in $\mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$ are precisely the spectra of ind-étale k-algebras.

($2 \iff 3$) All affine schemes are qcqs, so consider the other direction. Suppose that X is qcqs. A scheme is qcqs if and only if it admits a finite open affine cover $\{U_i \to X\}_{i=1}^n$ such that each $U_i \cap U_j$ for $1 \leq i, j \leq n$ is also affine. Since affines in $\mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$ have profinite underlying topological space (i.e., compact, Hausdorff, totally disconnected topological space), it follows that any qcqs X also has profinite underlying topological space (see the lemma below). Moreover, the structure sheaf of X has the form $V \mapsto \hom_{cont.}(V, k)$ since those of the U_i and $U_i \cap U_j$ have this form. Hence, it follows from Example 5 that if X is qcqs, it is the spectum of an ind-étale algebra.

($2 \iff 4$) This follows from the definition of $- \otimes S$ and the equivalence between the category of finite sets and the category of étale k-algebras. □

Lemma 7. *Suppose that X is a topological space admitting a finite open cover $\{U_i \to X\}_{i=1}^n$ such that all U_i and $U_i \cap U_j$ are compact, Hausdorff, totally disconnected topological spaces. Then show that X is also compact, Hausdorff, and totally disconnected.*

Proof. X is compact: Suppose that $\{V_j \to X\}_{j\in J}$ is an open covering. then each $\{V_j \cap U_i\}$ is an open covering. But each U_i is compact, so for each $i = 1, \dots, n$, there is a finite subset $J_i \subseteq J$ such that $U_i = \cup_{j\in J_i} U_i \cap V_j$. It follows that $X = \cup_{i=1}^n \cup_{j\in J_i} V_j$.

X is Hausdorff: Suppose that $x \neq y \in X$ are two points. Choose i_x, i_y such that $x \in U_{i_x}$ and $y \in U_{i_y}$ and set $U_x = U_{i_x}, U_y = U_{i_y}, U_{xy} = U_{i_x} \cap U_{i_y}$. If, say, $y \in U_{xy} \subseteq U_x$, then since U_x is Hausdorff, we can find opens $x \in V, y \in W$ such that $V \cap W = \varnothing$. So suppose that $x, y \notin U_{xy}$. Since U_{xy} is compact, and U_x, U_y are both profinite, U_{xy} is both closed and open in both U_x and U_y. In particular, $V = (U_x \setminus U_{xy}) \subseteq U_x$ and $W = (U_y \setminus U_{xy}) \subseteq U_y$ are also both closed and open in U_x, U_y respectively. This means that V and W are both open in X. By construction, $x \in V$ and $y \in W$ and $V \cap W = \varnothing$, so we are done.

X is totally disconnected: First recall that a subset $W \subseteq X$ is open (resp. closed) if and only if $W \cap U_i$ is open (resp. closed) for all i. Let us write $Y \Subset W$ to indicate that Y is both open and closed in W. Suppose that $W \subseteq X$ is a subset containing more than one point. We want to find a proper nonempty $Y \Subset W$. If $W \cap U_i$ has a single point, say w, for some i, then $\{w\}$ is open in W. But all U_i are totally disconnected, so $\{w\}$ is closed in all U_i, and therefore closed in X, and therefore closed in W. Hence, $Y = \{w\} \Subset W$ works.

So suppose each $W \cap U_i$ has more than one point. Since the U are totally disconnected, for each i there is some proper nonempty $Y_i \Subset W \cap U_i$. For any other j, we then have that $Y_i \cap U_j \Subset (W \cap U_i) \cap U_j$. Now as above, since $U_i \cap U_j$ is quasicompact, $U_i \cap U_j \Subset U_j$, so, $W \cap U_i \cap U_j \Subset W \cap U_j$, and we find that in fact, $Y_i \cap U_j \Subset W \cap U_i \cap U_j \Subset W \cap U_j$. Now define T_i inductively by setting $T_0 = W$. If one of $T_{i-1} \cap Y_i$ or $T_{i-1} \cap (W \cap U_i \setminus Y_i)$ are nonempty then choose one and set T_i to be this nonempty intersection. If both are empty, then define $T_i^a = T_{i-1}^a$. Now note that since $Y_i \cap U_j \Subset W \cap U_j$ for every i, j, it follows that each $T_i \Subset W \cap U_j$ for every $1 \leq j \leq i$. In particular, $T_n \Subset W \cap U_j$ for all j, and therefore $T_n \Subset W$. It is nonempty and proper by construction. $\square$

Write $\mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ for the fullsubcategory of $\mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$ of those objects satisfying the equivalent conditions of the previous lemma.

