0 Introduction

In the introduction to the pro-étale topology we argued that adding (filtered) limits to the site made $\mathbb{Z}_l$- and $\mathbb{Q}_l$-sheaves work better. One of the technical reasons that the category works better is that schemes morally become "locally contractible". More concretely, every scheme X admits a pro-étale covering $\{U_i \to X\}_{i\in I}$ such that each U_i is *weakly contractible* in the sense that for every pro-étale covering $\{V_{ij} \to U_i\}$ the morphism $\amalg_j V_{ij} \to U_i$ has a section $\amalg_j V_{ij} \leftrightarrows U_i$.[1]

The goal of this lecture is to show the following:

Today's Goal. Every affine scheme X admits a surjective pro-étale morphism $U \to X$ such that U is affine and weakly contractible.

This happens in roughly four steps:

1. (Zariski version) Build a surjective pro-Zariski morphism $X^Z \to X$ with X^Z weakly contractible for the Zariski topology.

   Set theoretically, X^Z is the disjoint union $\sqcup_{x\in X} \operatorname{Spec}(\mathcal{O}_{X,x})$ of the localisations of X at every point, but X^Z is equipped with a coarser topology than $\sqcup_{x\in X} \operatorname{Spec}(\mathcal{O}_{X,x})$. The closed points of X^Z are in bijection with the points of X.

2. (Profinite set case) Build a surjective morphism $T \to (X^Z)^c$ to the set $(X^Z)^c$ of closed points of X^Z from a profinite set[2] T which is weakly contractible as a compact Hausdorff topological space.

3. (Dimension zero scheme case) Give T a structure of affine scheme X_0 such that all residue fields are separably closed.

4. Henselise along $X_0 \to X$ to produce the desired $U \to X$.

   The closed points of U will be in bijection with the points of T, and all local rings of U will be strictly hensel local rings.

If X has finitely many points (e.g., $X = \operatorname{Spec}(R)$ with R a discrete valuation ring, or more generally, a localisation of a Dedekind domain at finitely many primes) then X^Z in Step 1 *is* just the disjoint union $\amalg_{x\in X} \operatorname{Spec}(\mathcal{O}_{X,x})$ of the localisations at each point of X, Step 2 is unnecessary because $(X^Z)^c$ is finite, Step 3 just chooses separable closures $k(x)^{sep}$ for each $k(x)$, and Step 4 produces the disjoint union of the strict henselisations $\sqcup_{x\in X} \operatorname{Spec}(\mathcal{O}^{sh}_{X,x}) \to X$.

The general case is not so easy because, for example, $\amalg_{x\in X} x$ is not affine if X has infinitely many points (all affine schemes are quasi-compact). However, instead of just making things more complicated, things actually become very interesting. The scheme $X^Z \to X$ that is produced is in a precise since the "smallest" Zariski covering of X. As a set, X^Z is the disjoint union $\amalg_{x\in X} \operatorname{Spec}(\mathcal{O}_{X,x})$

[1] This property implies that the pro-étale cohomology of any sheaf vanishes; $H^n_{proet}(U_i, F) = 0$, for $n > 0$. Hence the name "weakly contractible".

[2] There is some material on profinite sets in Section A at the end of these notes.

of the localisations of X but the topology is coarser: the subset $(X^Z)^c$ of closed points of X^Z is also affine, and has the curious property of being homeomorphic to the set of points of X equipped with the constructible topology.

Step 2 is clearly necessary for the following reason: strictly hensel local schemes X have the property that given any étale morphism $U \to X$, any lift of the closed point $X^c \to U \to X$ can be extended to a section $X \to U \to X$. So if our set of closed points is not weakly contractible, there is not much hope for X to be.

Step 3 and Step 4 are classical, but Bhatt-Scholze's approach to separably closing the residue fields in Step 3 quite amusing; they take the fibre product of *all* finite presentation surjective étale morphisms. Of course, this produces something huge, but certainly also produces something which has all residue fields separably closed.

1 Pro-Zariski covers of affine schemes

In this section, we consider the question: is there a "smallest" open cover, and what does it look like?

The category of *spectral spaces* is the image of the funcor

$$(\mathrm{Spec}(-))_{top} : (Ring)^{op} \to Top$$

which sends a ring to it's set of primes equipped with the Zariski topology. Of course, different rings can give rise to the same space (e.g, fields, discrete valuation rings, noetherian dimension one schemes, ...) but none-the-less, if we are only interested in the Zariski topology, all we are concerned with is the underlying topological space.

Let $\mathcal{S}$ be the category of spectral spaces (i.e., spaces of the form $\mathrm{Spec}(R)$ for some ring R), with spectral maps (i.e., maps of the form $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ for some ring homomorphism $S \to R$).

Definition 1 ([BS, Def.2.1.1], [Stacks project, 096A]). *A spectral space X is* w-local *if it satisfies:*

1. *All open covers split, i.e., for every open cover $\{U_i \hookrightarrow X\}$, the map $\sqcup_i U_i \to X$ has a section.*

2. *The subspace $X^c \subset X$ of closed points is closed.*

A map $f : X \to Y$ of w-local spaces is w-local if f is spectral and $f(X^c) \subset Y^c$. Let $i : \mathcal{S}^{wl} \hookrightarrow \mathcal{S}$ be the subcategory of w-local spaces with w-local maps.

exer:2.1.2 **Exercise 1** (Exa.2.1.2). Show that the following spectral spaces are w-local.

