

In this lecture we show how the category  $\mathbf{Shv}_{\text{et}}(X)$  of étale sheaves on a scheme  $X$  can be reconstructed from  $\mathbf{Shv}_{\text{et}}(Z)$  sheaves on a closed subscheme  $Z \subseteq X$  and  $\mathbf{Shv}_{\text{et}}(U)$  sheaves on its open complement  $U = X - Z$ , see Theorem 16.

For easy of exposition, all presheaves will be presheaves of abelian groups,<sup>1</sup> and all sites small.<sup>2</sup> All schemes are assumed to be quasi-compact quasi-separated (e.g., Noetherian and separated), and we recall that étale morphisms are by definition locally of finite presentation (this means finite type if working only with Noetherian schemes).<sup>3</sup>

## 1 Presheaf adjunctions

$Y \rightarrow X \in \text{TOP}$   
 e.g.  
 $C = \mathcal{O}_X(Y) \xrightarrow{\pi} \mathcal{O}_X(X) = C'$

**Definition 1.** Suppose that  $\pi : C' \rightarrow C$  is a functor. We denote the functor induced by composition as

$$\pi_p : \text{PreShv}(C) \rightarrow \text{PreShv}(C'); \quad F \mapsto F \circ \pi.$$

**Exercise 1.** If  $\pi' : C'' \rightarrow C'$  and  $\pi : C' \rightarrow C$  are two functors, show that  $(\pi \circ \pi')_p = \pi_p \circ \pi'_p$ .

**Definition 2.** Give a presheaf  $F \in \text{PreShv}(C')$  and  $X \in C$  define

$$(\pi^p F)(X) = \varinjlim_{X \rightarrow \pi(Y)} F(Y)$$

where the colimit is over the comma category  $(X \downarrow \pi)$  whose objects are morphisms  $X \rightarrow \pi(Y)$  in  $C$ , and  $\text{hom}(X \rightarrow \pi(Y), X \rightarrow \pi(Y')) = \{f : Y \rightarrow Y' \text{ s.t. the triangle } \begin{array}{ccc} \pi(Y) & \rightarrow & \pi(Y') \\ \swarrow & & \nearrow \\ X & & \end{array} \text{ commutes} \}$ .

**Remark 3.** There is also a right adjoint to  $\pi_p$  defined in an analogous way, but we will not use it.

**Exercise 2.** Using the universal property of the colimit, show that a morphism  $X \rightarrow X'$  in  $C$  induces a morphism  $(\pi^p F)(X') \rightarrow (\pi^p F)(X)$ , and that this makes  $\pi^p F$  into a presheaf on  $C$ .

**Exercise 3 (Advanced).** Given an object  $W \in C$  we write  $h_W(-) = \text{hom}_C(-, W)$  for the presheaf represented by  $W$ .

1. Show that  $\pi^p h_Y = h_{\pi Y}$  for any  $Y \in C'$ .
2. Show that there is canonical isomorphism  $\text{hom}(\pi^p h_Y, G) \cong \text{hom}(h_Y, \pi_p G)$ .  
 Note: the right side is isomorphic to  $(\pi_p G)(Y)$ .

<sup>1</sup>Sheaves of sets work just as well.

<sup>2</sup>This is to ensure that the colimits defining  $\pi^p$  are well-defined. In practice, there are many functors between large categories for which these colimits are still well-defined.

<sup>3</sup>These finiteness assumptions ensure that  $\text{Et}(X)$  are small (enough) categories (so that the left adjoints  $\pi^p$  existence is guaranteed), but otherwise are basically only used in the proof of Proposition 12.

3. Show that for any presheaf  $F \in \text{PreShv}(C')$ , we have  $F \cong \varinjlim h_Y$  where  $h_Y = \text{hom}_{C'}(-, Y)$  is the presheaf represented by  $Y$ , and the colimit is over the category  $\int_C F$  whose objects are pairs  $(Y, s)$  with  $Y \in C'$  and  $s \in F$  and morphisms  $(Y, s) \rightarrow (Y', s')$  are morphisms  $Y \rightarrow Y'$  of  $C'$  such that  $F(Y') \rightarrow F(Y)$  sends  $s'$  to  $s$ .
4. Show that  $\pi^p$  preserves any colimits of presheaves.
5. Deduce that for any  $F \in \text{PreShv}(C')$  (not just representable presheaves) there is a canonical isomorphism  $\text{hom}_{\text{PreShv}(C)}(\pi^p F, G) \cong \text{hom}_{\text{PreShv}(C')}(F, \pi_p G)$ .

