(Pro)Etale Cohomology
Lecture 11. Homological Algebra II

Reference: [BS] Bhatt, Scholze, “The pro-étale topology for schemes”.

In this lecture we consider replete topoi. This is a nice class of topoi that include the pro-étale topos, in which inverse limits work well. In particular, for any object $K \in D(\mathcal{X})$ in the derived category of a replete topos $\mathcal{X}$, the chain complex K is the derived inverse limit of its truncations

$$K \cong R \varprojlim \tau^{-n} K,$$

where $\tau^{-n}K = (\dots \to 0 \to 0 \to K^n/d(K^{n-1}) \to K^{n+1} \to K^{n+2} \to \dots)$. This allows us to prove things about unbounded chain complexes using bounded below chain complexes (e.g., we will do this in the proof of Proposition 34).

In Section 3 we will see that, for any topos $\mathcal{X}$, there is a canonical way to complete its derived category with respect to inverse limits $D(\mathcal{X}) \to \widehat{D}(\mathcal{X})$. In later lectures we will see that in the case $\mathcal{X} = \mathsf{Shv}_{et}$, the left completion is canonically equivalent to a subcategory of $D(\mathsf{Shv}_{\text{proét}})$

$$\widehat{D}(\mathsf{Shv}_{\text{et}}) \subseteq D(\mathsf{Shv}_{\text{proét}}).$$

One of the motivations we gave at the beginning of the course for the pro-étale topology was that it gives a better way of constructing the derived categories of l-adic sheaves. Instead of an ad-hoc 2-limit $D(\mathsf{Shv}_{\text{et}}(X, \mathbb{Z}_l)) :=$ 2-$\varprojlim_n D(\mathsf{Shv}_{\text{et}}(X, \mathbb{Z}/l^n))$, we would like the derived category of l-adic sheaves to be just that: the category of sheaves of $\mathbb{Z}_l$-modules on a site. If we consider sheaves of $\mathbb{Z}_l$-modules on the pro-étale site, we get a bigger category than $D(\mathsf{Shv}_{\text{et}}(X, \mathbb{Z}_l))$. In Section 4 we discuss the notion of *derived complete* objects in $D(\mathcal{X})$. It is the subcategory $D_{\text{comp}}(\mathsf{Shv}_{\text{proét}}(X, \mathbb{Z}_l))$ of derived complete objects which will be equivalent to $D(\mathsf{Shv}_{\text{et}}(X, \mathbb{Z}_l))$.

$$2\text{-}\varprojlim_n D(\mathsf{Shv}_{\text{et}}(X, \mathbb{Z}/l^n)) \cong D_{\text{comp}}(\mathsf{Shv}_{\text{proét}}(X, \mathbb{Z}_l)) \subseteq D(\mathsf{Shv}_{\text{proét}}(X, \mathbb{Z}_l)).$$

1 Replete topoi

Definition 1. *A* topos *is a category equivalent to a category of the form* $\mathsf{Shv}_\tau(\mathcal{C})$ *for some category* $\mathcal{C}$ *and some Grothendieck topology* τ *on* $\mathcal{C}$*. Given an object*

$X \in \mathcal{C}$, we write h_X for the sheaf represented by X. I.e., the sheafification of the presheaf $\hom_{\mathcal{C}}(-, X)$.

Example 2. Given a topos $\mathcal{X}$, the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences $(\dots, F_2, F_1, F_0)$ of objects in $\mathcal{X}$ is also a topos.[1] The category $\mathcal{X}^{\mathbb{N}}$ of towers $(\dots \to F_2 \to F_1 \to F_0)$ of morphisms in $\mathcal{X}$ is also a topos.[2]

Definition 3. *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a topos. A morphism $F \to G$ of objects of $\mathcal{X}$ is* surjective *if for every object $X \in \mathcal{C}$ and $s \in G(X)$, there exists a covering $\{U_i \to X\}_{i \in I}$ such that $s|_{U_i}$ is in the image of $F(U_i) \to G(U_i)$ for each $i \in I$.*

Remark 4. It can be shown that a morphism $F \to G$ of sheaves is surjective if and only if for every sheaf H the induced map $\hom(G, H) \to \hom(F, H)$ is injective. I.e., if and only if $F \to G$ is surjective in the categorical sense. Another equivalent condition for $F \to G$ to be surjective is asking that $im(F \to G) \to G$ become an isomorphism (of presheaves) after sheafification. Here, by $im(F \to G)$ we mean the *presheaf* image, i.e., $im(F \to G)(U) = im(F(U) \to G(U))$ (this is not necessarily a sheaf).

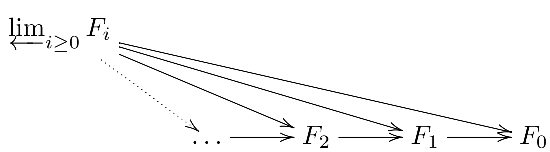
exer:epi **Exercise 1.** Note that:

(∗) For any presheaf F with sheafification aF, object $X \in \mathcal{C}$, and section $s \in aF(X)$, there exists a covering $\{U_i \to X\}_{i \in I}$ such that each $s|_{U_i}$ is in the image of $F(U_i) \to aF(U_i)$.

Let $\{V_i \to Y\}_{i \in I}$ be a covering family in a site $\mathcal{C}$. Using the axioms of a Grothendieck topology, and (∗) show that $\amalg_{i \in I} h_{V_i} \to h_Y$ is a surjective morphism of sheaves.

Remark 5. In the SGA definition of a covering family, the converse is also true: a family $\{V_i \to Y\}_{i \in I}$ is a covering family if and only if the induced morphism of sheaves $\amalg_{i \in I} h_{V_i} \to h_Y$ is a surjective.

Definition 6 ([BS, Def.3.1.1])**.** *A topos is* replete *(充実した) if for every tower of surjective morphisms $\cdots \to F_2 \to F_1 \to F_0$ the induced morphisms*



are surjective for all n.

[1] It is the category of sheaves on the disjoint union $\amalg_{n \in \mathbb{N}} \mathcal{C}$ equipped with the coarsest topology such that the inclusions $\mathcal{C} \to \amalg_{n \in \mathbb{N}} \mathcal{C}$ send covers to covers.

[2] It is the category of sheaves on the category $\mathbb{N} \times \mathcal{C}$ whose objects are pairs (n, X) consisting of an $n \in \mathbb{N}$ and an object $X \in \mathcal{C}$. Morphisms are $\hom((n, X), (m, Y)) = \varnothing$ if $n > m$ and $\hom(X, Y)$ otherwise. Again, the topology is the coarsest topology such that the inclusions $\mathcal{C} \to \mathbb{N} \times \mathcal{C}$ send covers to covers.

rema:limits **Remark 7.** Note: the inclusion $\mathsf{Shv}_\tau(\mathcal{C}) \subseteq \mathsf{PreShv}(\mathcal{C})$ preserves limits (but not all colimits). That is, $(\varprojlim_{i\in I} F_i)(X) = \varprojlim_{i\in I}(F_i(X))$ for any diagram of sheaves $I \to \mathsf{Shv}_\tau(\mathcal{C})$ and $X \in \mathcal{C}$ (to calculate colimits of sheaves, one takes the colimit in the category of presheaves and then sheafifies).

Exercise 2.

1. Show that the category of sets is replete. (Note, this is a topos: Set is the category of sheaves on the category $\overset{\mathrm{id}}{\curvearrowright}\!\!\!\!\!\! *$ with only one object equipped with the trivial Grothendieck topology).
2. Let $\mathcal{C}$ be a category equipped with the trivial Grothendieck topology,[3] so every presheaf is a sheaf. Show that $\mathsf{PreShv}(\mathcal{C})$ is replete.
3. Let G be a (discrete) group. Deduce that the category of G-sets is replete. Note: G-sets is the category of presheaves on the category BG which has one object, one morphism for every element of G, and composition is defined by the multiplication in G.

exam:etNotReplete **Example 8.** Let k be a field such that k^{sep}/k is not finite. Then the category $\mathsf{Shv}_{et}(k)$ of étale sheaves on k is not replete: Since k^{sep}/k is not finite there exists a tower $\dots/L_2/L_1/L_0 = k$ of nontrivial finite separable field extensions. Since each $\mathrm{Spec}(L_n) \to \mathrm{Spec}(L_{n-1})$ is a covering, each morphism in the tower induces a surjective morphism of sheaves. However,

$$\varprojlim_i h_{\mathrm{Spec}(L_i)} \to h_{\mathrm{Spec}(k)} \tag{1}$$ equa:idSurjEq

cannot be surjective.