1. Any profinite set[3] (with the profinite topology).

[3] There is some material on profinite sets in Section A at the end of these notes.

2. The topological space $\mathrm{Spec}(\mathcal{O}_{X,x})$ underlying any local ring of any scheme, i.e., a topological space with a unique closed point.

3. Any finite disjoint union of w-local spaces.

picloc **Exercise 2.**

1. Show that if X is w-local then every connected component has at most one closed point.

2. Using the facts that connected components of topological spaces are closed, and closed subpaces of spectral spaces are spectral, show that if X is w-local then every connected component has at least one closed point.

3. Deduce that when X is w-local, the set $\pi_0 X$ of connected components is in canonical bijection with the set X^c of closed points; $X^c \xrightarrow{\sim} \pi_0 X$.

4. Using [Stacks project, 097C], show that $X^c \xrightarrow{\sim} \pi_0 X$ is a homeomorphism.[4]

$----$ picture: triangles over the cantor set $C \times \mathrm{Spec}(\mathcal{O}_{X,x})$ $----$

Definition 2 (Def.2.1.12)**.** *A map $f : W \to V$ of spectral spaces is a* Zariski localization *if $W = \sqcup_{j=1}^{n} U_j$ with the $U_j \to V$ open immersions (automatically quasi-compact because W is spectral). A* pro-(Zariski localization) *is a cofiltered limit of such maps.*

Example 3.

1. For any profinite set $I = \varprojlim I_\lambda$ the map $\varprojlim \prod_{I_\lambda} V \to V$ is a pro-(Zariski localisation).

   $----$ picture: cantor set of copies of some space $----$

2. For any point $v \in V$, the map $\cap_{v \in U} U \to V$ is a pro-(Zariski localisation), where the intersection is over quasicompact opens containing v.

   $----$ picture: triangle representing a localisation at a point $----$

3. We can combine the above two to describe every pro-(Zariski localisaton). Suppose that $W \to V$ is a pro-(Zariski localisaton), so $W = \varprojlim_{i \in I} \sqcup_{j=1}^{n_i} U_{ij}$ with each $U_{ij} \to V$ a quasicompact open immersion. Then $W \subseteq \varprojlim_{i \in I} \sqcup_{j=1}^{n_i} V$. As a set, $\varprojlim_{i \in I} \sqcup_{j=1}^{n_i} V$ is a disjoint union of copies of V indexed by the

[4] Here, $\pi_0 X$ is equipped with the quotient topology induced by the canonical projection $\pi : X \to \pi_0 X$ (a subset $U \subseteq \pi_0 X$ is open if and only if $\pi^{-1} U$ is open), and X^c is equipped with the topology induced by $\iota : X^c \to X$ (a subset $U \subseteq X^c$ is open if there is an open $V \subseteq X$ satisfying $X^c \cap V = U$). Show that the bijection $X^c \xrightarrow{\sim} \pi_0 X$ is actually a homeomorphism.

profinite set $\varprojlim_{i\in I}\{1,\dots,n_i\}$. Given an element $(j_i)_{i\in I} \in \varprojlim_{i\in I}\{1,\dots,n_i\}$, the intersection W with the $(j_i)_{i\in I}$th copy of V is $\bigcap_{i\in I} U_{ij_i}$.

$- - - -$ picture: $U \subseteq \mathbb{N}^* \times \operatorname{Spec}\mathbb{Z}$ $- - - -$

Lemma 4 (Lem.2.1.10)**.** *The inclusion $i : \mathcal{S}^{wl} \to \mathcal{S}$ admits a right adjoint $(-)^Z : \mathcal{S} \to \mathcal{S}^{wl}$. The counit $X^Z \to X$ is a surjective pro-(Zariski localisation) for all X, and the composite $(X^Z)^c \to X$ is a homeomorphism for the constructible topology on X.*

The Z probably stands for "Zariski".

Remark 5. The family of *constructible* subsets of an affine scheme $\operatorname{Spec}(A)$ is the smallest family closed under finite intersection, finite union, complement, and containing the closed subsets $V(I)$ for every *finitely generated* ideal I (a ring is noetherian if and only if every ideal is finitely generated; we will often need non-noetherian rings). The constructible topology is a (usual)[5] topology on X whose opens are the constructible subsets.

Exercise 3. Describe the constructible subsets of $\operatorname{Spec}(\mathbb{Z})$ and $\operatorname{Spec}(\mathbb{C}[x,y])$.

Sketch of proof. The idea is that every spectral space is an inverse limit of finite spectral spaces, and for finite spectral spaces, X^Z is the disjoint union of the localisations at each point $X^Z = \amalg_{x\in X} X_x$.

$- - - -$ picture: X^Z for a dvr $- - - -$

□

Exercise 4. Prove that if X is a finite spectral space, then any map $Y \to X$ from a w-local space factors through $\amalg_{x\in X} X_x \to X$. Hint: use the fact that each $X_x \to X$ is an open immersion, and in fact, $\{X_x \to X\}_{x\in X}$ is an open covering.

rema:spectralLimit **Remark 6** (Omitted from lecture)**.** In fact, every spectral space is the inverse limit of it's constructible partitions: Let $X = \cup_{i\in I} X_i$ be a partition of X into constructible sets, and consider the canonical projection $X \to I$ to the *components* of the partition (so $x \in X$ is sent to the i such that $x \in X_i$). We equip I with the coarsest topology which makes this map continuous (so a subset $J \subset I$ is open iff it's preimage is). Then the topological space of X is the inverse limit over these projections.

