Algebra II

§0 Introduction

Want to work with ind-étale algebras instead of just étale algebras, so that "$\mathbb{Z}_\ell$-adic type objects" become representable.

Doing this, schemes become "locally contractible".
I.e., every scheme X admits a pro-étale covering $\{U_i \to X\}$ s.t. each U_i is <u>weakly contractible</u> in the sense that for every pro-étale covering $\{V_{ij} \to U_i\}$, $\exists$ j, and a section $V_{ij} \xrightarrow{F} U_i$. ($s$)

<u>Today's goal:</u> Every affine scheme X admits a surjective pro-étale morphism $U \to X$ s.t. U is affine and weakly contractible.

1. (Zariski case) Build a surjective "pro-Zariski morphism" $X^Z \to X$ with X^Z weakly contractible for the Zariski topology.

2. (Profinite case) Build a surjective morphism $T \to (X^Z)^c$ to the set $(X^Z)^c$ of closed points of X^Z from a profinite set T which is weakly contractible as a compact Hausdorff topological space.

Profinite set $= \underbrace{\lim_{\leftarrow \Lambda} S_\lambda}_{\text{limit topology.}}$ Λ cofiltered, S_λ finite discrete top. spaces

e.g. $S_\lambda = \operatorname{Spec} \prod_{s \in S_\lambda} \mathbb{C}$

e.g. $1 \quad 2 \quad 3 \quad 4 \quad 5 \; 6 \cdots \infty$

$\operatorname{Spec}\left(\operatorname{colim}_{\to \Lambda} \prod_{s \in S_\lambda} \mathbb{C} \right)$

3. (Dim. zero case) Give T a structure of affine scheme X_0 s.t. all residue fields are separable closed.

4. Henselise along

$$X_0 \to (X^Z)^c \hookrightarrow X^Z \to X$$

to produce $\mathcal{U} \to X$.

X finite has $\mathcal{U} = \coprod_{x \in X} \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$

$A \to B$

$\mathrm{Hens}_A(-) : \mathrm{Ind}(B_{et}) \leftrightarrows \mathrm{Ind}(A_{et}) : B \otimes_A -$

$\mathrm{Hens}_A(B_0) = \mathrm{colim}\, A'$

$\underbrace{A \to A'}_{\text{étale}} \to B_0$

§1 Pro-Zarish:

Def. The category $\mathcal{S}$ of spectral spaces is image of the functor

$$\mathrm{Spec} : (\mathrm{Ring})^{op} \to \mathrm{Top}.$$

$X \in \mathcal{S}$ if $\exists A$, $\mathrm{Spec}(A) \cong X$ NB: A is not unique!

$f: Y \to X \in \mathcal{S}$ if $\exists \phi : A \to B$, $\mathrm{Spec}(\phi) = f$ ϕ

Def A spectral space X is w-local if:

1) All open covers split. I.e., $\forall$ open covering $\{U_i \to X\}_{i \in I}$ $\coprod_{i \in I} U_i \to X$ has a section. $\coprod U_i \rightleftarrows X$

2) The subspace $X^c \subset X$ of closed points is closed.

A map of w-local spaces is w-local if its spectral, and sends closed points to closed points.

Exercise 1. Show the following spaces are w-local:

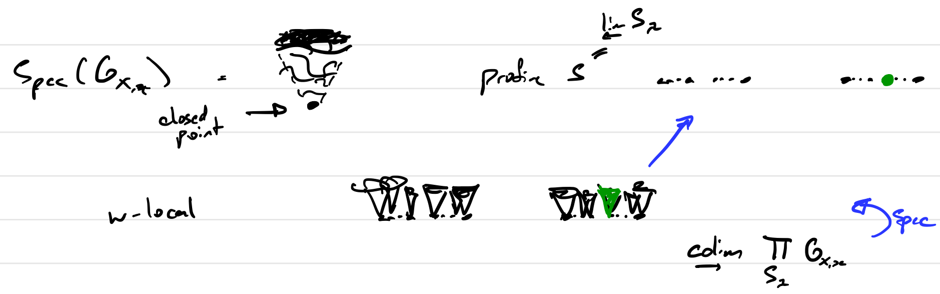
1) Any profinite set.

2) $\mathrm{Spec}(\mathcal{O}_{X,x})$

3) Any finite disjoint union of w-local spaces.

Exercise 2

Show that if X is w-local, then every connected component has at most one closed point.



Def. A map $f: W \to V$ of spectral spaces is a <u>Zariski localisation</u> if $W = \coprod_{i=1}^{n} U_i$ with the $U_i \to V$ open immersions.
A <u>pro-Zariski localisation</u> is a cofiltered limit of such maps.

Example 1) For any profinite set $S = \varprojlim S_\lambda$ and spectral space V, then $\underbrace{\varprojlim \coprod_{S_\lambda} V}_{\prod_S V =} \to V$ is a pro-Zariski localisation.

2) Given $v \in V$, the map $\bigcap_{v \in U} U \to V$ is a pro-Zariski localisation
↖ intersection of quasi-compact opens $v \in U \subseteq V$.

3) If $W \to V$ is a pro-Zariski localisation, so $W = \varprojlim_{i \in I} \coprod_{j=1}^{n_i} U_{ij}$
with $U_{ij} \subseteq V$ quasi-compact open.
Then $W \subseteq \varprojlim_{i \in I} \coprod_{j=1}^{n_i} V$

$$\downarrow$$

$$(j_i) = i \in \varprojlim_{i \in I} \{1, \dots, n_i\} \leftarrow \text{profinite set}$$

fibre of W over i is $\bigcap_{j_i} U_{ij}$

w. local spectral spaces

<u>Lemma</u> The inclusion $i: \mathcal{S}^{wl} \to \mathcal{S}$ admits a right adjoint $(-)^Z: \mathcal{S} \to \mathcal{S}^{wl}$. The counit $X^Z \to X$ is a surjective pro-Zariski localisation for all X, and the composite $(X^Z)^c \to X$ is a homeomorphism for the constructible topology on X.