Exercise 3. By evaluating on $X = \mathrm{Spec}(k)$ and considering $s = \mathrm{id}_k$ prove the claim that Example 8(1) is not surjective. Hint: recall that the coverings of $\mathrm{Spec}(k)$ are of the form $\{\mathrm{Spec}(K_j) \to \mathrm{Spec}(k)\}_{j\in J}$ with K_j/k finite separable field extensions.

Example 9 ([BS, Example 3.1.7])**.** The category of affine schemes with the fpqc topology[4] is replete. Suppose $\dots \to F_2 \to F_1 \to F_0$ is a tower of surjective morphisms, and consider some affine scheme $X = \mathrm{Spec}(A)$ and some $s \in F_0(X)$. Since $F_1 \to F_0$ is surjective, there is a faithfully flat morphism $A \to B_0$ such that $s|_{B_0}$ is in the image of $F_1(B_0) \to F_0(B_0)$. That is, there is some $s_1 \in F_1(B_0)$ mapping to $s|_{B_0}$. Repeating the argument, we find a tower of faithfully flat morphisms $A \to B_0 \to B_1 \to B_2 \to \dots$ and elements $s_i \in F_i(B_{i-1})$ such that s_i maps to $s_{i-1}|_{B_{i-1}}$. Set $B = \varinjlim B_i$. Now $A \to B$ is again a faithfully flat morphism, and the sequence $(s_n \in F_n(B_{n-1}))$ induces a sequence $(t_n \in F_n(B))$ such that $t_n \mapsto t_{n-1}$ for all n. In other words, it induces an element $t \in (\varprojlim F_i)(B)$. By construction, $s|_B = t$, and so we deduce that $\varprojlim F_i \to F_0$ is surjective. The same argument shows each $\varprojlim F_i \to F_n$ is surjective.

[3] I.e., the only covering families are families of the form $\{X \overset{\mathrm{id}}{\to} X\}$.

[4] I.e., the topology whose coverings are families $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)\}_{i\in I}$ such that each $A \to B_i$ is flat, and $\amalg \mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ is surjective.

Remark 10. Note that the reason the fpqc site is replete and the étale site is not replete is precisely because limits of coverings exist in the category, and are still coverings.

Our first goal is to show that countable products are exact in replete topoi. This is Proposition 14. Knowing that products are exact, makes derived limits easy to calculate, cf. Remark 15. The first step is the following lemma.

lem:3.1.8 **Lemma 11** ([BS, Lem.3.1.8])**.** *Let*

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 \end{array}$$

be morphisms in a replete topos, and suppose $F_i \to G_i$ *and* $F_{i+1} \to F_i \times_{G_i} G_{i+1}$ *are surjective for all* i*. Then* $\varprojlim F_i \to \varprojlim G_i$ *is surjective.*

Exercise 4. Prove Lemma 11 when the topos is the category of sets.

Exercise 5. This exercise shows that limits do not preserves surjections in general. So the hypotheses of Lemma 11 are really necessary.

1. Show that $(\dots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}) \to (\dots \to \mathbb{Z}/l^3 \to \mathbb{Z}/l^2 \to \mathbb{Z}/l)$ does not satisfy the hypotheses of Lemma 11.

2. Show that the limit of the above morphism of towers is $\mathbb{Z} \to \mathbb{Z}_l$. Show that this is not surjective.

exer:surjective **Exercise 6.** Suppose that $(f_i : B_i \to C_i)_{i \in \mathbb{N}}$ is a sequence of surjections. Show that the conditions of Lemma 11 are satisfied for $F_n = \prod_{0 \leq i \leq n} B_i$, and $G_n = \prod_{0 \leq i \leq n} C_i$. Note: $X \times_Y (Y \times Y') \cong X \times Y'$ so

$$(B_0 \times \dots \times B_n) \times_{(C_0 \times \dots \times C_n)} (C_0 \times \dots \times C_{n+1}) \cong (B_0 \times \dots \times B_n \times C_{n+1})$$

Deduce that $\prod_{i \in \mathbb{N}} f_i : \prod_{i \in \mathbb{N}} B_i \to \prod_{i \in \mathbb{N}} C_i$ is surjective.

exer:EpiLemma **Exercise 7.** Suppose that $\dots \to F_2 \stackrel{t_2}{\to} F_1 \stackrel{t_1}{\to} F_0$ is a tower of surjections in a replete topos $\mathcal{X}$.

1. Show that each map

$$\prod_{i=0}^{n+1} F_i \stackrel{t-\mathrm{id}}{\longrightarrow} \prod_{i=0}^{n} F_i;$$

$$(s_{n+1}, \dots, s_2, s_1, s_0) \mapsto (t_{n+1}s_{n+1} - s_n, \dots, t_2 s_2 - s_1, t_1 s_1 - s_0)$$

is surjective.

2. Using $B_n = \prod_{i=0}^{n+1} F_i$ and $C_n = \prod_{i=0}^{n} F_i$ and Exercise 6, show that

$$\prod_{\mathbb{N}} F_i \xrightarrow{t-\mathrm{id}} \prod_{\mathbb{N}} F_i$$

is surjective where $t-$id is the morphism $(\dots, c_2, c_1, c_0) \mapsto (\dots, tc_3-c_2, tc_2-c_1, tc_1-c_0)$.

Definition 12. *If $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ is a topos we write $D(\mathcal{X}) = D(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))$ for its derived category.*

Recall that if $\mathcal{X}$ is a topos, then the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences of objects and the category $\mathcal{X}^{\mathbb{N}}$ of towers of morphisms are also topoi. We can this consider the right derived functors associated to product and limit

$$R\Pi : D(\prod_{\mathbb{N}} \mathcal{X}) \to D(\mathcal{X}),$$

$$R\varprojlim : D(\mathcal{X}^{\mathbb{N}}) \to D(\mathcal{X}).$$

We prove the following proposition in an appendix to this lecture.

prop:rlimDef **Proposition 13** (See Proposition 43 below)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category with products and $(\dots \to C_2 \xrightarrow{t} C_1 \xrightarrow{t} C_0)$ a tower of chain complexes (the t's are different, but we ommit the indices). Then there is a isomorphism in $D(\mathcal{A})$*

$$R\varprojlim C_n \cong \mathrm{Cone}\left(R\Pi C_n \xrightarrow{t-\mathrm{id}} R\Pi C_n\right)[-1]$$

where $t - \mathrm{id}$ is the morphism $(\dots, c_2, c_1, c_0) \mapsto (\dots, tc_3-c_2, tc_2-c_1, tc_1-c_0)$.

One of the reasons we are interested in replete topoi is that limits work very well.

prop:derivedProd **Proposition 14.** *Let $\mathcal{X}$ be a replete topos. Then the functor $\Pi : \prod_{\mathbb{N}} \mathcal{X} \to \mathcal{X}$ preserves injections and surjections. In particular, Π preserves quasi-isomorphism of chain complexes and so induces a well-defined functor*

$$\Pi : D(\prod_{\mathbb{N}} \mathcal{X}) \to D(\mathcal{X})$$

which is just Π on each object.