Remark 7 (Rem.2.1.11, Omitted from lecture)**.** The space X^Z can be alternatively described as:

$$X^Z = \varprojlim_{\{X_i \hookrightarrow X\}} \sqcup_i \widetilde{X_i},$$

[5] I.e., not a Grothendieck topology.

where the limit is indexed by the cofiltered category of constructible stratifications $\{X_i \hookrightarrow X\}$, and $\widetilde{X_i}$ denotes the set of all points of X specializing to a point of X_i. As a set, X^Z is the disjoint union of the localisations at each point $X^Z = \amalg_{x \in X} X_x$, but the topology is coarser.

Example 8. Let X be the topological space associated to a curve (e.g., $\mathrm{Spec}(\mathbb{Z})$ or $\mathrm{Spec}(k[t])$), so X has one generic point η, every other point is closed, and the non-empty proper closed subsets are finite sets of closed points. Then X^Z as a set is $\{\eta\} \sqcup \amalg_{x \in X^c}\{x, \eta_x\}$. The sets $\{x, \eta_x\}$ and $\{\eta_x\}$ are open, as well as $\{\eta\} \sqcup \amalg_{x \in S}\{x, \eta_x\}$ for any cofinite set $S \subseteq X^c$ of closed points. These open sets generate the topology. Note that $\{\eta\}$ is closed, but not open. Note also that the topology induced on $\{\eta\} \cup X^c$ is the constructible topology of X.

From a ring point of view,

$$\mathrm{Spec}(\mathbb{Z})^Z = \mathrm{Spec}\left(\varinjlim_{\text{primes } p} \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(5)} \times \cdots \times \mathbb{Z}_{(p)} \times (\mathbb{Z}[\tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{p}]) \right)$$

The transition morphisms are the identities on the $\mathbb{Z}_{(p)}$ factors, and the diagonal $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}] \to \mathbb{Z}_{(p')} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}, \frac{1}{p'}]$ on the end.

The following lemma is used in Lem.2.2.8 to transfer a scheme structure from T to S.

Lemma 9 (Lem.2.1.14)**.** *Any map $f : X \to Y$ in $\mathcal{S}^{wl}$ admits a canonical factorization $X \to Z \to Y$ in $\mathcal{S}^{wl}$ with $Z \to Y$ a pro-(Zariski localization) and $X \to Z$ inducing a homeomorphism $X^c \simeq Z^c$.*

Proof. (Omitted from lecture). Take $Z = \pi_0(X) \times_{\pi_0(Y)} Y$. Here $\pi_0(X), \pi_0(Y)$ are the sets of connected components of X, Y. The morphism $\pi_0(X) \to \pi_0(Y)$ is a morphism of profinite sets, and therefore a pro-(Zariski localisation). Hence, its pullback $\pi_0(X) \times_{\pi_0(Y)} Y \to Y$ is also a pro-(Zariski localisation). The map $X \to \pi_0(X)$ induces a homeomorphism $X^c \to \pi_0(X)$ because each connected component of X has a unique closed point. For the same reason $(\pi_0(X) \times_{\pi_0(Y)} Y)^c \to \pi_0(\pi_0(X) \times_{\pi_0(Y)} Y)$ is a homeomorphism. But $\pi_0(\pi_0(X) \times_{\pi_0(Y)} Y) \cong \pi_0(X)$, so it follows that $X^c \to Z^c$ is a homeomorphism. □

2 Rings

Definition 10 (Def.2.2.1)**.** *Fix a ring A.*

1. *A is* w-local *if* $\mathrm{Spec}(A)$ *is w-local.*

2. *A is* w-strictly local *if A is w-local, and every faithfully flat étale map $A \to B$ has a section.*

3. *A map $f : A \to B$ of w-local rings is w-local if* $\mathrm{Spec}(f)$ *is w-local.*

4. *A map $f : A \to B$ is called a* Zariski localization *if $B = \prod_{i=1}^n A[\frac{1}{f_i}]$ for some $f_1, \dots, f_n \in A$. An* ind-(Zariski localization) *is a filtered colimit of Zariski localizations.*

5. *A map $f : A \to B$ is called* ind-étale *if it is a filtered colimit of étale A-algebras.*

Example 11.

1. We have already seen an example of an ind-(Zariski localisation) above:

$$\varinjlim_{\text{primes } p} \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(5)} \times \cdots \times \mathbb{Z}_{(p)} \times (\mathbb{Z}[\tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{p}])$$

2. Let k be a field, and k^{sep} a separable closure. Then $k \to k^{sep}$ is étale if and only if k^{sep}/k is a finite extension, but it is always ind-étale.

3. Any field k and any profinite set $T = \varprojlim_{i \in I} T_i$, gives rise to a k-algebra $\varinjlim_{i\in I} \prod_{t \in T_i} k \subseteq \prod_{t \in T} k$. The algebra $\varinjlim_{i \in I} \prod_{t \in T_i} k$ has the property that each residue field is isomorphic to k, and it's topological space is homeomorphic to the profinite set T.

Definition 12. *A ring B is called* absolutely flat *if B is reduced with Krull dimension* 0 *(or, equivalently, that B is reduced with* Spec(B) *Hausdorff).*

Absolutely flat rings are also characterised as those rings such that every module is flat [Stacks project, 092F].

So Spec(B) of an absolutely flat ring has a profinite set as its topological space,[6] and all local rings are fields.[7] So it is some kind of "profinite product" of fields.

Example 13.

