Homological Algebra II

Motivation: $\mathcal{X}$ topos (i.e., $\mathcal{X} = \mathrm{Shv}_\tau(\mathcal{C})$)

$K \in D(\mathcal{X})$

$\mapsto \quad \tau^{\geq n} K = (\cdots \to 0 \to K^n/dK^{n-1} \to K^{n+1} \to K^{n+2} \to \cdots)$

$\mapsto$

$$(\cdots \to \tau^{\geq n} K \to \tau^{\geq n+1} K \to \cdots)$$

$$(*) \qquad K \to R\varprojlim \tau^{\geq n} K$$

is this a weak equivalence?

$\mathcal{X} = \mathrm{Shv}_{et}$ No. Not in general.

$\mathcal{X} = \mathrm{Shv}_{pro\acute{e}t}$ Yes! So can reduce arguments to the bounded below case.

If $(*)$ is always a w.e., $D(\mathcal{X})$ is called <u>left complete</u>.

There always exists a "left completion" $D(\mathcal{X}) \to \widehat{D}(\mathcal{X})$

$$\widehat{D}(\mathrm{Shv}_{\acute{e}t}) \subseteq D(\mathrm{Shv}_{pro\acute{e}t})$$

full subcategory of "complete" objects.

$$2\text{-}\varprojlim D(\mathrm{Shv}_{\acute{e}t}(X, \mathbb{Z}/\ell^n)) \cong D_{comp}(\mathrm{Shv}_{pro\acute{e}t}(X, \mathbb{Z}_\ell))$$

$$\subseteq D(\mathrm{Shv}_{pro\acute{e}t}(X, \mathbb{Z}_\ell))$$

§1 Replete topoi

Def A *topos* is a category of the form $\mathrm{Shv}_\tau(\mathcal{C})$ for some category $\mathcal{C}$ and Grothendieck topology τ.

Given $X \in \mathcal{C}$, write $h_X :=$ sheafification of $\hom_{\mathcal{C}}(-, X)$

Example Given a topos $\mathcal{X}$, the category

$\prod_{\mathbb{N}} \mathcal{X}$ of sequences $(\dots, F_2, F_1, F_0)$ of objects in $\mathcal{X}$ is a topos.

The category $\mathcal{X}^{\mathbb{N}}$ of sequences $(\dots \to F_2 \to F_1 \to F_0)$ of morphisms in $\mathcal{X}$ is a topos.

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d] \\
\dots \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & E_2 \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & E_1 \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & E_0
\end{tikzcd}$$

Def Let $\mathcal{X} = \mathrm{Shv}_\tau(\mathcal{C})$ be a topos. A morphism $F \to G$ of objects is *surjective* if for every $X \in \mathcal{C}$ and $s \in G(X)$, there exists a covering $\{U_i \to X\}_{i \in I}$ such that $s|_{U_i} \in \mathrm{Im}(F(U_i) \to G(U_i))$ for all $i \in I$.

Exercise 1. Let $\{V_i \to Y\}_{i \in I}$ is a covering in $\mathcal{C}$. Show that $\coprod_{i \in I} h_{V_i} \to h_Y$ is a surjective morphism of sheaves.

Definition A topos is *replete* (充実した) if for every sequence of surjective morphisms $\dots \to F_2 \to F_1 \to F_0$ the induced morphisms

$$\begin{tikzcd}
& \varprojlim_{i \geq 0} F_i \arrow[dl] \arrow[d] \arrow[dr] & \\
\dots \arrow[r] F_2 & F_1 \arrow[r] & F_0
\end{tikzcd}$$

are surjective for all n.

Exercise 2

1) Show that the category of sets is replete
2) Show that $PSh(\mathcal{C})$ is replete for any category $\mathcal{C}$
3) Let G be a discrete group. Deduce that the category of G-sets is replete.

Example Let k be a field such that k^{sep}/k is not finite. Then $Shv_{et}(k)$ of étale sheaves on k is not replete:

$\exists$ $k^{sep} \ldots / L_2 / L_1 / L_0 = k$ infinite tower of finite separable non trivial field extensions.