Proof. We want to show that if $(f_i : F_i \to G_i)_{i\in\mathbb{N}}$ is a sequence of morphisms in $\mathcal{X}$ which is injective (resp. surjective) then $\prod f_i$ is injective (resp. surjective). It is automatically injective because limits always preserve monomorphisms. The surjective case is exactly Exercise 6. □

rema:derivedLimCalc **Remark 15.** Proposition 14 (combined with Prop.13) shows that in a replete topos, given a tower $(\dots \to K_2 \to K_1 \to K_0)$ in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))^{\mathbb{N}}$ of chain complexes of sheaves of abelian groups, we could *define* $R\varprojlim K_i$ as

$$\mathrm{Cone}(\prod K_i \to \prod K_i)[-1]$$

(where the products take place termwise in $\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})$). We will use this description in the future.

Proposition 16 ([BS, 3.1.10])**.** *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a replete topos and suppose $\cdots \to F_2 \xrightarrow{t} F_1 \xrightarrow{t} F_0$ is a tower of* surjective *morphisms in $\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})$. Then we have $\varprojlim F_i \cong R\varprojlim F_i$ in $D(\mathcal{X})$.*

Proof. Since each $F_{i+1} \to F_i$ is surjective, the morphism $t - \mathrm{id}$ is surjective by Exercise 7. So, we have a short exact sequence

$$0 \to \varprojlim F_i \to \prod F_i \xrightarrow{t-\mathrm{id}} \prod F_i \to 0.$$

Since products are automatically derived by Proposition 13 and Lemma 14 , we have

$$\varprojlim F_i \cong \mathrm{Cone}\left(\prod F_i \xrightarrow{\mathrm{shift}-\mathrm{id}} \prod F_i\right)[-1].$$

□

2 Locally weakly contractible topoi

coherentSite

Definition 17 (Cf.[Johnstone, Topos theory, Thm.7.35])**.** *A topos is said to be* coherent *if there is a site (C, τ) such that C has finite limits and for every covering $\{U_i \to X\}_{i\in I}$ there is a finite set $\{i_1, \ldots, i_n\} \subseteq I$ such that $\{U_{i_j} \to X\}_{j=1}^n$ is also a covering. We will call (C, τ) a* coherent site of definition.

Exercise 8. Let SCH be the category of all schemes and AFF the category of all affine schemes (both equipped with the Zariski topology).

1. Show that the canonical restriction functor $\mathsf{Shv}_{\mathsf{Zar}}(SCH) \to \mathsf{Shv}_{\mathsf{Zar}}(AFF)$ induces an equivalence of categories.

2. Show that $(C, \tau) = (AFF, \mathsf{Zar})$ satisfies the conditions of Definition 17.

3. Show that $(C, \tau) = (SCH, \mathsf{Zar})$ does not satisfy the conditions of Definition 17, but that none-the-less, $\mathsf{Shv}_{\mathsf{Zar}}(SCH)$ is a coherent topos.

Definition 18 ([Bs, Def.3.2.1])**.** *An object F of a topos $\mathcal{X}$ is* weakly contractible *if every surjection $G \to F$ has a section. Suppose $\mathcal{X}$ is a coherent topos with coherent site of definition (C, τ). We say that $\mathcal{X}$ is* locally weakly contractible *if every object $X \in \mathcal{X}$ admits a surjection $\amalg_{i\in I} Y_i \to X$ with Y_i weakly contractible objects, which are representable by objects of C.*

Exercise 9. 1. Suppose that $\mathcal{X} = \mathsf{Shv}_\tau(C)$ is a topos such that C is small. Show that for any sheaf F, the canonical morphism

$$\amalg_{X\in C, s\in F(X)} \hom_C(-, X) \to F$$

is surjective.

2. Suppose that $\mathcal{X} = \mathsf{Shv}_\tau(C)$ is a topos such that C is small, and suppose that $C' \subseteq C$ is a full subcategory such that C' is a coherent site of definition for $\mathcal{X}$ (e.g., $C = SCH, C' = AFF$ with the Zariski topology[5]). Show that $\mathcal{X}$ is locally weakly contractible if and only if for each $X \in C$, there is a covering family $\{U_i \to X\}_{i \in I}$ such that each U_i is in C', and is weakly contractible.

3. Let Sch_R be the category of schemes of finite presentation over a ring R equipped with the Zariski topology. Using the results about w-local spaces from the lecture on commutative algebra, show that $\mathsf{Shv}_{\mathsf{Zar}}(Sch_R)$ is locally weakly contractible.

Example 19. The pro-étale site that we define in the next lecture is locally weakly contractible.

Proposition 20 ([BS, Prop.3.2.3)**.** *] Let $\mathcal{X}$ be a locally weakly contractible topos. Then $\mathcal{X}$ is replete, and for any object $K \in D(\mathcal{X})$ we have $R\lim_n \tau^{\geq n} K \cong K$ where*

$$\tau^{\geq n} K = (\cdots \to 0 \to (K^n / dK^{n-1}) \to K^{n+1} \to K^{n+2} \to \dots).$$

Remark 21. The property $R\lim_n \tau^{\geq n} K \cong K$ means that all the information of K is contained in its truncations. This lets us deduce properties of unbounded complexes from bounded below complexes.

Sketch of proof. Since $\mathcal{X}$ is locally weakly contractible, a morphism f in $\mathcal{X}$ is an isomorphism (resp. surjection) if and only if evaluating on each weakly contractible object of $\mathcal{C}$ is an isomorphism (resp. surjection). It follows that $\mathcal{X}$ is replete.

Similarly, for any complex of sheaves K and weakly contractible object U we have $(H^i K)(U) = H^i(K(U))$. It follows that $R\lim_n \tau^{\geq n} K \cong K$. ☐

3 Truncation completing derived categories

sec:TruncationCompletion

Recall that if $\mathcal{X}$ is a category, then $\mathcal{X}^{\mathbb{N}}$ is the category of towers $(\dots \to F_2 \to F_1 \to F_0)$ of morphisms in $\mathcal{X}$. If $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ is a topos, then $D(\mathcal{X}^{\mathbb{N}})$ is the derived category of the abelian category $\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})^{\mathbb{N}}$.

Now that we are working with towers of chain complexes, we will have two indices: an upper index for the terms in the chain complex, and a lower index for the terms in the tower.

[5]The categories SCH and AFF are not small, but we can instead consider a variant such as: choose an uncountable strong limit cardinal κ and only consider those schemes that can be build using sets of size $< kappa$. Another alternative is to choose some base ring R, and consider the categories Sch_R and Aff_R of R-schemes (resp. affine R-schemes) locally of presentation.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\
\dots & \longrightarrow & K_2^{i+1} & \xrightarrow{t} & K_1^{i+1} & \xrightarrow{t} & K_0^{i+1} \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\
\dots & \longrightarrow & K_2^{i} & \xrightarrow{t} & K_1^{i} & \xrightarrow{t} & K_0^{i} \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\
\dots & \longrightarrow & K_2^{i-1} & \xrightarrow{t} & K_1^{i-1} & \xrightarrow{t} & K_0^{i-1} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

Here, the d's and t's should have indices too, but we did not write them. Note that $Ch(\mathcal{A})^{\mathbb{N}} = Ch(\mathcal{A}^{\mathbb{N}})$. That is, we can think about objects in this category as towers of chain complexes $\xrightarrow{t_2} \begin{pmatrix}\vdots\end{pmatrix} \xrightarrow{t_1} \begin{pmatrix}\vdots\end{pmatrix} \xrightarrow{t_0} \begin{pmatrix}\vdots\end{pmatrix}$ or chain complexes of towers

$$
\begin{array}{c}
\uparrow{\scriptstyle d^{i+1}} \\
(\dots) \\
\uparrow{\scriptstyle d^{i}} \\
(\dots) \\
\uparrow
\end{array}
$$

Definition 22 ([BS, 3.3.1]). *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a topos. We define the* left-completion $\widehat{D}(\mathcal{X})$ *of $D(\mathcal{X})$ as the full subcategory of $D(\mathcal{X}^{\mathbb{N}})$ spanned by the projection systems $(\dots \to K_2 \to K_1 \to K_0)$ in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})^{\mathbb{N}})$ such that*

1. *$K_n \in D^{\geq -n}(\mathcal{X})$. That is, $H^i K_n = 0$ for $i < -n$.*
2. *The canonical map $\tau^{\geq -n} K_{n+1} \to K_n$ is an equivalence. In other words, the map $H^i K_{n+1} \to H^i K_n$ is an isomorphism for all $i \geq -n$.*

We say that $D(\mathcal{X})$ is left-complete *if the map*

$$\tau : D(\mathcal{X}) \to \widehat{D}(\mathcal{X}); \qquad K \mapsto \{\tau^{\geq -n} K\}$$

is an equivalence.