1. For any perfect field k, the tensor product $\overline{k} \otimes_k \overline{k}$ is absolutely flat. Note that $\mathrm{Spec}(\overline{k} \otimes_k \overline{k}) \cong Gal(\overline{k}/k)$ as *topological spaces.*

2. Let $\zeta_{p^n} := e^{2\pi i/p^n}$ and consider the colimit

$$A := \varinjlim \big(\mathbb{Q} \to \mathbb{Q}\times\mathbb{Q}(\zeta_p) \to \mathbb{Q}\times\mathbb{Q}(\zeta_p)\times\mathbb{Q}(\zeta_{p^2}) \to \dots\big)$$

[6]Recall that profinite sets are precisely the quasi-compact Hausdorff totally disconnected spaces. Every affine scheme is quasi-compact. We observed above that absolutely flat rings have Hausdorff spectrum. Suppose $\mathrm{Spec}(B/I) \subseteq \mathrm{Spec}(B)$ is a connected component. We can assume $I = \sqrt{I}$, and so Spec(B/I) is again reduced of Krull dimension zero, so we can in fact, assume $B = B/I$. Since Spec(B) is Hausdorff, for any two points $\mathfrak{m}_0, \mathfrak{m}_1$, there are disjoint opens containing each of them. Shrinking, we can assume the opens are of the form $\mathrm{Spec}(B_{f_0})$, $\mathrm{Spec}(B_{f_1})$. The complement of union of these two opens is the closed subset of the ideal $\langle f_0, f_1 \rangle$. Since every module is flat however, the support of this ideal is open, and we deduce that Spec(B) is a disjoint union of three opens, two of which are nonempty by virtue of containing the closed points $\mathfrak{m}_0, \mathfrak{m}_1$ from the beginning. This contradicts Spec(B) being connected.

[7]Indeed, since all modules are flat, for any maximal ideal $\mathfrak{m}$, the module $A/\mathfrak{m}$ is flat. It follows that $A/\mathfrak{m} = A_{\mathfrak{m}}$.

where the transition maps $\prod_{k=0}^{n} \mathbb{Q}(\zeta_{p^k}) \to \prod_{k=0}^{n} \mathbb{Q}(\zeta_{p^k})$ are identities on the first n factors and the diagonal $\mathbb{Q}(\zeta_{p^n}) \to \mathbb{Q}(\zeta_{p^n}) \times \mathbb{Q}(\zeta_{p^{n+1}})$ on the end. Then A is absolutely flat with topological space the one point compactification $\mathbb{N} \sqcup \{\infty\}$ of the natural numbers. The local ring at $n \in \mathbb{N}$ is $\mathbb{Q}(\zeta_{p^n})$ and the local ring at ∞ is $\cup_{n \in \mathbb{N}} \mathbb{Q}(\zeta_{p^n})$.

3. Let A be any ring and consider the colimit

$$A' := \varinjlim_{\mathrm{Spec}\, A = \sqcup_{i=1}^{n} W_i} \Pi_{i=1}^{n} \Gamma(W_i, \mathcal{O}_{\mathrm{Spec}(A)})$$

over finite partitions $\mathrm{Spec}\, A = \sqcup_{i=1}^{n} W_i$ into constructible subschemes $W_i \subseteq \mathrm{Spec}\, A$. Then A' is absolutely flat, and the canonical map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is an isomorphism on prime ideals and induces isomorphisms on all residue fields. In fact, the underlying topological space of $\mathrm{Spec}(A')$ is $\mathrm{Spec}(A)$ equipped with the constructible topology.

exer:absFlatProEt **Exercise 5.** Suppose A is absolutely flat and $(B_\lambda)_{\lambda \in \Lambda}$ is a filtered system of étale A-algebras. Show that $\varinjlim B_\lambda$ is absolutely flat. Hint: pullback along the local rings $A_{\mathfrak{p}}$ of A.

exer:absFlatwLocal **Exercise 6.** Show that every absolutely flat ring is w-local. Cf.Exercise 1.

Lemma 14 (Lem.2.2.3)**.** *If A is w-local, then the Jacobson radical I_A ($= \cap_{maximal\ ideals} \mathfrak{m}$) cuts out $\mathrm{Spec}(A)^c \subset \mathrm{Spec}(A)$ with it's reduced structure. The quotient A/I_A is an absolutely flat ring.*

Proof. (Omitted from lecture). Let $J \subset A$ be the (radical) ideal cutting out $\mathrm{Spec}(A)^c \subset \mathrm{Spec}(A)$ with the reduced structure. Then $J \subset \mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Spec}(A)^c$, so $J \subset I_A$. Hence, $\mathrm{Spec}(A/I_A) \subset \mathrm{Spec}(A)^c$ is a closed subspace; we want the two spaces to coincide. If they are not equal, then there exists a maximal ideal $\mathfrak{m}$ such that $I_A \not\subset \mathfrak{m}$, which is impossible. ☐

Lemma 15 (Lem.2.2.4)**.** *The inclusion of the category w-local rings and maps inside all rings admits a left adjoint $A \mapsto A^Z$. The unit $A \to A^Z$ is a faithfully flat ind-(Zariski localization), and $\mathrm{Spec}(A)^Z = \mathrm{Spec}(A^Z)$ over $\mathrm{Spec}(A)$.*

Remark 16. See [Stacks Project, 096U] for a more detailed construction of A^Z.