**Corollary 4.** *The pair  $(\pi^p, \pi_p)$  is an adjunction  $\text{PreShv}(C) \rightleftarrows \text{PreShv}(C')$ .*

**Exercise 4.** Using Exercise 1, Corollary 4, and the uniqueness properties of adjunctions show that  $\pi'^p \circ \pi^p = (\pi \circ \pi')^p$ .

**Exercise 5.** Suppose that  $f : Y \rightarrow X$  is a morphism of topological spaces, and let  $\pi : \text{Op}(X) \rightarrow \text{Op}(Y); U \mapsto f^{-1}U$  be the induced functor between the categories of open subsets of  $X, Y$ . Show that  $\pi_p$  is the usual push-forward  $\text{PreShv}(Y) \rightarrow \text{PreShv}(X)$  and  $\pi^p$  is the usual inverse image of presheaves functor  $\text{PreShv}(X) \rightarrow \text{PreShv}(Y)$ .

**Exercise 6.** Suppose that the category  $C$  has a final object  $X$ , and let  $\pi : * \rightarrow C$  be the functor from the category with one morphism which sends the unique object to  $X$ . Show that  $\pi_p$  is the global sections functor  $F \mapsto F(X)$ , and  $\pi^p$  is the constant presheaf functor  $(\pi^p A)(U) = A$  for  $A \in \text{Ab} = \text{PreShv}(*)$ .

**Exercise 7.** Let  $Y \rightarrow X$  be an étale morphism of schemes, and consider the functors

$$\pi : \text{Et}(X) \rightarrow \text{Et}(Y); \quad U \mapsto Y \times_X U \quad (1)$$

and

$$\gamma : \text{Et}(Y) \rightarrow \text{Et}(X); \quad (V \rightarrow Y) \mapsto (V \rightarrow Y \rightarrow X) \quad (2)$$

Show that  $(\gamma, \pi)$  is an adjunction. Show that  $\gamma_p = \pi^p$ .

## 2 Sheaf adjunctions

**Definition 5.** *Suppose that  $C', C$  are sites, i.e., categories equipped with Grothendieck topologies. A functor  $\pi : C' \rightarrow C$  is called continuous if for every sheaf  $F$  on  $C$ , the presheaf  $\pi_p F$  is a sheaf.*

**Exercise 8.** Suppose  $\pi : C' \rightarrow C$  sends fibre products to fibre products. Show that  $\pi$  is continuous if it sends covers to covers.

*( $Y \in \text{Top}$  is continuous if  $f : \text{Subsets}(X) \rightarrow \text{Subsets}(Y)$  sends opens to opens)*

**Example 6.** If  $Y \rightarrow X$  is a continuous morphism topological spaces then the induced morphism of sites  $\text{Op}(X) \rightarrow \text{Op}(Y)$  is continuous.

**Example 7.** If  $f : Y \rightarrow X$  is a morphism of schemes, then  $\pi$  from Equation 1 is continuous. If  $f$  is an étale morphism of schemes then  $\gamma$  from Equation 2 is also continuous.

**Definition 8.** Suppose  $\pi : C' \rightarrow C$  is a continuous morphism of sites. The induced functor on sheaves is denoted

$$\pi_* : \text{Shv}(C) \rightarrow \text{Shv}(C').$$

The composition of  $\pi^p$  with sheafification  $a : \text{PreShv}(C) \rightarrow \text{Shv}(C)$  is denoted

from  $\rightarrow$   
Lecture 3

$$\pi^* = a \circ \pi^p : \text{Shv}(C') \rightarrow \text{Shv}(C).$$

**Exercise 9.** Suppose we are in the situation of Definition 8. Using the fact that sheafification  $a : \text{PreShv}(C) \rightarrow \text{Shv}(C)$  is the left adjoint to the canonical inclusion  $\iota : \text{Shv}(C) \rightarrow \text{PreShv}(C)$ , show that

$$\pi^* : \text{Shv}(C) \rightleftarrows \text{Shv}(C') : \pi_*$$

is an adjunction.