Each $Spec(L_n) \to Spec(L_{n-1})$ is a covering, but

$$\varprojlim h_{Spec(L_i)} \to h_{Spec(k)} \text{ is not surjective. } (*)$$

Exercise 3 Prove $(*)$. Hint: evaluate on $X = Spec(k)$ and consider $s = Id_k \in h_{Spec(k)}(Spec(k))$.

This sections goal: countable products are exact in replete topoi.

$\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n$

Note: 1) $\varprojlim A_n = \ker\left(\prod_{\mathbb{N}} A_n \xrightarrow{Id-\text{shift}} \prod_{\mathbb{N}} A_n\right)$

$(\ldots, a_2, a_1, a_0) \longmapsto (\ldots, a_2 - \varphi(a_3), a_1 - \varphi(a_2), a_0 - \varphi(a_1))$

2) $\prod_{\mathbb{N}} A_n = \varprojlim_{n \in \mathbb{N}} (A_n \times A_{n-1} \times \ldots \times A_1 \times A_0)$

Lemma Let

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d] \\ \cdots \arrow[r] & G_2 \arrow[r] & G_1 \arrow[r] & G_0 \end{tikzcd}$$

$$\begin{tikzcd} F_{i+1} \arrow[rr] \arrow[dr] \arrow[ddr] & & F_i \arrow[dd] \\ & F_i \times_{G_i} G_{i+1} \arrow[ur] \arrow[d] & \\ G_{i+1} \arrow[rr] & & G_i \end{tikzcd}$$

be morphisms in a replete topos, such that all $F_i \to G_i$ and $F_{i+1} \to F_i \times_{G_i} G_{i+1}$ are surjective. Then

$$\varprojlim F_i \to \varprojlim G_i \text{ is surjective.}$$

Exercise 4 Prove the above lemma in the category of sets.

Exercise 5 Show that $(\cdots \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z}) \to (\cdots \to \mathbb{Z}/\ell^3 \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell)$ does not satisfy the second condition, and

$$\varprojlim \mathbb{Z} \to \varprojlim \mathbb{Z}/\ell^n \quad \text{is not surjective.}$$

Exercise 6 Suppose that $(f_i : B_i \to C_i)_{i \in \mathbb{N}}$ is a sequence of surjections. Show that $F_n = B_n \times \cdots \times B_0$, $G_n = C_n \times \cdots \times C_0$ with the canonical morphisms satisfy the conditions of the lemma. Deduce that $\prod_{\mathbb{N}} B_n \to \prod_{\mathbb{N}} C_n$ is surjective.

Exercise 7 Suppose $\cdots \to F_2 \xrightarrow{t_2} F_1 \xrightarrow{t_1} F_0$ is a sequence of surjections in a replete topos $\mathcal{X}$.

1) Show that each map

$$\prod_{i=0}^{n+1} F_i \xrightarrow{id - t} \prod_{i=0}^{n} F_i \quad \text{is surjective.}$$

2) Using Exercise 6, show that $\prod_{\mathbb{N}} F_i \xrightarrow{id - t} \prod_{\mathbb{N}} F_i$ is surjective.

Recall: $F \xrightarrow{\phi} G \in Shv$ is surjective if
$\forall X \in \mathcal{C}$, $s \in G(X)$, $\exists$ covering $U \to X$, $\exists t \in F(U)$
s.t. $\phi(t) = s|_U$

$\prod_{i \in I} F_i \to \prod_{i \in I} G_i$ s.t. $F_i \to G_i$ surjective.

<u>Def</u> If $\mathcal{X} = Shv_\tau(\mathcal{C})$ is a topos, write $D(\mathcal{X}) := D(Shv_\tau(\mathcal{C}, Ab))$.