Proof. (Ommitted from lecture). This follows from Remark 2.1.11 (above). In more details, let $X = \mathrm{Spec}\, A$, and define a ringed space $X^Z \to X$ by equipping $(\mathrm{Spec}\, A)^Z$ with the pullback of the structure sheaf from X. Then Remark 2.1.11 presents X^Z as an inverse limit of affine schemes, so that $X^Z = \mathrm{Spec}(A^Z)$ is itself affine. ☐

Lemma 17 (Lem.2.2.6)**.** *Any w-local map $f : A \to B$ of w-local rings admits a canonical factorization $A \xrightarrow{a} C \xrightarrow{b} B$ with C w-local, $a : A \to C$ a w-local ind-(Zariski localization), and $b : C \to B$ a w-local map inducing $\pi_0(\mathrm{Spec}(B)) \simeq \pi_0(\mathrm{Spec}(C))$.*

Lemma 18 ([BS, Lem.2.2.7], [Stacks Project, 097R])**.** *For any absolutely flat ring A, there is an ind-étale faithfully flat map $A \to \overline{A}$ with $\overline{A}$ w-strictly local and absolutely flat. For a map $A \to B$ of absolutely flat rings, we can choose such maps $A \to \overline{A}$ and $B \to \overline{B}$ together with a map $\overline{A} \to \overline{B}$ of A-algebras.*

Remark 19. The proof in Bhatt-Scholze claims that the $T(A)$ defined in the proof already satisfies the requirements of $\overline{A}$ but this seems unlikely.

Proof. Choose a set F of isomorphism classes of faithfully flat étale A-algebras. For $I \subseteq F$ a finite subset set $A_I := \otimes_{i\in I} A_i$ and set $T(A) = \varinjlim_{I\subseteq F} A_I$. Now repeat this process to get $T^2(A) := T(T(A))$, $T^3(A) := T(T(T(A)))$, etc, and set $\overline{A} := \varinjlim(A \to T(A) \to T^2(A) \to \dots)$.

Then $\overline{A}$ is absolutely flat by Exercise 5. Furthermore, any faithfully flat étale morphism $\overline{A} \to C$ is of the form $C = \overline{A} \otimes_{T^n(A)} C'$ for some $n \in \mathbb{N}$ and some faithfully flat étale morphism $T^n(A) \to C'$, [Stacks project, 01ZM, 07RP, 081D]. By construction, there is a factorisation $T^n(A) \to C' \to T^{n+1}(A)$, and this induces a factorisation $\overline{A} \to C \to \overline{A}$.

For the second part, simply set $\overline{B}$ to be a w-strictly local faithfully flat ind-étale algebra over $\overline{A} \otimes_A B$. □

Remark 20. Note that one can construct the algebraic closure of a field k using this method. Taking $T(A)$ to be the tensor product of all finite flat A-algebras, then every residue field of $\overline{k}$ will be an algebraic closure of k.

Recall that in Exercise 2 we proced that if A is w-local then $\pi_0(\mathrm{Spec}(A))$ is canonically homeomorphic to the set $\mathrm{Spec}(A)^c$ of closed points of $\mathrm{Spec}(A)$.

Lemma 21 ([BS, Lem.2.2.8], [Stacks Project, 097D])**.** *For any ring A and a continuous map $T \to \pi_0(\mathrm{Spec}(A))$ of profinite sets, there is an ind-(Zariski localization) $A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives rise to the given map $T \to \pi_0(\mathrm{Spec}(A))$ on applying π_0. Moreover, the association $T \mapsto \mathrm{Spec}(B)$ is a limit-preserving functor.*

− − − − picture − − − −

Remark 22 (No number)**.** One may make the following more precise statement: for any affine scheme X, the functor $Y \mapsto \pi_0(Y)$ from affine X-schemes to profinite $\pi_0(X)$-sets has a fully faithful right adjoint $S \mapsto S \times_{\pi_0(X)} X$, the fibre product in the category of topological spaces ringed using the pullback of the structure sheaf on X. Moreover, the natural map $S \times_{\pi_0(X)} X \to X$ is a pro-(Zariski localisation) and pro-finite.

3 Henselisation

To pass from w-strictly local covers of absolutely flat rings to arbitrary rings, we use henselizations:

Definition 23 (Def.2.2.10)**.** *Given a map of rings $A \to B$, let* $\mathrm{Hens}_A(-) : \mathrm{Ind}(B_{\mathsf{et}}) \to \mathrm{Ind}(A_{\mathsf{et}})$ *be the functor right adjoint to the base change functor* $\mathrm{Ind}(A_{\mathsf{et}}) \to \mathrm{Ind}(B_{\mathsf{et}})$. *Explicitly, for $B_0 \in \mathrm{Ind}(B_{\mathsf{et}})$, we have* $\mathrm{Hens}_A(B_0) = \varinjlim A'$, *where the colimit is indexed by diagrams $A \to A' \to B_0$ of A-algebras with $A \to A'$ étale.*

Lemma 24 (Lem.2.2.13)**.** *Let A be a ring henselian along an ideal I. Then A is w-strictly local if and only if A/I is so.*

Corollary 25 (Cor.2.2.14)**.** *Any ring A admits an ind-étale faithfully flat map $A \to A'$ with A' w-strictly local.*

Proof. Set $A' := \mathrm{Hens}_{A^Z}(\overline{A^Z/I_{A^Z}})$, where $\overline{A^Z/I_{A^Z}}$ is a w-strictly local ind-étale faithfully flat A^Z/I_{A^Z}-algebra; then A' satisfies the required property by Lemma 2.2.13. □