Remark 23. The definition is equivalent to asking that when we take cohomology, we get the following picture:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
\dots & \xrightarrow{\cong} & H^2K_2 & \xrightarrow{\cong} & H^2K_1 & \xrightarrow{\cong} & H^2K_0 \\
\dots & \xrightarrow{\cong} & H^1K_2 & \xrightarrow{\cong} & H^1K_1 & \xrightarrow{\cong} & H^1K_0 \\
\dots & \xrightarrow{\cong} & H^0K_2 & \xrightarrow{\cong} & H^0K_1 & \xrightarrow{\cong} & H^0K_0 \\
\dots & \xrightarrow{\cong} & H^{-1}K_2 & \xrightarrow{\cong} & H^{-1}K_1 & \longrightarrow & 0 \\
\dots & \xrightarrow{\cong} & H^2K_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

Remark 24. The inclusion $\widehat{D}(\mathcal{X}) \subseteq D(\mathcal{X})$ is not an inclusion of triangulated categories (because $\widehat{D}(\mathcal{X}))$ is not preserve by the deshift $[-1]$ from $D(\mathcal{X})$.

We just state the main facts about completions without giving too many details.

Theorem 25. *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a topos.*

1. *[BS, Lem.3.3.2] The functor $R\varprojlim : \widehat{D}(\mathcal{X}) \to D(\mathcal{X}^{\mathbb{N}}) \to D(\mathcal{X})$ is the right adjoint of τ. In particular, if $D(\mathcal{X})$ is left-complete, then $K \cong R\varprojlim \tau^{-n}K$ for any $K \in Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))$.*

2. *[BS, Prop.3.3.3] If $\mathcal{X}$ is a replete topos then $D(\mathcal{X})$ is left-complete.*

3. *[BS, Exam.3.3.5] If $k = \mathbb{C}(x_1, x_2, \dots)$, then $D(\mathrm{Spec}(k)_{et})$ is not left-complete.*

4. *[BS, Prop.3.3.7] If $U \in \mathcal{C}$ is an object such that $\Gamma(U, -)$ is exact then for any $K \in D(\mathcal{X})$ we have $R\Gamma(U, K) \cong R\varprojlim R\Gamma(U, \tau^{-n}K)$.*

5. *[BS, Prop.3.3.7] If for each $K \in D(\mathcal{X})$ and $U \in \mathcal{C}$ there exists some $d \in \mathbb{N}$ such that $H^p(U, \underline{H}^q K) = 0$ for $p > d$, then $D(\mathcal{X})$ is left-complete.*

Example 26. The finiteness condition of [BS, Prop.3.3.7] above is satisfied for the étale sites of $\mathrm{Spec}(\mathbb{F}_q)$, and X when X is a smooth affine variety over an algebraically closed field.

Example 27 ([BS, 3.3.4, 3.3.5])**.** Let $k := \mathbb{C}(x_1, x_2, x_3, \dots)$ be a field of countable transcendence degree over $\mathbb{C}$. Then $D(\mathsf{Shv}(k_{\mathsf{et}}, \mathrm{Ab}))$ is not left complete. We discuss this example at length in Section B at the end of this pdf.

4 ℓ-adically completing objects

sec:l-adic

Prior to the pro-étale cohomology, the most widely spread triangulated category to work with the six functors $f^*, f_*, f_!, f^!, \otimes, \underline{\text{hom}}$ on constructible ℓ-adic étale cohomology was developed by Ekedahl. Ekedahl's category corresponds to a full subcategory of *derived complete* complexes of proétale sheaves of $\mathbb{Z}_\ell$-modules satisfying a certain finiteness condition. In this section we discuss this notion of derived completeness.

Through-out this section we work with the discrete valuation ring $\mathbb{Z}_\ell$. We also fix a replete topos $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$, and now our derived category will always be the derived category of sheaves of $\mathbb{Z}_\ell$-modules

$$D(\mathcal{X}, \mathbb{Z}_\ell) = D(\mathsf{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell)).$$

Definition 28. *We say that $M \in \mathrm{Mod}_{\mathbb{Z}_\ell}$ is* classically complete *if $M \cong \varprojlim M/\ell^n M$. We write $\mathrm{Mod}_{\mathbb{Z}_\ell,\mathrm{comp}} \subseteq \mathrm{Mod}_{\mathbb{Z}_\ell}$ for the full subcategory of classically complete modules.*

Exercise 10. Suppose that $M \in \mathrm{Mod}_{\mathbb{Z}_\ell}$ is ℓ-torsion free. That is, the multiplication by ℓ map $\ell : M \to M;\ m \mapsto \ell m$ is injective.

Show that $M \cong \varprojlim M/\ell^n M$ if and only if both $\varinjlim(\dots \xrightarrow{\ell} M \xrightarrow{\ell} M)$ and $\varinjlim^1(\dots \xrightarrow{\ell} M \xrightarrow{\ell} M)$ are zero,[6]

Hint: Consider the short exact sequences $0 \to M \xrightarrow{\ell^n} M \to M/\ell^n \to 0$.

Exercise 11. Show that $\mathbb{Z}_l$ and $\mathbb{Z}/l^n$ are classically complete $\mathbb{Z}_l$-modules but $\mathbb{Q}_\ell$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell \cong \varinjlim(\mathbb{Z}/\ell\mathbb{Z} \to \mathbb{Z}/\ell^2\mathbb{Z} \to \dots)$ are not. (The group homomorphisms $\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^{n+1}\mathbb{Z}$ in the colimit send $\overline{1}$ to $\overline{\ell}$).

Definition 29. *Given a complex $K \in D(\mathcal{X}, \mathbb{Z}_\ell)$ we define*

$$T(K) := R\varprojlim(\dots \xrightarrow{\ell} K \xrightarrow{\ell} K \xrightarrow{\ell} K).$$

We say K is derived complete *if $T(K) \cong 0$ in $D(\mathcal{X}, \mathbb{Z}_\ell)$ where the transition maps are multiplication ℓ. We use the notation $D_{\mathrm{comp}}(\mathcal{X}, \mathbb{Z}_\ell) \subseteq D(\mathcal{X}, \mathbb{Z}_\ell)$ for the full subcategory of derived complete objects.*

Remark 30. Since we are assuming that $\mathcal{X}$ is replete, by Proposition 14 we have

$$T(K) = \mathrm{Cone}\left(\prod_{\mathbb{N}} K \xrightarrow{\mathrm{id} - \ell} \prod_{\mathbb{N}} K\right)[-1].$$

Remark 31. Later on we will see that $T(K) = 0$ if and only if the canonical map $K \to R\varprojlim(K \otimes^L_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^n)$ is a quasi-isomorphism.

[6] Recall that $\varinjlim^1$ is a functor such that for any short exact sequence of towers $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ induces a long exact sequence $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \varprojlim^1 A_n \to \varprojlim^1 B_n \to \varprojlim^1 C_n \to 0$

exer:derivedCompleteCone **Exercise 12.**

1. Show that if K is derived complete then so is $K[n]$ for any n.

2. Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of chain complexes in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$. Using the fact that products in a replete topos are exact, show that $0 \to TA \to TB \to TC \to 0$ is also a short exact sequence. Deduce that if two of A, B, C are derived complete, then so is the third.

3. Consider a morphism $K \to L$ in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$ and define $C = Cone(K \to L)$. Use the second part above to show that if two of K, L, C are derived complete then the third is also derived complete.

exer:derivedCompleteLim **Exercise 13.** Using the fact that for any *double* sequence of chain complexes $(K_{n,m})$ we have a canonical isomorphism $R\varprojlim_n R\varprojlim_m K_{n,m} \cong R\varprojlim_m R\varprojlim_n K_{n,m}$, show that if $(\dots \to K_2 \to K_1 \to K_0)$ is a sequence of derived complete chain complexes then $R\varprojlim K_n$ is derived complete.