Corollary 8. *If k is a separably closed field, there is an equivalence of categories*

$$\begin{aligned} \mathsf{ProFinSet} &\cong \mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}} \\ S &\mapsto \mathrm{Spec}(k) \otimes S \\ X(k) &\leftarrow\!\shortmid X \end{aligned}$$

Under this identification, coverings of $\mathrm{Spec}(k) \otimes S$ are precisely the jointly surjective families of profinite sets $\{S_i \to S\}_{i\in I}$ that admit a jointly surjective finite subfamily $\{S_{i_j} \to S\}_{j=1}^n$.

Example 9. If S is any nonfinite profinite set then the family $\{s \to S\}_{s \in S}$ of inclusions of its points is *not* a covering family.

The following is basically a version of the equivalence we saw in Galois theory between étale k-algebras and finite G-sets.

Proposition 10. *Let k be any field, choose a separable closure k^{sep}/k, and let $G = Gal(k^{sep}/k)$. There is an equivalence of categories between profinite sets equipped with a continuous G-action and the affine objects in* $\mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$.

$$G\text{-}ProFinSet \cong \mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$$

Under this identification, coverings are precisely the jointly surjective families $\{S_i \to S\}_{i \in I}$ that admit a jointly surjective finite subfamily $\{S_{i_j} \to S\}_{j=1}^{n}$.

Sketch of proof. In one direction, we use the functor

$$\mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}} \xrightarrow{k^{sep} \otimes_k -} \mathrm{Spec}(k^{sep})^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$$

and the equivalence

$$\mathrm{Spec}(k^{sep})^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}} \cong \mathsf{ProFinSet}$$

The G-action is induced by the canonical G-action on $\mathrm{Spec}(k^{sep})$. In the other direction, given a pro-finite set S equipped with a continuous G-action, we take $\mathrm{Spec}(\hom_{cont}(S, k^{sep})^G)$, i.e., the spectrum of the ring of those continuous functions which are invariant for the action of G acting via its action on S. □

3 The pro-étale topos

Definition 11 ([Def.4.2.1])**.** *Let X be a scheme. An object $U \in X_{\mathsf{pro\acute{e}t}}$ is called a* pro-étale affine *if it is of the form $U = \varprojlim U_i$ for some small filtered diagram $(U_i)_{i \in I}$ of (absolutely) affine schemes $U_i = \mathrm{Spec}(A_i)$ in X_{et}. The expression $U = \varprojlim U_i$ is called a* presentation *of U. The full subcategory of $X_{\mathsf{pro\acute{e}t}}$ spanned by pro-étale affines is denoted $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$. We make it a site by saying a family in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ is a covering in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ if it is a covering in $X_{\mathsf{pro\acute{e}t}}$.*

Lemma 12. *For X a scheme, every scheme $Y \in X_{\mathsf{pro\acute{e}t}}$ admits a pro-étale covering $\{Y_i \to Y\}$ such that each Y_i is in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$.*

Proof. Choose an open affine covering $\{\mathrm{Spec}(A_i) \to X\}_{i \in I}$ of X, and for each i, choose an open affine covering $\{\mathrm{Spec}(B_{ij}) \to \mathrm{Spec}(A_i) \times_X Y\}_{i \in J_i}$ of the preimage of $\mathrm{Spec}(A_i)$ in Y. Now by [Thm.2.3.4], since the morphisms $A_i \to B_{ij}$ are weakly étale for each i, j, there is a faithfully flat ind-étale morphism $B_{ij} \to C_{ij}$ such that $A_i \to C_{ij}$ is ind-étale. Consequently, $\{\mathrm{Spec}(C_{ij}) \to Y\}_{i \in I, j \in J_i}$ is a covering of the desired form. □

Corollary 13 ([Lem.4.2.4, Rem.4.2.5])**.** *For any scheme X, the canonical restriction functor induces an equivalence of categories of sheaves*

$$\mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}) \xrightarrow{\sim} \mathsf{Shv}(X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}).$$

Proof. This is a general fact about Grothendieck sites. Consider any site (C, τ) and full subcategory $D \subseteq C$ equipped with the induced topology. If every object of C has a covering by objects of D, then there is an equivalence $\mathsf{Shv}(C) \cong \mathsf{Shv}(D)$. $\square$

Proposition 14 ([Prop.4.2.8]). *For any scheme X, the topos $\mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ is locally weakly contractible [Def.3.2.1]. In particular, it is replete [Def.3.1.1], and so $D(X_{\mathsf{pro\acute{e}t}})$ is left-complete [Def.3.3.1].*