4 Weakly étale versus pro-étale

Definition 26 (Def.2.3.1)**.** *A morphism $A \to B$ of commutative rings is called* weakly étale *if both $A \to B$ and the multiplication morphism $B \otimes_A B \to B$ are flat.*

p:PropWeaklyEtale **Proposition 27** (Prop.2.3.3)**.** *Fix maps $f : A \to B$, $g : B \to C$, and $h : A \to D$ of rings.*

1. *If f is ind-étale, then f is weakly étale.*
2. *If f is weakly étale and finitely presented, then f is étale.*
3. *If f and g are weakly étale (resp. ind-étale), then $g \circ f$ is weakly étale (resp. ind-étale). If $g \circ f$ and f are weakly étale (resp. ind-étale), then g is weakly étale (resp. ind-étale).*
4. *If h is faithfully flat, then f is weakly étale if and only if $f \otimes_A D : D \to B \otimes_A D$ is weakly étale.*

Theorem 28 (Thm.2.3.4)**.** *Let $f : A \to B$ be weakly étale. Then there exists a faithfully flat ind-étale morphism $g : B \to C$ such that $g \circ f : A \to C$ is ind-étale.*

Proof. Lemma 2.3.7 (omitted from the lecture) gives a diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ B & \longrightarrow & B' \end{array}$$

with f' a w-local map of w-strictly local rings, and both horizontal maps being ind-étale and faithfully flat. The map f' is also weakly étale since all other maps in the square are so. Then f' is a ind-(Zariski localization) by Lemma 2.3.8 which says that any w-local weakly étale map of w-local rings from a w-strictly local ring is an ind-(Zariski localisation). Setting $C = B'$ then proves the claim. □

5 Local contractibility

Definition 29 (Def.2.4.1)**.** *A ring A is* w-contractible *if every faithfully flat ind-étale map $A \to B$ has a section.*

Lemma 30 (Lem.2.4.2)**.** *A w-contractible ring A is w-local (and thus w-strictly local).*

Lemma 31 (Lem.2.4.3)**.** *Let A be a ring henselian along an ideal I. Then A is w-contractible if and only if A/I is so.*

Definition 32 (Def.2.4.4)**.** *A profinite set is* extremally disconnected *if the closure of every open is open.*

Remark 33. We are interested in extremally disconnected spaces because they are weakly local. In fact, a theorem of Gleason from 1958 says that they are exactly the projective objects in the category of all compact Hausdorff spaces: any continuous surjection $Y \to X$ from a compact Hausdorff space to an extremally disconnected profinite set has a section. Cf.[Stacks Project, 08YN].

Example 34 (Exa.2.4.6)**.** Every compact Hausdorff space X admits a continuous surjection from an extremally disconnected space. (Proof ommitted). See [Stacks Project, 090D].

Proof: Let $\delta(X)$ denote X with the discrete topology, and consider it's Stone-Cech compactification $\beta(\delta(X))$, i.e., the universal map to a compact Hausdorff space. (Aside: as $\delta(X)$ is discrete, $\beta(\delta(X))$ is the profinite completion, i.e., the inverse limit over all maps $X \to S$ towards a finite set S.) As X is compact Hausdorff, by the universal property defining the Stone-Cech compactification we get a factorisation $\delta(X) \to \beta(\delta(X)) \to X$, showing that $\beta(\delta(X)) \to X$ is surjective, and it remains to see that $\beta(\delta(X))$ is extremally disconnected. A theorem of Gleason [Gle58] says that extremally disconnected spaces are exactly the projective objects in the category of all compact Hausdorff spaces, i.e., those X for which every continuous surjection $Y \to X$ splits. But $\delta(X) \to \beta(\delta(X))$ lifts through any continuous surjection $Y \to \beta(\delta(X))$, and again using the universal property of $\beta(\delta(X))$, the lifting extends to $\beta(\delta(X))$.

Lemma 35 (Lem.2.4.8)**.** *A w-strictly local ring A is w-contractible if and only if $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.*

Proof. Suppose A is w-contractible and $T \to \pi_0(\mathrm{Spec}(A))$ is a continuous surjection of profinite sets. Then $\mathrm{Spec}(T) \times_{\pi_0(\mathrm{Spec}(A))} T \to \mathrm{Spec}(A)$ is a pro-(Zariski-localisation) and therefore has a section by w-contractibility. Composing with $\mathrm{Spec}(A)^c \to \mathrm{Spec}(A)$, we get a factorisation $\mathrm{Spec}(A)^c \to T \to \pi_0(\mathrm{Spec}(A))$, but $\mathrm{Spec}(A)^c \to \pi_0(\mathrm{Spec}(A))$ is an isomorphism (by w-locality of A) so we have found a section to $T \to \pi_0(\mathrm{Spec}(A))$.