**Exercise 10.** Using Exercise 1 and Exercise 4, show that if  $C, C', C''$  are equipped with Grothendieck topologies, and  $\pi, \pi'$  are continuous, then  $(\pi \circ \pi')_* = \pi_* \circ \pi'_*$  and  $\pi'^* \circ \pi^* = (\pi \circ \pi')^*$ .

$$\pi : \mathcal{E}t(X) \rightarrow \mathcal{E}t(Y)$$

**Definition 9.** If  $f : Y \rightarrow X$  is a morphism of schemes, and  $\pi$  the pullback functor from Equation (1), we write

$$f^* := \pi^*, \quad f_* := \pi_*.$$

If  $f$  is étale, so  $\pi$  has a left adjoint  $\gamma$  from Equation 2 then we write

$$f_! := \gamma^*$$

Note that since  $\gamma_* = \pi^*$ , cf. Exercise 7, in addition to the adjunction  $(f^*, f_*)$ , we have another adjunction  $(f_!, f^*)$ .

**Lemma 10.** Let  $f : Y \rightarrow X$  be a morphism of schemes. Then  $f^*$  preserves exact sequences.

*Proof.* It automatically commutes with colimits because it is a left adjoint. On the other hand,  $\pi^p$  commutes with finite limits because limits of presheaves are calculated object wise, and  $\pi^p$  is defined using filtered colimits, which commute with finite limits. To deduce that  $f^* = \pi^*$  commutes with finite limits from  $\pi^p$  commuting, we just recall that sheafification is exact, so  $\pi^* = a \circ \pi^p$  is a composition of two functors which commute with finite limits.  $\square$

**Lemma 11.** Let  $f : Y \rightarrow X$  and  $X' \rightarrow X$  be morphisms of schemes. Let  $h'_X$  denote the étale sheaf of sets  $h_{X'} = \text{hom}_X(-, X') \in \text{Shv}_{\text{et}}(X)$ , and similarly,  $h_{Y \times_X X'} = \text{hom}_Y(-, Y \times_X X') \in \text{Shv}_{\text{et}}(Y)$ . We have

$$f^* h_{X'} = h_{Y \times_X X'}.$$

*Proof.* By Yoneda, it suffices to produce isomorphisms

$$\mathrm{hom}_{\mathrm{Shv}}(f^*h_{X'}, F) \cong \mathrm{hom}_{\mathrm{Shv}}(h_{Y \times_X X'}, F)$$

for each  $F \in \mathrm{Shv}_{\mathrm{et}}(Y)$ , which are natural in  $F$ . But we have

$$\begin{aligned} \mathrm{hom}_{\mathrm{Shv}}(f^*h_{X'}, F) &\cong \mathrm{hom}_{\mathrm{Shv}}(h_{X'}, f_*F) && \text{adjunction} \\ &\cong (f_*F)(X') && \text{Yoneda} \\ &\cong F(Y \times_X X') && \text{definition} \\ &\cong \mathrm{hom}_{\mathrm{Shv}}(h_{Y \times_X X'}, F) && \text{Yoneda.} \end{aligned}$$

□

**Exercise 11.** Using the same argument as in Lemma 11 show that if  $f : Y \rightarrow X$  is an étale morphism of schemes, and  $Y' \rightarrow Y$  any morphism then

$$f_! h_{Y'} = h_{Y'}$$

where the left  $Y'$  is considered as a  $Y$ -scheme, and the right one as an  $X$ -scheme.

### 3 Immersions

**Exercise 12.** Suppose that  $j : U \rightarrow X$  is an open immersion. Show that in this case,  $\gamma : \mathrm{Et}(U) \rightarrow \mathrm{Et}(X)$  from Equation 2 is the inclusion of a **full** subcategory. Show that since this subcategory is full, the functor  $j^* : \mathrm{Shv}(X) \rightarrow \mathrm{Shv}(U)$  is none-other-than the restriction functor

$$j^*F = F|_{\mathrm{Et}(U)}.$$

$$\begin{array}{ccc} & \mathrm{Ab} & \\ \text{---} j^*F \text{---} & \uparrow F & \\ \mathrm{Et}(U)^{\mathrm{op}} & \subset & \mathrm{Et}(X)^{\mathrm{op}} \end{array}$$

Show that  $j_! : \mathrm{Shv}(U) \rightarrow \mathrm{Shv}(X)$  is “extension by zero” in the sense that for any  $H \in \mathrm{Shv}(U)$ ,

$$(j_!H)(V) = \begin{cases} H(V) & V \in \mathrm{Et}(U) \\ 0 & V \notin \mathrm{Et}(U) \end{cases}$$

Show that

$$(j_*H)(V) = H(V) \text{ if } V \in \mathrm{Et}(U),$$

but give an example where  $V \notin \mathrm{Et}(U)$ , and  $(j_*H)(V) \neq 0$ .