The relationship between classical complete and derived complete is the following.

Proposition 32 ([BS, Prop.3.4.2])**.** *An $\mathbb{Z}_\ell$-module $M \in \mathrm{Mod}_{\mathbb{Z}_\ell}$ is classically complete if and only if it is ℓ-adically separated*[7] *and derived complete.*

Constructing derived complete modules which are not classically complete is not so easy, however they do arise quite naturally.

Example 33 ([Stacks project, 0G3F], [SAG])**.** Let $\mathbb{Z}_\ell\langle t\rangle$ be the ℓ-adic completion of the $\mathbb{Z}_\ell$-module $\mathbb{Z}_\ell[t]$. Then the cokernel of the ring homomorphism

$$\lambda : \mathbb{Z}_\ell\langle t\rangle \to \mathbb{Z}_\ell\langle t\rangle; \qquad t \mapsto \ell t$$

is derived complete, but not ℓ-adically separated.

Indeed, $\mathbb{Z}_\ell\langle t\rangle$ is classically complete by definition, so it is derived complete, and derived complete-ness is preserved under cokernels, so $coker(\lambda)$ is derived complete. On the other hand, the element $\sum_{n\geq 0} \ell^n t^n \in \mathbb{Z}\langle t\rangle$ is not in the image of λ, so it is non-zero in $coker(\lambda)$. For every m, the element $1 + \ell t + \dots + \ell^m t^m$ is in the image of λ, so $\sum_{n\geq 0} \ell^n t^n \sim \sum_{n>m} \ell^n t^n$ in $coker(\lambda)$, so $\sum_{n\geq 0} \ell^n t^n \in \cap_{m\geq 0} \ell^m coker(\lambda)$.

In particular, for classical $\mathbb{Z}_\ell$-modules, classical completeness is strictly stronger than derived completeness.

We omit the proof of 3.4.2 as it is not used elsewhere.

BS344 **Proposition 34** ([BS, Prop.3.4.4])**.** *An $\mathbb{Z}_\ell$-complex $K \in D(\mathcal{X}, \mathbb{Z}_\ell)$ is derived complete if and only if each $H^i K \in \mathsf{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell)$ is derived complete.*

[7] ℓ-adically separated means that $\cap_{n\in\mathbb{N}} \ell^n M = 0$.

Recall that for a chain complex K we define

$$\tau^{\geq n}K = [\cdots \to 0 \to 0 \to (K^n/dK^{n-1}) \to K^{n+1} \to K^{n+2} \to \dots]$$

$$\tau^{\leq n}K = [\cdots \to K^{n-2} \to K^{n-1} \to (\ker\ d) \to 0 \to 0 \to \dots]$$

Exercise 14. Show that $H^i\tau^{\leq n}K = H^iK$ for $i \leq n$ and $H^i\tau^{\leq n}K = 0$ for $i > n$. Similarly, show that $H^i\tau^{\geq n}K = H^iK$ for $i \geq n$ and $H^i\tau^{\geq n}K = 0$ for $i < n$.

Proof. Suppose that each H^iK is derived complete. We will show that K is derived complete. For any $i \in \mathbb{N}, n \in \mathbb{Z}$ we have

$$\text{Cone}\left(\tau^{\leq n+i}\tau^{\geq n}K \to \tau^{\leq n+i+1}\tau^{\geq n}K\right) \underset{\sim}{\overset{q.i.}{\to}} H^{i+1}K$$

so by induction on i, and Exercise 12, each $\tau^{\leq n+i}\tau^{\geq n}K$ is derived complete. Now we are assuming that $\mathcal{X}$ is replete, so in particular, we have

$$\tau^{\leq m}K \cong R\varprojlim_{n\in\mathbb{N}} \tau^{-n}\tau^{\leq m}K.$$

So by Exercise 13, we find that $\tau^{\leq m}K$ is derived complete. Now consider the short exact sequence of complexes

$$0 \to K \to \text{Cone}\left(\tau^{\leq m}K {\to} K\right) \to \tau^{\leq m}K[1] \to 0$$

By Exercise 12 the functor T takes short exact sequence to short exact sequences. Since $\tau^{\leq m}K$ is derived complete, we deduce that

$$TK \overset{q.i.}{\to} T\,\text{Cone}\left(\tau^{\leq m}K \to K\right)$$

But

$$\text{Cone}\left(\tau^{\leq m}K \to K\right) \overset{q.i.}{\to} \tau^{\geq m+1}K$$

so

$$TK \overset{q.i.}{\to} T\tau^{\geq m+1}K$$

Finally, from the definition we see that $(T\tau^{\geq m+1}K)^i = 0$ for $i < m$. Since this is valid for any m, we deduce that $H^iTK = 0$ for all i. $\square$

Definition 35. *Suppose that $K \in Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$ is a chain complex. Then we define*

$$K \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n := Cone(K \overset{\ell^n}{\to} K).$$

Remark 36. The functor $- \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n$ that we defined above actually calculates the left derived functor of $- \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n$ where here $\otimes_{\mathbb{Z}_\ell}$ is the usual tensor product. Since we only need the derived product in this case, we just take this as the definition.

exer:derCompletion **Exercise 15.**

1. Show that there is a canonical morphism of sequences of chain complexes from $(\dots \xrightarrow{\ell} K \xrightarrow{\ell} K \xrightarrow{\ell} K)$ to $(\dots \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K)$

2. Deduce that there is a canonical morphism from $(\dots \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K)$ to $(\dots \to K \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^2 \to K \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell \to K)$.

3. Show that there is a short exact sequence

$$0 \to K \to \widehat{K} \to TK \to 0$$

where

$$\widehat{K} := R\varprojlim(K \overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n).$$

Deduce that K is derived complete if and only if the morphism $K \to \widehat{K}$ is a quasi-isomorphism.

Proposition 37 ([BS, Lem.3.4.9, Prop.3.5.1])**.** *The functor sending K to $\widehat{K}$ defines a left adjoint to the inclusion $D_{\mathrm{comp}}(\mathcal{X}, \mathbb{Z}_\ell) \subseteq D(\mathcal{X}, \mathbb{Z}_\ell)$.*

Sketch of proof. By Exercise 15 we see that $\widehat{K}$ is derived complete. Suppose that $L \in D(\mathcal{X}, \mathbb{Z}_\ell)$ is also derived complete. Then we want to show that

$$\hom_{D(\mathcal{X},\mathbb{Z}_\ell)}(\widehat{K}, L) \to \hom_{D(\mathcal{X},\mathbb{Z}_\ell)}(K, L)$$

is an isomorphism. By the short exact sequence in Exercise 15(3) it suffices to show that

$$\hom_{D(\mathcal{X},\mathbb{Z}_\ell)}(TK, L) = 0$$

(this uses some homological algebra that we have not covered, but it is not difficult homological algebra). Now we make two claims.

Claim 1. [BS, Lem.3.4.7] We have that $\hom(M, L) = 0$ for all $M \in D(\mathcal{X}, \mathbb{Q}_\ell)$.

Claim 2. [BS, Lem.3.4.8] We have that TK is in the essential image of the canonical functor $D(\mathcal{X}, \mathbb{Q}_\ell) \to D(\mathcal{X}, \mathbb{Z}_\ell)$.

The proof of these claims is not difficult, but is omitted. □

Definition 38. *We define a tensor product on $D_{\mathrm{comp}}(\mathcal{X}, \mathbb{Z}_\ell)$ using the tensor product on $D(\mathcal{X}, \mathbb{Z}_\ell)$:*

$$K \widehat{\otimes}_{\mathbb{Z}_\ell} L := \widehat{K \overset{L}{\otimes}_{\mathbb{Z}_\ell} L}.$$

Here, $\overset{L}{\otimes}_{\mathbb{Z}_\ell}$ is the derived tensor product on $D(\mathcal{X}, \mathbb{Z}_\ell)$.