Recall, associated to a topos $\mathcal{X}$, we had the topoi:

$\prod_{\mathbb{N}} \mathcal{X}$ " = " $\{ (\dots, F_2, F_1, F_0) \}$ $\xrightarrow{\Pi} \mathcal{X}$

$(\dots, F_2, F_1, F_0) \longmapsto \prod F_i$

$\mathcal{X}^{\mathbb{N}}$ " = " $\{ (\dots \to F_2 \to F_1 \to F_0) \}$ $\xrightarrow{\lim} \mathcal{X}$

these lead to derived functors

$$R\Pi : D(\prod_{\mathbb{N}} \mathcal{X}) \to D(\mathcal{X})$$
$$R\lim : D(\mathcal{X}^{\mathbb{N}}) \to D(\mathcal{X})$$

<u>Proposition</u> Let $\mathcal{A}$ be a Grothendieck abelian category with countable products and let $(\dots \to C_2 \xrightarrow{t_2} C_1 \xrightarrow{t_1} C_0)$ be a tower of chain complexes C_i. Then there is an isomorphism in $D(\mathcal{A})$

$$R\varprojlim C_n \cong \mathrm{Cone}\left(R\Pi C_n \xrightarrow{\mathrm{Id}-t} R\Pi C_n\right)[-1]$$

Proof in Appendix to lecture notes.

cf. $\varprojlim C_n = Eq\left(\prod C_n \rightrightarrows \prod C_n\right)$
$= \mathrm{hofib}\left(\prod C_n \xrightarrow{\mathrm{Id}-t} \prod C_n\right) = \mathrm{Cone}\left(\xrightarrow{\mathrm{Id}-t}\right)[-1]$

<u>Proposition</u> Let $\mathcal{X}$ be a replete topos. Then the functor
$\Pi : \prod_{\mathbb{N}} \mathcal{X} \to \mathcal{X}; (\dots, F_1, F_0) \longmapsto \prod_{\mathbb{N}} F_i$ preserves injections and surjections. In particular, Π preserves quasi-isomorphisms of chain complexes so induces a functor $\Pi : D(\prod \mathcal{X}) \to D(\mathcal{X})$ which is just Π on objects. In other words, $R\Pi \cong \Pi$

$(f_i : F_i \to G_i)$

Pf. Limits always preserve injections.
Surjections are preserved by Exercise 6. $\square$

Proposition (BS, 3.1.10) Let $\mathcal{X} = \mathrm{Shv}_\tau(\mathcal{C})$ be a replete topos and suppose $\cdots \to F_2 \to F_1 \to F_0$ is a sequence of surjective morphisms in $\mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Ab})$. Then $\varprojlim F_i \simeq R\varprojlim F_i$ in $D(\mathcal{X})$.

(Using the canonical functor $\mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Ab}) \to D(\mathcal{X})$
$\mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Ab})^{\mathbb{N}} \to D(\mathcal{X}^{\mathbb{N}})$)

Proof Since each $F_{i+1} \to F_i$ is surjective, $\mathrm{Id} - t$ is surjective by Exercise 7. So

$$0 \to \varprojlim F_i \to \prod F_i \xrightarrow{\mathrm{Id}-t} \prod F_i \to 0$$

is exact. So

$$\begin{aligned} R\varprojlim F_i &= \mathrm{Cone}(R\prod F_i \to R\prod F_i)[-1] \\ &= \mathrm{Cone}(\prod F_i \to \prod F_i)[-1] \\ &\simeq \varprojlim F_i \end{aligned}$$

$\square$

Remark The two propositions on the previous page show that in a replete topos we could define $R\varprojlim F_i$ as $\mathrm{Cone}(\prod F_i \to \prod F_i)[-1]$.

§2 Locally weakly contractible topoi

Def A topos is said to be *coherent* if there is a small site $(\mathcal{C}, \tau)$ such that $\mathcal{C}$ has finite limits and for every covering $\{U_i \to X\}_{i \in I}$ there is a finite set $\{i_1, \ldots, i_n\} \subset I$ such that $\{U_{i_1} \to X, \ldots, U_{i_n} \to X\}$ is also a covering.

Def {BS, Def. 3.2.1} An object F of a topos $\mathcal{X}$ is <u>weakly contractible</u> if every surjection $G \to F$ has a section
$$\begin{tikzcd} G \arrow[r] & F \arrow[l, dashed, bend left] \end{tikzcd}.$$
Suppose $\mathcal{X}$ is a coherent topos with coherent site of definition $(\mathcal{C}, \tau)$. We say $\mathcal{X}$ is <u>locally weakly contractible</u> if every object $X \in \mathcal{X}$ admits a surjection $\coprod_{i \in I} Y_i \to X$ with Y_i weakly contractible and representable.