The converse reduces the dimension zero (i.e., absolutely flat) case. In this case since the residue fields of A are all separable closed, every ind-étale faithfully flat A-algebra B is indued by a continuous surjection of profinite sets $T \to \mathrm{Spec}(A)$. Then we just apply that $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected. □

lem:cwcontractiblecover **Lemma 36** (Lem.2.4.9)**.** *For any ring A, there is an ind-étale faithfully flat A-algebra A' with A' w-contractible.*

Proof. By Lemmas 2.1.10, 2.2.3, and 2.2.4, the pro-finite set $\mathrm{Spec}(A^Z/I_{A^Z})$ is homeomorphic to $\mathrm{Spec}(A)$ with the constructible topology, where I_{A^Z} is the Jacobson radical. In particular, it is a compact Hausdorff space. Choose a continuous surjection $T \to \mathrm{Spec}(A^Z/I_{A^Z})$ from an extremally disconnected profinite set as mentioned in Example 2.4.6. Using Lemma 2.2.8, choose an algebra $A^Z/I_{A^Z} \to B$ such that $T = \pi_0(B)$. Using Lemma 2.2.7, we find an ind-étale faithfully flat A^Z/I_{A^Z}-algebra A_0 with A_0 w-strictly local and $\mathrm{Spec}(A_0)$ an extremally disconnected profinite set. Let $A' = \mathrm{Hens}_{A^Z}(A_0)$. Then A_0 is w-contractible because $\pi_0(\mathrm{Spec}(A_0))$ is extremally disconnected (Lem.2.4.8), and A' is w-contractible because A_0 is (Lemma 2.4.3). The map $A \to A'$ is faithfully flat and ind-étale since both $A \to A^Z$ and $A^Z \to A'$ are so individually. □

A Some point-set topology

profinset

A.1 Profinite sets

Definition 37. *A* profinite set *is a filtered inverse limit $T = \varprojlim T_i$ of finite sets T_i, equipped with the limit topology. So the open sets are (possibly infinite) unions of sets of the form $\pi_i^{-1}(t)$ where $\pi_i : T \to T_i$ is the canonical projection and $t \in T_i$.*

Example 38.

1. Any finite set is a profinite set.
2. The set $\{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \dots\}$ is profinite.
3. The Cantor set is profinite.
4. Any product of profinite sets is profinite.

− − − − *** add picture *** − − − −

Proposition 39 ([Stacks Project, Tag 08ZY])**.** *A topological space is a profinite set if and only if it is compact*[8] *Hausdorff*[9] *and totally disconnected*[10]*.*

This proposition gives a canonical choice for the filtered limit.

[8] Compact means every open cover admits a finite subcover. This property is called *quasi-compact* when talking about schemes.

[9] Hausdorff means for every pair of distinct points $x \neq y$ there are open sets U, V with $x \in U, y \in V$ and $U \cap V = \varnothing$. When talking about schemes, we say quasi-compact for this same property.

[10] Totally disconnected means that every subset $V \subseteq T$ containing more than one point can be written as a disjoint union $V = V_1 \amalg V_2$ of nonempty sets $V_1, V_2 \subseteq T$ both of which are both open *and* closed.

Lemma 40. *Let T be a profinite set. Then $T = \varprojlim_{\sqcup_{i \in I} T_i} I$ were the limit is over the finite partitions $T = \sqcup_{i \in I} T_i$ of T into a disjoint union of subsets T_i which are both open and closed.*

Note that any closed open U is compact (any covering of U extends to a covering of T by adjoining $T \setminus U$, and T is compact). On the other hand, any compact open U is a finite union of basic opens[11] and these are closed, so U is closed.

So, to summarise:

$$\left\{ \begin{array}{c} \text{basic} \\ \text{opens} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{compact} \\ \text{opens} \end{array} \right\} = \left\{ \begin{array}{c} \text{closed} \\ \text{opens} \end{array} \right\}$$

A.2 Finite sober spaces

A topological space X is called *sober*[12] if every irreducible closed subset has a unique generic point. I.e., if a closed subset Z is irreducible, then Z is the closure $Z = \overline{\{\eta\}}$ of some unique point η). The reader is probably already familiar with finite sober spaces in a different form:

Proposition 41. *The category of finite sober spaces is equivalent to the category of finite partially ordered sets.*

Proof. Given a partially ordered set P we define $X(P)$ to be the topological space whose points are the points of P, and whose opens are the "upwards closed" subsets. I.e., subsets U satisfying

$$x \in U, x \leq y \Rightarrow y \in U.$$

Exercise: check that the U really form a topology.

Conversely, given a topological space X we define $P(X)$ to be the partially ordered set whose points are the points of X, and relation

$$x \leq y \text{ iff } y \in \bigcap_{U \ni x} U$$

where the intersection is over all opens U containing x.

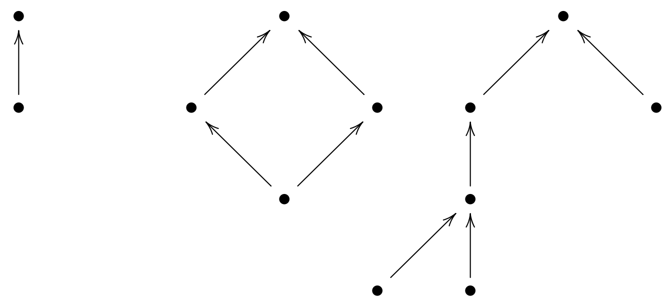
Exercise: Check that we have $x \leq y$ and $y \leq z \Rightarrow x \leq z$, that we have $x \leq x$, and that when X is sober we have $x \leq y$ and $y \leq x \Rightarrow x = y$. Note that an equivalent condition for $x \leq y$ is that $x \in \bigcap_{Z \ni y} Z$ where the intersection is over all closed subsets containing y (its straightforward to prove $y \notin \cap_{U \ni x} U \iff x \notin \cap_{Z \ni y} Z$).

Finally, one should check that $X(P(X)) = X$ and that $P(X(P)) = P$, and also that the operations $X \mapsto P(X)$ ad $P \mapsto X(P)$ send continuous morphisms (resp. morphisms of partially ordered sets) to morphisms of partially ordered sets (resp. continuous morphisms). □

[11] By *basic* we mean the preimage of some point $t \in T_i$ for some presentation $T = \varprojlim_{i \in I} T_i$.