Proof. [Prop.3.2.3] says that a locally weakly contractible topos is replete. [Prop.3.3.3] says that the derived category of a replete topos is left-complete. It suffices to show that for every scheme $Y \in X_{\mathsf{pro\acute{e}t}}$ there is a covering $\{Y_i \to Y\}_{i \in I}$ with $Y_i \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ locally weakly contractible. Lemma 12 says that every scheme admits a pro-étale affine covering. So it remains only to see that affine schemes have locally weakly contractible coverings. This was the main result of the Algebra II lecture. $\square$

On the pro-étale site, one can define interesting "constant" sheaves associated to topological spaces.

Lemma 15 ([Lem.4.2.12]). *Suppose X is a scheme and T is a topological space. Then the presheaf*

$$F_T : X^{op}_{\mathsf{pro\acute{e}t}} \to Set; \qquad U \mapsto Map_{cont}(U, T)$$

which sends a scheme U to the set of continuous maps from the underlying topological space of U to T is a sheaf.

Sketch of proof. This uses [Lem.4.2.6] which we did not do. It says that a presheaf F on $X_{\mathsf{pro\acute{e}t}}$ is a sheaf if and only if it satisfies the sheaf condition for Zariski covers, and surjective maps in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$. In the category of topological spaces, any representable presheaf is a sheaf for the topology generated by usual open coverings of topological spaces, and surjective morphisms $Y \to X$ such that X has the quotient topology induced from Y. Hence, in our setting, it suffices to check that for any surjective morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ in $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ a subset $U \subseteq \mathrm{Spec}(A)$ is open if and only if f^{-1} is open. This is proved in a really neat way using the constructible topology, and the fact that a subset of a scheme is open if and only if it is constructible and closed under generisation. $\square$

4 Addendum

We did not have time for the following comments. There are of course many more details in Bhatt, Scholze.

Let k be a field, k^{sep} a separable closure, and $G = \mathrm{Gal}(k^{sep}/k)$. Recall that we had an equivalence of categories

$$\mathsf{Shv}(k_{\mathsf{et}}, \mathrm{Ab}) \cong G\text{-mod}$$

between the category of étale sheaves on k, and discrete G-modules. A consequence of this was that for any discrete G-module M with associated sheaf F_M, the group cohomology of M is isomorphic to the étale sheaf cohomology of F_M,

$$H^n_{\mathsf{et}}(k, F_M) \cong H^n(G, M).$$

The pro-étale site allows us to upgrade this, although things become more technical and complicated.

Recall that we have already seen an equivalence of categories

$$k^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}} \cong G\text{-ProFinSet}$$

between the subcategory of affine objects in $k_{\mathsf{pro\acute{e}t}}$ and the category of profinite sets equipped with a continuous action. The covering families in the left side are just surjective families.

Definition 16. *Given an arbitrary profinite group G, we define a topology on the category G-ProFinSet whose covering families are surjective families.*

Definition 17. *Let G-Spc be the category of topological spaces equipped with a continuous G-action. Let $G\text{-}Spc_{cg} \subseteq G\text{-}Spc$ be the full subcategory of $X \in G\text{-}Spc$ whose underlying topological space can be written as a quotient of a disjoint union of compact Hausdorff spaces. These spaces are called* compactly generated.

Lemma 18 ([Lem.4.3.2])**.** *The association $T \mapsto \hom_{cont,G}(-, T)$ produces a functor $G\text{-}Spc \to \mathsf{Shv}(G\text{-}ProFinSet)$. This functor is fully faithful on $G\text{-}Spc_{cg}$, admits a left adjoint (everywhere), and its essential image generates $\mathsf{Shv}(G\text{-}ProFinSet)$ under colimits.*

Definition 19. *We write G-Mod for the category of topological abelian groups equipped with a continuous G-action. We write $G\text{-}Mod_{cg}$ for the full subcategory whose underlying space is compactly generated (i.e., lies in $G\text{-}Spc_{cg}$).*

As above, given $M \in G$-Mod, we get an abelian sheaf $F_M : X \mapsto \hom_{cont,G}(-, M)$ on G-ProFinSet.

We did not define continuous cohomology, but the main result about it is the following.

Lemma 20 ([Lem.4.3.9])**.** *For a large class of "nice" $M \in G$-Mod we have*

$$H^n_{cont}(G, M) \cong H^n_{\mathsf{pro\acute{e}t}}(G\text{-}ProFinSet, F_M).$$