Rmk. Y weakly contractible, representable, $\hom(Y, -)$ commutes with all colimits and limits.

$$F \in PSh \underset{i}{\overset{a}{\rightleftarrows}} Shv, \qquad \check{H}^0(Y, F) = \operatorname*{colim}_{W \to Y} eq\left(F(W) \rightrightarrows F(W \times_Y W)\right) = F(Y)$$

so $(aF)(Y) = F(Y)$

$F_- : I \to Shv$,

$$\left(\operatorname{colim}^{Shv} F_i\right)(Y) = \left(a \operatorname{colim}^{PSh} F_i\right)(Y) = \left(\operatorname{colim}^{PSh} F_i\right)(Y) = \operatorname{colim}\left(F_i(Y)\right)$$

$$\hom_{Shv}(Y, \operatorname{colim} F_i) \qquad \qquad \operatorname{colim} \hom_{Shv}(Y, F_i)$$

Prop. Let $\mathcal{X}$ be a locally weakly contractible topos. Then $\mathcal{X}$ is replete, and for any $K \in D(\mathcal{X})$ we have $R\lim \tau^{\geq n} K \cong K$.

Here $\tau^{\geq n} K = (\cdots \to 0 \to K^n / dK^{n-1} \to K^{n+1} \to K^{n+2} \to \cdots)$

Sketch of proof. Since $\mathcal{X}$ is loc. weakly contractible, a morphism f in $\mathcal{X}$ is an iso. (resp. surjection) if and only if evaluating on each weakly contractible object is iso. (surj.). It follows that $\mathcal{X}$ is replete.

Similarly, for $K \in D(\mathcal{X})$, U weakly contractible, we have $(H^iK)(U) = H^i(K(U))$. It follows that $K \to R\lim \tau^{\geq n} K$ is a quasi-isomorphism. $\square$

$\Rightarrow$ $F \xrightarrow{\phi} G$ epi, $\Rightarrow$ $\forall$ $X \in \mathcal{C}$, $s \in G(X)$ $\exists$ U, $t \in F(U)$ s.t. $s|_U = \phi(t)$. If X is weakly contr. then $\exists$ $X \to U \to X$, so $s|_U|_X = \phi(t|_X)$ so $s \in \operatorname{Im}(F(X) \to G(X))$.

$\Leftarrow$ $X \in \mathcal{C}$, $s \in G(X)$, choose $\{U_i \to X\}_{i \in I}$ covering U_i weakly contractible then $\prod F(U_i) \to \prod G(U_i)$ is surjective.

① $K \to R\lim \tau^{\geq n} K$ is a quasi-iso

② $\underline{H}^i K \simeq \underline{H}^i R\lim \tau^{\geq n} K$ in $Shv_\tau(\mathcal{C}, Ab)$

③ $(\underline{H}^i K)(U) \simeq (\underline{H}^i R\lim \tau^{\geq n} K)(U)$ for U weakly contractible