A Derived limits

In this subsection we consider a Grothendieck abelian category $\mathcal{A}$ that admits products (in other words, satisfies Grothendieck's axiom (AB3*)). We are concerned with the derived functors

$$R\Pi : D(\prod_{\mathbb{N}} \mathcal{A}) \to D(\mathcal{A})$$

$$R\varprojlim : D(\mathcal{A}^{\mathbb{N}}) \to D(\mathcal{A})$$

associated to product $\Pi : \prod_{\mathbb{N}} \mathcal{A} \to \mathcal{A}$ and limit $\varprojlim : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$. Note that $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ are again Grothendieck abelian categories (since they are functor categories from a small category to a Grothendieck abelian category).

Recall from the lecture Homological Algebra I that for a general left exact functor between Grothendieck abelian categories $F : \mathcal{B} \to \mathcal{B}'$, the derived functor $RF : D(\mathcal{B}) \to D(\mathcal{B}')$ can be calculated as follows. If $C \in Ch^+(\mathcal{B})$ is a bounded below chain complex, then there exists a quasi-isomorphism $C \to I$ with I a bounded below chain complex of injective objects,[8] and $RF(C) \cong F(I)$ in $D(\mathcal{B}')$. More generally, for any chain complex $C \in Ch(\mathcal{B})$, there exists a quasi-isomorphism $C \to Q$ to a fibrant chain complex,[9] and $RF(C) \cong F(Q)$ in $D(\mathcal{B}')$.

lemm:injProd

Lemma 39. *An object $(I_i)_{i\in\mathbb{N}}$ in $\prod_{\mathbb{N}} \mathcal{A}$ is injective if and only if each I_i is injective in $\mathcal{A}$.*

Exercise 16. Prove Lemma 39.

lemm:injectiveSequences

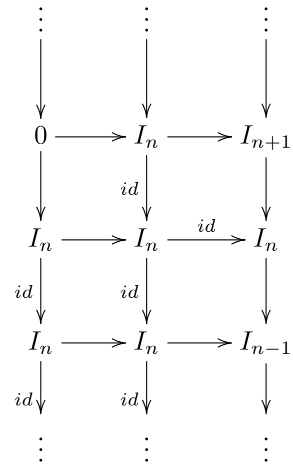
Lemma 40. *An object $(\cdots \to A_2 \to A_1 \to A_0)$ in $\mathcal{A}^{\mathbb{N}}$ is injective if and only if each A_i is injective and each $A_{i+1} \to A_i$ is a split surjection.*

Proof. Suppose $\mathcal{I}_\bullet = (\cdots \to I_2 \to I_1 \to I_0)$ is an injective object in $\mathcal{A}^{\mathbb{N}}$. Let $\lambda_n : \mathcal{A} \to \mathcal{A}^{\mathbb{N}}$ be the functor sending $A \in \mathcal{A}$ to $(\cdots \to 0 \to \underbrace{A \stackrel{id}{\to} \dots \stackrel{id}{\to} A}_{n \text{ morphisms}})$. Then λ_n is exact and a left adjoint to the "evaluation at n" functor Ev_n (which sends $(\cdots \to B_2 \to B_1 \to B_0)$ to B_n). Since Ev_n has an exact left adjoint it sends injectives to injectives, and hence, each $I_n = Ev_n \mathcal{I}_\bullet$ is injective in $\mathcal{A}$. To see that each $I_{n+1} \to I_n$ is split surjective, consider the canonical monomorphism $\lambda_n I_n \to \lambda_{n+1} I_n$. Since $\mathcal{I}_\bullet$ is injective, the canonical morphism $\lambda_n I_n \to \mathcal{I}_\bullet$ factors as $\lambda_n I_n \to \lambda_{n+1} I_n \to \mathcal{I}_\bullet$. The degree $n+1, n, n-1$ piece of this

[8] Recall that an object $I \in \mathcal{B}$ is injective if for every monomorphism $A \to B$, every morphism $A \to I$ factors through $A \to B$.

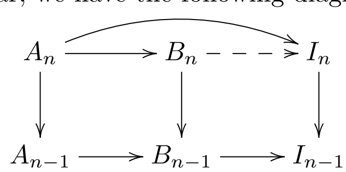
[9] Recall that a chain complex $Q \in Ch(\mathcal{B})$ is fibrant if for every monomorphic quasi-isomorphism $A \to B$ of chain complexes, every morphism $A \to Q$ factors through $A \to B$.

factorisation is



So $I_{n+1} \to I_n$ is split surjective.

Conversely, suppose that $I_\bullet = (\cdots \to I_2 \to I_1 \to I_0)$ is an object of $\mathcal{A}^{\mathbb{N}}$ such that each I_n is injective in $\mathcal{A}$, and each $I_{n+1} \to I_n$ is split surjective. Suppose that $A_\bullet = (\ldots \to A_2 \to A_1 \to A_0) \to (\ldots \to B_2 \to B_1 \to B_0) = B_\bullet$ is a monomorphism in $\mathcal{A}^{\mathbb{N}}$, and that $A_\bullet \to I_\bullet$ is some morphism. We will show by induction that it factors through $A_\bullet \to B_\bullet$. In degree 0, this follows from the fact that I_0 is injective: $A_0 \to B_0$ is a monomorphism and I_0 injective so $A_0 \to I_0$ factors as $A_0 \to B_0 \to I_0$. Suppose that we have factorisations $A_i \to B_i \to I_i$ for all $0 \leq i < n$ which are compatible with the transition morphisms of $A_\bullet, B_\bullet, I_\bullet$ respectively. In particular, we have the following diagram



and we are looking for the dashed morphism which makes the diagram commute. By hypothesis, $I_n \to I_{n-1}$ is split surjective. That is, $I_n \cong I_{n-1} \oplus J$ for some J, which is also injective as it is a direct summand of the injective object I_n. As J is injective, the induced morphism $A_n \to J$ factors as $A_n \to B_n \stackrel{a}{\to} J$. On the other hand, we have the morphism $b : B_n \to B_{n-1} \to I_{n-1}$ from the above diagram. Then we define the dashed morphism to be $(b, a) : B_n \to I_{n-1} \oplus J \cong I_n$. On checks that this makes the diagram commute. $\square$

Now that we consider chain complexes in $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ we will have two indices, (an upper) one for the chain complex direction, and (a lower) one for the $\prod_{\mathbb{N}} \mathcal{A}$, $\mathcal{A}^{\mathbb{N}}$ direction. We will implicitly use the canonical equivalences of categories $Ch(\prod_{\mathbb{N}} \mathcal{A}) \cong \prod_{\mathbb{N}} Ch(\mathcal{A})$ and $Ch(\mathcal{A}^{\mathbb{N}}) \cong Ch(\mathcal{A})^{\mathbb{N}}$.

注意 Beware, however, that the canonical inclusions $Ch^+(\prod_{\mathbb{N}} \mathcal{A}) \subseteq \prod_{\mathbb{N}} Ch^+(\mathcal{A})$ and $Ch^+(\mathcal{A}^{\mathbb{N}}) \subseteq Ch^+(\mathcal{A})^{\mathbb{N}}$ are *not* essentially surjective.