[12] From Johnstone's Topos Theory book, p.230: "If we regard two distinct points having the same closure as an instance of double vision (and an irreducible closed set with no generic point as a species of pink elephant!), then the reason for the term 'sober space' will be apparent."

So it's quite easy to produce examples of finite sober spaces.



A.3 Spectral spaces

Definition 42. *A* spectral space *is a topological space of the form* Spec(R) *for some ring* R.

Spaces of the form Spec(R) satisfy a number of nice properties.

1. Each Spec(R) is sober.
2. Each Spec(R) is quasi-compact, i.e., every open cover $\{U_i \to \mathrm{Spec}(R)\}_{i \in I}$ has a finite subcover.[13]
3. The topology on Spec(R) is generated by the opens of the form $D(f) = \mathrm{Spec}(R[\frac{1}{f}])$; note that these are also quasi-compact.
4. In fact an open $U \subset \mathrm{Spec}(R)$ is quasi-compact if and only if $\mathrm{Spec}(R) \backslash U = V(I)$ with I a finitely generated ideal. Consequently, intersections of quasi-compact opens of Spec(R) are also quasi-compact.

It is an old theorem of Hochster that the above four properties *characterise* spectral spaces: I.e., A topological space X is (i) sober, (ii) quasi-compact, (iii) has topology generated by quasi-compact opens, and (iv) has it's set of quasi-compact opens preserved by finite intersection if and only if there is some ring R with Spec(R) homeomorphic to X. Moreover, a continuous morphism of spectral spaces is spectral if and only if the inverse image of any quasicompact open is quasicompact.

In particular, we can deduce from this that for every finite sober space X, there is some ring R with $X = \mathrm{Spec}(R)$. In light of the above proposition, an equivalent statement is: every partially ordered set is the partially ordered set of primes of some ring.

On the other hand, there is another characterisation of spectral spaces that is useful for us.

[13]This follows from a partition of unity type argument: If $V(I_i)$ are the closed complements of the U_i, then $\cap V(I_i) = \varnothing$, but $\cap V(I_i) = V(\sum_i I_i)$ and this is empty if and only if $1 \in \sum_i I_i$, so $1 = a_{i_1} + \cdots + a_{i_n}$ for some $a_{i_j} \in I_j$. But then $V(I_{i_1} + \cdots + I_{i_n})$ is also empty, so $\{U_{i_1}, \ldots, U_{i_n}\}$ is also an open cover.

Proposition 43. *A topological space is spectral if and only if it is a filtered inverse limit of finite sober spaces.*

B Constructible sets

Just as any profinite set X is the filtered inverse limit of it's partitions into closed-open sets (see above), any spectral space is the inverse limit of it's partitions into constructible sets.

Definition 44. *Suppose that X is a spectral space. The family of* constructible sets *of X is the smallest family of subsets closed under finite intersection, finite union, complement, and containing the quasi-compact open subsets of X.*

– – – – *** add picture *** – – – –

Constructible sets generate a new topology on X: the *constructible topology.* The opens of the constructible topology are (possibly infinite) unions of constructible sets (which, of course, may not be open in the original topology). Since the original topology on our spectral space X is generated by quasi-compact opens, we see that the constructible topology is finer than the original topology. In fact, equipped with the constructible topology, X becomes a profinite set!

Proposition 45 ([Stacks Project, Tag 0901])**.** *Let X be a spectral space. The constructible topology on X is compact, Hausdorff, and totally disconnected. In other words, it is profinite.*

The claim about being totally disconnected is not in the Stacks Project statement, but follows easily from the fact that the constructible topology is Hausdorff, and that it has a basis consisting of sets which are both open and closed.

On the other hand, since the compact opens of a profinite set are precisely the closed opens, for a profinite set the constructible topology is the same as the original topology.

So changing a spectral spaces topology to the constructible topology is a kind of "profinitification".

Construction 46. Suppose that $X = \mathrm{Spec}(R)$ is a spectral space and $X = \sqcup_{P \in p} X_p$ is a decomposition into constructible subsets. Sending $x \in X$ to the index of the X_p that contains it defines a map $\pi : X \to P$. Then we give P the induced topology. So a subset $U \subset P$ is open if and only if $\pi^{-1}U$ is open.

Since the constructible open sets are the quasicompact ones, one sees that the map $X \to P$ is spectral. That is, there is a ring homomorphism $S \to R$ such that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induces the map of topological spaces $X \to P$ (this R may be different from the R in the construction, even though it gives rise to the same topological space).

Lemma 47. *Suppose that X is a spectral space. Then X is the inverse limit of it's finite constructible decompositions $X = \varprojlim_{X=\sqcup_{p\in P} X_p} P$.*

Proof. If we equip each P with the discrete topology, then we get the morphism $\pi : X \to \varprojlim_{X=\sqcup_{p\in P} X_p} P$ from the profinite version of this lemma, which we have already proven is injective and surjective. So it suffices to show that $U \subseteq X$ is open if and only if $\pi(U)$ is open. Suppose U is a quasicompact open. Then U induces a constructible partition $X = U \amalg (X \setminus U)$ and so $\pi(U)$ is open. Since all opens of X are unions of quasicompact opens, this shows that if U is open, $\pi(U)$ is open. Conversely, if $\pi(U)$ is open, then U must be open by continuity and bijectivity of π. □