Deduce that

$$j^*j_! = \mathrm{id} = j^*j_*$$

*this will give gluing data*

**Exercise 13.** Let  $X$  be a scheme, and  $\iota : \bar{x} \rightarrow X$  a geometric point. Show that

$$\iota^*F = F_{\bar{x}}$$

*i.e.  $\bar{x} = \mathrm{Spec}(k)$   
separably closed field*

is the stalk of  $F$  at  $\bar{x}$ , where we implicitly use the identification  $\mathrm{Shv}(\bar{x}) \cong \mathrm{Ab}$ .

**Proposition 12** (Milne Cor.II.3.5). Let  $i : Z \rightarrow X$  be the inclusion of a closed immersion,  $\bar{x} \rightarrow X$  a geometric point, and  $G \in \text{Shv}(Z)$ . Then

$$(i_*G)_{\bar{x}} = \begin{cases} G_{\bar{x}} & \text{im}(\bar{x}) \in Z \\ 0 & \text{im}(\bar{x}) \notin Z \end{cases} \quad \begin{array}{l} \text{"doesn't change on } Z\text{"} \\ \text{"zero everywhere else"} \end{array}$$

If  $j : U \rightarrow X$  is the open complement of  $Z$ , then we have  $j^*i_* = 0$ .

**Remark 13.** We are abusing the notation a bit here in the case  $\text{im}(\bar{x}) \in Z$ . When writing  $(i_*G)_{\bar{x}}$ , we are considering  $\bar{x}$  as a geometric point of  $X$ , so the colimit is over factorisations through  $\text{Et}(X)$ . But when writing  $G_{\bar{x}}$ , we are considering  $\bar{x}$  as a geometric point of  $Z$ , and so the colimit is over factorisations through  $\text{Et}(Z)$ .

*Easy parts of the proof (Omitted from the lecture).* The second claim follows from the first claim, since a sheaf is zero if and only if all its stalks are zero, and the stalks of  $j^*i_*$  are all zero by Exercises 10, 12, and 13.

Certainly, if  $\text{im}(\bar{x}) \notin Z$ , then  $(i_*G)_{\bar{x}} = \varinjlim_{\bar{x} \rightarrow V \rightarrow X} G(V) = 0$ , since each  $\bar{x} \rightarrow V \rightarrow X$  is refinable by some  $\bar{x} \rightarrow V' \rightarrow X$  with  $Z \times_X V' = \emptyset$  (e.g.,  $V' = U \times_X V$ ), and for such  $V'$  we have  $(i_*G)(V') = G(Z \times_X V') = G(\emptyset) = 0$ .

The difficult part is to show that for any  $\bar{x} \rightarrow V \rightarrow Z$  with  $V \in \text{Et}(Z)$ , there is some  $\bar{x} \rightarrow V' \rightarrow X$  and a factorisation  $\bar{x} \rightarrow Z \times_X V' \rightarrow V \rightarrow Z$ . If we know this, then the system defining  $(i_*G)_{\bar{x}}$  is cofinal in the system defining  $G_{\bar{x}}$ , and the colimits will be the same.

The proof of this claim is omitted. See Milne Thm.II.3.2(b) for details. Really, check it out. Its a very neat argument using properties of limit schemes from EGA, in particular, EGA IV, Part 3, Cor.8.13.2.  $\square$

**Exercise 14.** Suppose that  $F \in \text{Shv}_{\text{et}}(U)$  is a constant sheaf (that is, there is an abelian group such that for each connected  $V \in \text{Et}(U)$ , we have  $F(V) = A$ ). If  $i : Z \rightarrow X$  is a nowhere dense closed immersion with open complement  $j : U \rightarrow X$ , using the fact that étale morphisms send generic points to generic points, show that  $j_*F$  and  $i^*j_*F$  are constant sheaves on  $X$  and  $Z$  respectively.

## 4 The localisation sequences

**Definition 14.** Let  $i : Z \rightarrow X$  be a closed immersion of schemes, and  $j : U \rightarrow X$  the open complement. Define  $T(X)$  to be the category<sup>4</sup> whose objects are triples  $(F_1, F_2, \phi)$  consisting of two objects  $F_1 \in \text{Shv}(Z)$ ,  $F_2 \in \text{Shv}(U)$ , and a morphism  $\phi : F_1 \rightarrow i^*j_*F_2$ . Morphisms  $(F_1, F_2, \phi) \rightarrow (F'_1, F'_2, \phi')$  are pairs of morphisms

$\Delta$   
 $\uparrow$   
 glueing  
 instructions

<sup>4</sup>This is just the comma category  $(\text{Shv}(Z) \downarrow i^*j_*)$ .