Lemma 41. *A chain complex $(Q_i^\bullet)_{i\in\mathbb{N}}$ in $Ch(\prod_{\mathbb{N}} \mathcal{A})$ is fibrant if and only if each $Q_i^\bullet$ is fibrant in $Ch(\mathcal{A})$.*

Proof. It suffices to note that a morphism $(A_i^\bullet)_{i\in\mathbb{N}} \to (B_i^\bullet)_{i\in\mathbb{N}}$ is a monomorphic quasi-isomorphism if and only if each $A_i^\bullet \to B_i^\bullet$ is a monomorphic quasi-isomorphism. □

lemm:seqFibrant **Lemma 42.** *If a chain complex $(Q_i^\bullet)_{i\in\mathbb{N}}$ in $Ch(\mathcal{A}^{\mathbb{N}})$ is fibrant then each $Q_i^\bullet$ is fibrant in $Ch(\mathcal{A})$.*

Proof. As in the proof of Lemma 40, the "evaluation at n" functor $Ev_n : Ch(\mathcal{A}^{\mathbb{N}}) \to Ch(\mathcal{A})$ has a left adjoint $\lambda_n : Ch(\mathcal{A}) \to Ch(\mathcal{A}^{\mathbb{N}})$ which preserves monomorphisms and quasi-isomorphisms. Consequently, Ev_n sends fibrant objects to fibrant objects. □

prop:limTriangle **Proposition 43.** *Suppose that $\mathcal{A}$ is a Grothendieck abelian category with products. Then for any object $(\dots \to C_2^\bullet \to C_1^\bullet \to C_0^\bullet)$ in $Ch(\mathcal{A}^{\mathbb{N}})$, there is an isomorphism*

$$R\varprojlim C_n^\bullet \cong \operatorname{Cone}\left(R\Pi C_n^\bullet \stackrel{id-shift}{\longrightarrow} R\Pi C_n^\bullet\right)$$

in $D(\mathcal{A})$.

Proof. In order to calculate $R\varprojlim C_n^\bullet$, replace $(\dots \to C_2^\bullet \to C_1^\bullet \to C_0^\bullet)$ with a quasi-isomorphic fibrant complex $(\dots \to Q_2^\bullet \to Q_1^\bullet \to Q_0^\bullet)$ in $Ch(\mathcal{A}^{\mathbb{N}})$. Recall that every fibrant chain complex is a chain complex of injective objects (the converse is true if the complex is bounded below). In particular, for each i the sequence $(\dots \to Q_2^i \to Q_1^i \to Q_0^i)$ is injective in $\mathcal{A}^{\mathbb{N}}$, and therefore by Lemma 40, the morphisms $Q_{n+1}^i \to Q_n^i$ are split surjective. We will use this fact later.

Now by Lemma 42 each $Q_n^\bullet$ is fibrant. Hence, $(Q_\bullet^\bullet)$ can also be used to calculate the derived products as well. That is,

$$R\varprojlim C_n^\bullet \cong \varprojlim Q_n^\bullet, \qquad R\Pi C_n^\bullet \cong \Pi Q_n^\bullet$$

So it suffices to show that the canonical morphism

$$\varprojlim Q_n^\bullet \to \operatorname{Cone}\left(\Pi Q_n^\bullet \stackrel{id-\text{shift}}{\longrightarrow} \Pi Q_n^\bullet\right)[-1]$$

is a quasi-isomorphism. But since each $Q_{n+1}^i \to Q_n^i$ is split surjective, it follows that each $\Pi Q_n^i \stackrel{id-\text{shift}}{\longrightarrow} \Pi Q_n^i$ is surjective. So the sequence

$$0 \to \varprojlim Q_n^\bullet \to \Pi Q_n^\bullet \stackrel{id-\text{shift}}{\longrightarrow} \Pi Q_n^\bullet \to 0$$

is exact, and therefore the left term is quasi-isomorphic to the shifted cone of the right morphism. □

B A worked example

example

Example 44 ([BS, 3.3.4, 3.3.5])**.** Let $k := \mathbb{C}(x_1, x_2, x_3, \dots)$ be a field of countable transcendence degree over $\mathbb{C}$. We will show that $\mathsf{Shv}(k_{\mathsf{et}}, \mathrm{Ab})$ is not left complete.

First we describe a setup in a general topos $\mathsf{Shv}_\tau(\mathcal{C})$. Let $F_1, F_2, \dots$ be a sequence of sheaves. Define $K = \oplus_{n\geq 1} F_n[n]$. That is, K is the chain complex of sheaves

$$K = (\dots \to F_n \stackrel{0}{\to} \dots \stackrel{0}{\to} F_1 \to 0 \to \dots)$$

with all differentials zero. We will find a criterion for $K \to \widehat{K}$ to not be a weak equivalence and then give a concrete choice of F_n in the case $\mathsf{Shv}(k_{\mathsf{et}}, \mathrm{Ab})$ fulfilling this criterion.

First notice that the completion $\widehat{K}$ of K is $(R\prod)F_n[n]$. This can be seen by noticing that $\tau^{\geq -n}K = \oplus_{i=1}^n F_n[n] = \prod_{i=1}^n F_n[n]$, taking injective resolutions $F_n \to I_n^\bullet$, and noticing that the sequence

$$\prod_{i\geq 1} I_i^\bullet[i] \stackrel{\phi}{\to} \prod_{n\geq 1}\prod_{i=1}^{n} I_n^\bullet[n] \stackrel{\mathrm{id}-\mathrm{shift}}{\to} \prod_{n\geq 1}\prod_{i=1}^{n} I_n^\bullet[n]$$

is a short exact sequence of chain complexes of sheaves.[10]

So our claim is that

$$K = \oplus_{n\geq 1} F_n[n] \to (R\prod_{n\geq 1})F_n[n] = \widehat{K}$$

is not a weak equivalence for well chosen F_n. To prove this choose a filtered system of objects $(P_\lambda)_{\lambda\in\Lambda}$ in the site $\mathcal{C}$ such that the functor

$$\Phi : \mathsf{PreShv}(\mathcal{C}) \to \mathrm{Ab}$$

defined by

$$\Phi : F \mapsto \varinjlim_{\Lambda} F(P_\lambda)$$

factors through $\mathsf{PreShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$. That is, choose a fibre functor. (In the case of $\mathsf{Shv}(k_{\mathsf{et}})$, the system (P_λ) will be the system $(\mathrm{Spec}(L))_{k\subseteq L\subseteq k^{sep}}$ of all finite Galois sub-extensions of k^{sep}/k.)

Note also that since Φ is defined by a filtered colimit, it commute with finite limits and all colimits, and in particular, preserves quasi-isomorphisms. So, extending Φ to chain complexes in the obvious way, there is a canonical factorisation

$$\mathsf{Comp}(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})) \to D(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})) \to D(\mathrm{Ab}).$$

[10] Here, shift is the map induced by the projection $\prod_{i=1}^n \to \prod_{i=1}^{n-1}$, the map ϕ is the product (in n) of the projections $\prod_{i\geq 1} \to \prod_{i=1}^n$. The map ϕ has retraction $\rho : \prod_{n\geq 1}\prod_{i=1}^n \to \prod_{n\geq 1}\prod_{i=n}^n \cong \prod_{i\geq 1}$. The induced section to $\mathrm{id}-\mathrm{shift}$ is $\mathrm{id}+\mathrm{deshift}$ where deshift is induced by the inclusions $\prod_{i=1}^{n-1} \to \prod_{i=1}^n$ adding zero in the nth component.

To show that $K \to \widehat{K}$ is not an equivalence in $D(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))$ it suffices to show that $\Phi(K) \to \Phi(\widehat{K})$ is not a weak equivalence in $D(\mathrm{Ab})$. For this, it suffices to show that $H^0\Phi(K) \to H^0\Phi(\widehat{K})$ is not an isomorphism. Since $H^0\Phi(K)$ is zero (because Φ does not need to be derived and commutes with all sums) it suffices to produce a nonzero element in $H^0\Phi(\widehat{K}) = H^0\Phi((R\prod_{n\geq 1})F_n[n])$. First let's get a description of it.