<u>Exercise 3</u> Describe the constructible subsets of $\operatorname{Spec}(\mathbb{Z})$ and $\operatorname{Spec} \mathbb{C}[x,y]$.

<u>Sketch of proof</u> Every spectral space is an inverse limit of finite spectral spaces. If X is a finite spectral space,

$$X^Z = \coprod_{x \in X} X_x \quad \text{where} \quad X_x = \bigcap_{x \in U} U = \{y \in X \mid y \text{ specialises to } x\}.$$

$\mathfrak{q} \subseteq \mathfrak{p}$

cf. $A_\mathfrak{p} = \varinjlim_{f \notin \mathfrak{p}} A[f^{-1}]$

<u>Exercise 4</u> Prove that if X is a finite spectral space, then any map $Y \to X$ from a w-local space Y factors through $\coprod_{x \in X} X_x$.

<u>Rmk</u> As a set, $X^Z = \coprod_{x \in X} X_x$.

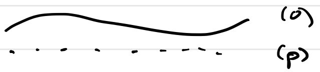
<u>Example</u>

$$\operatorname{Spec}(\mathbb{Z})^Z = \operatorname{Spec}\left(\operatorname{colim}_{\text{primes } p} \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \times \cdots \times \mathbb{Z}_{(p)} \times \mathbb{Z}[\tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{p}]\right)$$

transition maps are identity on the $\mathbb{Z}_{(\ell)}$ and diagonal

$$\mathbb{Z}[\tfrac{1}{2}, \ldots, \tfrac{1}{p}] \to \mathbb{Z}_{(p')} \times \mathbb{Z}[\tfrac{1}{2}, \ldots, \tfrac{1}{p}, \tfrac{1}{p'}]$$

Spec $\mathbb{Z}$ =



$\mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \times \dots \times \mathbb{Z}_{(p)} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}]$

$(\operatorname{Spec} \mathbb{Z})^2$

Lemma Any map $f: X \to Y$ in $\mathcal{S}^{wl}$ admits a canonical factorisation $X \to Z \to Y$ in $\mathcal{S}^{wl}$ with $Z \to Y$ is a pro-Zariski localisation and $X \to Z$ induces a homeomorphism $X^c \xrightarrow{\sim} Z^c$.

Remark proof uses $\pi_0(X) \cong X^c$.

§2 Rings

Definition Fix a ring A.

1. A is <u>w-local</u> if $\operatorname{Spec}(A)$ is w-local
2. A is <u>strictly w-local</u> if A is w-local and every faithfully flat étale morphism $A \to B$ has a retraction $A \to B$.
3. A map $f: A \to B$ of w-local rings is w-local if $\operatorname{Spec}(f)$ is w-local.
4. A map $f: A \to B$ is called a <u>Zariski localisation</u> if $B = \prod_{i=1}^{n} A[f_i^{-1}]$. Its called an ind-Zariski localisation if its a filtered colimit of Zariski localisations.

5. A map $f: A \to B$ is called <u>ind-étale</u> if its a filtered colimit of étale A-algebras.

<u>Example</u>

1) $\mathbb{Z} \to \underset{\substack{\longrightarrow \\ \text{primes } p}}{\operatorname{colim}} \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \times \dots \times \mathbb{Z}_{(p)} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}]$

2) If k is a field, k^{sep}/k a seperable closure, $k \to k^{sep}$ is étale only if its finite, but its always ind-étale.

3) k a field, $S = \varprojlim S_i$ profinite set, $k \to \underset{\longrightarrow}{\operatorname{colim}} \prod_{S_i} k$ is an ind-Zariski localisation.

<u>Def</u> B is called <u>absolutely flat</u> if its reduced and has Krull dimension zero, or equivalently, if every B-module is flat.

<u>Example</u> $k^{sep} \otimes_k k^{sep}$

<u>Lemma</u> If A is w-local, then the Jacobson ideal $I_A := \bigcap_{\text{maximal}} m$ cuts out $\operatorname{Spec}(A)^c \subset \operatorname{Spec}(A)$ with it's reduced structure. The quotient A/I_A is absolutely flat.

<u>Lemma</u> The inclusion of the category of w-local rings and maps into all rings admits a left adjoint $A \mapsto A^Z$. The unit $A \to A^Z$ is a faithfully flat ind-Zariski localisation. and $\operatorname{Spec}(A^Z) = \operatorname{Spec}(A)^Z$ over $\operatorname{Spec}(A)$.

Lemma Any w-local map $f: A \to B$ of w-local rings admits a canonical factorisation $A \xrightarrow{a} C \xrightarrow{b} B$ with C w-local, $a: A \to C$ a w-local ind-Zariski localisation, and $b: C \to B$ a w-local map inducing $\pi_0(\operatorname{Spec} B) \cong \pi_0(\operatorname{Spec} C)$.

Lem 2.2.7 BS

Lemma A absolutely flat, $\exists\ A \to \overline{A}$ ind-étale, faithfully flat, $\overline{A}$ w-strictly local and absolutely flat.

Proof

$$\overline{A} := \bigotimes_{I} A_i \qquad I = \left\{ \begin{array}{l} \text{faithfully flat} \\ \text{étale } A \to A_i \end{array} \right\}$$

$$= \operatorname*{colim}_{\substack{J \subset I \\ \text{finite}}} \bigotimes_{J} A_i$$