$\quad\; \|$

$H^i(K(U))$

③ $\Rightarrow$ ② $\Rightarrow$ ①

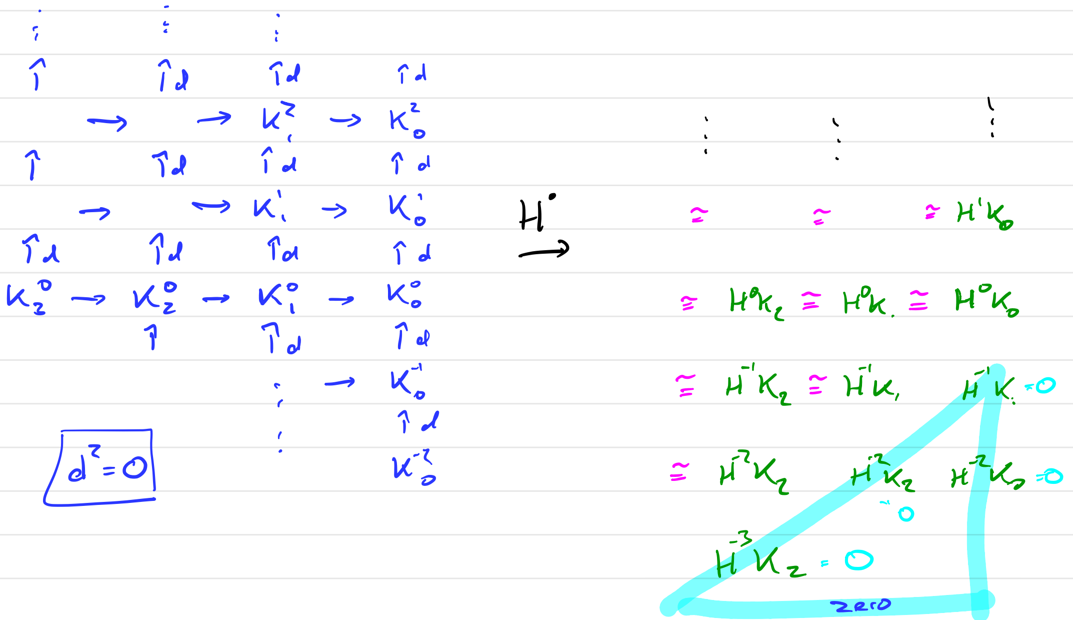
§3 Truncation completing derived categories

Def Let $\mathcal{X} = \mathrm{Shv}_\tau(\mathcal{C})$ be a topos. We define the *left completion* $\hat{D}(\mathcal{X})$ of $D(\mathcal{X})$ as the full subcategory of $D(\mathcal{X}^{\mathbb{N}})$ spanned by the towers $(\cdots \to K_2 \to K_1 \to K_0)$ in $\mathrm{Ch}(\mathrm{Shv}_\tau(\mathcal{C}, Ab)^{\mathbb{N}})$ such that.

1) $K_n \in D^{\geq -n}(\mathcal{X})$, that is $H^i K_n = 0$ for $i < -n$

2) $\tau^{\geq -n} K_{n+1} \to K_n$ is a weak equivalence.
That is, $H^i K_{n+1} \to H^i K_n$ is iso for $i \geq -n$

We say that $D(\mathcal{X})$ is *left complete* if the map
$$\tau : D(\mathcal{X}) \to \hat{D}(\mathcal{X}); \quad K \mapsto \{\tau^{\geq n} K\}$$
is an equivalence.



Theorem Let $\mathcal{X} \overset{=\,Shv_\tau(\mathcal{C})}{}$ be a topos.

1. [BS, 3.3.2] $\hat{D}(\mathcal{X}) \hookrightarrow D(\mathcal{X}^{\mathbb{N}}) \xrightarrow{R\lim} D(\mathcal{X})$ is the right adjoint of $\tau : D(\mathcal{X}) \to \hat{D}(\mathcal{X})$
$$K \mapsto \{\tau^{\geq n} K\}$$
In particular, if $D(\mathcal{X})$ is left complete, then $K \xrightarrow{\sim} R\lim \tau^{n} K \quad \forall \quad K$.

2. [BS, 3.3.3] If $\mathcal{X}$ is replete, $D(\mathcal{X})$ is left complete

3. [BS, 3.3.5] $D(Spec(\mathbb{C}(x_1, x_2, \ldots))_{et})$ is not left complete.

4. [BS 3.3.7] If $U \in \mathcal{C}$ is such that $\Gamma(U, -)$ is exact, then $\forall$ K,
$$R\Gamma(U,K) \cong R\varprojlim R\Gamma(U, \tau^{n} K)$$

5. [3.3.7] If for each $K \in D(\mathcal{X})$ and $U \in \mathcal{C}$ $\exists$ $d \in \mathbb{N}$ such that $H^p(U, \underline{H}^q K) = 0$ for $p > d$ then $D(\mathcal{X})$ is left complete.

Example (5.) is satisfied for $Spec(\mathbb{F}_q)_{et}$ and X_{et} when X is a smooth affine variety over an alg. cl. field.