We claim that

$$H^0\Phi((R\prod_{n\geq 1})F_n[n]) = \varinjlim_{\Lambda} \prod H^n_\tau(P_\lambda, F_n).$$

To see this, choose injective resolutions for the F_n. These can be used to calculate $R\prod$. Then we have the following. We write H^n_{Ab} to emphasise that we are taking the cohomology of an object of $D(\mathrm{Ab})$. Note also that product of sheaves commutes with evaluation at an object. So we don't need brackets for expressions such as $\prod I^\bullet_n[n](P_\lambda)$.

$$\begin{aligned}
H^0_{\mathrm{Ab}}\Phi\left((R\prod)F_n[n]\right) &= H^0_{\mathrm{Ab}}\Phi\left(\prod I^\bullet_n[n]\right)\\
&= H^0_{\mathrm{Ab}} \varinjlim_{\Lambda} \prod I^\bullet_n[n](P_\lambda)\\
&= \varinjlim_{\Lambda} H^0_{\mathrm{Ab}}\left(\prod I^\bullet_n[n](P_\lambda)\right)\\
&= \varinjlim_{\Lambda} \left(\prod H^0_{\mathrm{Ab}}(I^\bullet_n[n](P_\lambda))\right)\\
&= \varinjlim_{\Lambda} \left(\prod H^n_{\mathrm{Ab}}(I^\bullet_n(P_\lambda))\right)\\
&= \varinjlim_{\Lambda} \prod H^n_\tau(P_\lambda, F_n)
\end{aligned}$$

Without loss of generality, we can assume Λ has an initial object 0. Here are our criterion:

(1) There exist classes $\alpha_n \in H^n_\tau(P_0, F_n)$ which are exactly p^n-torsion. That is, $p^n\alpha_n = 0$ but $(p^n - 1)\alpha_n \neq 0$.

(2) There exist "transfer" maps $H^n_\tau(P_\lambda, F_n) \to H^n_\tau(P_0, F_n)$ such that for all $n \geq 1$, the composition

$$H^n_\tau(P_0, F_n) \to H^n_\tau(P_\lambda, F_n) \to H^n_\tau(P_0, F_n)$$

is d_λ times the identity for some $d_\lambda \in \mathbb{N}$.

Lemma 45. *If the above criterion (1) and (2) are satisfied, then $K \to \widehat{K}$ is not a weak equivalence.*

Proof. As discussed above, it suffices to show that $H^0\Phi((R\prod_{n\geq 1})F_n[n])$ is nonzero. The sequence $\alpha = (\alpha_1, \alpha_2, \dots) \in \prod H^n_\tau(P_0, F_n)$ determines an element of

$$H^0\Phi((R\prod_{n\geq 1})F_n[n]) = \varinjlim_{\Lambda} \prod H^n_\tau(P_\lambda, F_n).$$

This element α becomes zero in the colimit if and only if there is some λ such that α is sent to zero under $\prod H^n_\tau(P_0, F_n) \to \prod H^n_\tau(P_\lambda, F_n)$. Using the transfer maps (2) we see that $d_\lambda \alpha = 0$. So $d_\lambda \alpha_n = 0$ for all n. but this contradicts (1), since for any $p^n - 1 > d_\lambda$ we have $(p^n - 1)\alpha_n \neq 0$. $\square$

Now we discuss a way to construct F_n, α_n in the case that our topos is $\mathsf{Shv}_\tau(\mathcal{C}) = \mathsf{Shv}(\mathbb{C}(x_1, x_2, \dots)_{\mathsf{et}})$. The sheaves F_n will be the constant sheaves associated to the abelian groups $\mathbb{Z}/p^n$.

To construct the α_n, recall from the first part of this course that $\mathsf{Shv}(k_{\mathsf{et}}, \mathrm{Ab})$ is canonically equivalent to the category of continuous G-modules where $G := Gal(k^{sep}/k)$. In particular, we can calculate étale cohomology of k_{et} as the group cohomology of G. Note that there is a canonical surjection[11]

$$Gal(k^{sep}/k) \to \prod_{\mathbb{N}} \mathbb{Z}_p.$$

We will use the following facts. Given an abelian group A (with trivial G-action and discrete topology), we have $H^1(G, A) \cong \hom_{cont.}(G, A)$ [Weibel, 6.1.5, 6.4.1, 6.4.2, 6.11.15]. In particular, for each $n \geq 1$ we have a canonical class $\alpha'_n \in H^1(\mathbb{Z}_p, \mathbb{Z}/p^n)$ which is exactly p^n-torsion.

Next, given a product of groups $G_1 \times G_2$ and abelian groups A_1, A_2, we have a cohomology cross-product [Brown, Cohomology of groups Section V.2],

$$H^i(G_1, A_1) \otimes H^i(G_1, A_1) \to H^{i+j}(G_1 \times G_2, A_1 \otimes A_2).$$

So our classes α'_n can be combined to a class $\alpha''_n \in H^n(\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \mathbb{Z}/p^n)$ where there are n copies of $\mathbb{Z}_p$. Using the surjection $Gal(k^{sep}/k) \to \prod_{i \geq 1} \mathbb{Z}_p \to \prod_{i=1}^n \mathbb{Z}_p$ we get a class

$$\alpha_n \in H^n(Gal(k^{sep}/k), \mathbb{Z}/p^n) = H^n_{\mathsf{et}}(k, \mathbb{Z}/p^n).$$

As we mentioned above, we take (P_λ) to be the system $(\mathrm{Spec}(L))_{k \subseteq L \subseteq k^{sep}}$ of all finite Galois sub-extensions of k^{sep}/k. The transfer morphisms can be deduced from the theory of group cohomology [Brown, Cohomology of groups, Prop.III.9.5] or from the étale theory as follows: Given a finite Galois extension L/k, since $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is an étale covering, for every sheaf F there is an exact sequence

$$0 \to F(k) \to F(L) \to \oplus_{Gal(L/k)} F(L)$$

where we use the fact that $L \otimes_k L \cong \prod_{Gal(L/k)} L$. So $F(k) \subseteq F(L)$ is the set of Galois invariant sections. Sending a section $s \in F(L)$ to the Galois invariant

[11]Indeed, for each $m \in \mathbb{N}$ and $i = (i_1, \dots, i_m) \in \mathbb{N}^m$ consider the field extension $k_i := \mathbb{C}(x_1^{1/p^{i_1}}, \dots, x_m^{1/p^{i_m}}, x_{m+1}, \dots)$ of k. This is an abelian Galois extension with group $\mathbb{Z}/p^{i_1} \times \cdots \times \mathbb{Z}/p^{i_m}$ (the group $\mathbb{Z}/p^{i_j}$ acts by sending $x_j^{1/p^{i_j}}$ to $\zeta^a_{p^{i_j}} x_j^{1/p^{i_j}}$ for $a \in \mathbb{Z}/p^{i_j}$ where $\zeta_{p^{i_j}} = e^{2\pi i/p^{i_j}}$ is a primitive p^{i_j}th root of unity). Hence, by the Fundamental Theorem of Profinite Galois Theory [Weibel, 6.11.5], there exist a sequence of surjections $Gal(k^s/k) \to \mathbb{Z}/p^{i_1} \times \cdots \times \mathbb{Z}/p^{i_m}$ fitting together into a tower. Taking the limit over i gives the surjection $Gal(k^s/k) \to \varprojlim_{m,i} \mathbb{Z}/p^{i_1} \times \cdots \times \mathbb{Z}/p^{i_m} \cong \prod_{n \in \mathbb{N}} \mathbb{Z}_p$.

section $\sum_{g \in Gal(L/k)} g^* s$ defines a map $F(L) \to F(k)$ such that the composition $F(k) \to F(L) \to F(k)$ is $[L : k]$ times the identity. This is natural in F, so by Yoneda, it induces a map $\mathbb{Z}\operatorname{Spec}(k) \to \mathbb{Z}\operatorname{Spec}(L)$ of representable étale sheaves of abelian groups. More directly, $\hom(\mathbb{Z}\operatorname{Spec}(k), \mathbb{Z}\operatorname{Spec}(L)) = (\mathbb{Z}\operatorname{Spec}(L))(k)$, each k-automorphism $g \in Gal(L/k)$ determines a k-morphism $g : L \to L$, and our transfer map $\mathbb{Z}\operatorname{Spec}(k) \to \mathbb{Z}\operatorname{Spec}(L)$ corresponds to the Galois invariant section $\sum g \in (\mathbb{Z}\operatorname{Spec}(L))(L)$.

In any case, since $H^n_{\mathsf{et}}(-, F) \cong \hom_{D(k_{\mathsf{et}})}(\mathbb{Z}-, F[n])$, we obtain the desired transfer maps.