$(F_1 \xrightarrow{\psi_1} F'_1, F_2 \xrightarrow{\psi_2} F'_2)$  such that the square commutes.

$$\begin{array}{ccc} F_1 & \xrightarrow{\phi} & i^* j_* F_2 \\ \psi_1 \downarrow & & \downarrow i^* j_* \psi_2 \\ F'_1 & \xrightarrow{\phi'} & i^* j_* F'_2 \end{array}$$

We define a functor  $t : \mathbf{Shv}(X) \rightarrow T(X)$  by

$$F \mapsto (i^* F, j^* F, i^*(F \xrightarrow{\eta} j_* j^* F))$$

where  $\eta : \text{id} \rightarrow j_* j^*$  is the unit of the adjunction  $(j^*, j_*)$ .

**Remark 15.** Given  $V \in \text{Et}(X)$ , we have  $j_* j^* F(V) = F(U \times_X V)$ , and  $\eta$  is the canonical morphism  $F(V) \rightarrow F(U \times_X V)$  induced by the canonical morphism  $U \times_X V \rightarrow V$ .

**Theorem 16** (Milne Thm.II.3.10). The functor  $t : \mathbf{Shv}(X) \rightarrow T(X)$  is an equivalence of categories.

*Proof.* Given a triple  $(F_1, F_2, \phi)$  in  $T(X)$  define

$$s(F_1, F_2, \phi) := \ker \left( \begin{array}{ccc} i_* F_1 & \oplus & j_* F_2 \\ \underbrace{\quad} \quad \underbrace{\quad} & \xrightarrow{i_* \phi \oplus \eta} & \underbrace{\quad} \\ \underbrace{\quad} \quad \underbrace{\quad} & & \underbrace{\quad} \end{array} \right).$$

*“overlap”*

Here,  $\eta : \text{id} \rightarrow i_* i^*$  is the unit of the adjunction  $(i^*, i_*)$ . Notice that every morphism of  $T(X)$  induces a morphism in  $\mathbf{Shv}(X)$  in a way that defines a functor

$$s : T(X) \rightarrow \mathbf{Shv}(X).$$

So it suffices to check that  $st \cong \text{id}$  and  $ts \cong \text{id}$ . Consider  $stF$ . By definition, this is

$$stF = \ker \left( i_* i^* F \oplus j_* j^* F \xrightarrow{i_* \phi \oplus \eta} i_* i^* j_* j^* F \right).$$

This comes equipped with a canonical morphism  $F \rightarrow stF$ . This morphism is an isomorphism if and only if the sequence

$$0 \rightarrow F \rightarrow \underbrace{i_* i^* F} \oplus \underbrace{j_* j^* F} \xrightarrow{i_* \phi \oplus \eta} \underbrace{i_* i^* j_* j^* F}$$

is exact. One can check exactness on stalks, so consider a geometric point  $\bar{x} \rightarrow X$ . If  $\text{im}(\bar{x}) \in U = X \setminus Z$  then our sequence becomes

$$0 \rightarrow F_{\bar{x}} \rightarrow 0 \oplus F_{\bar{x}} \rightarrow 0.$$

If  $\text{im}(\bar{x}) \in Z = X \setminus U$  then our sequence becomes

$$0 \rightarrow F_{\bar{x}} \rightarrow F_{\bar{x}} \oplus (j_* j^* F)_{\bar{x}} \rightarrow (j_* j^* F)_{\bar{x}}.$$

Hence, we have confirmed exactness, and  $F \xrightarrow{\sim} stF$ .

Now consider  $ts(F_1, F_2, \phi)$ . We have

$$\begin{aligned} i^*s(F_1, F_2, \phi) &= i^* \ker \left( i_*F_1 \oplus j_*F_2 \longrightarrow i_*i^*j_*F_2 \right) \\ &= \ker \left( i^*i_*F_1 \oplus i^*j_*F_2 \longrightarrow i^*i_*i^*j_*F_2 \right) \\ &= \ker \left( F_1 \oplus i^*j_*F_2 \longrightarrow i^*j_*F_2 \right) \\ &= F_1 \end{aligned}$$

One similarly checks that  $j^*s(F_1, F_2, \phi) \cong F_2$ , and that the canonical morphism  $i^*s(F_1, F_2, \phi) \rightarrow i^*j_*j^*s(F_1, F_2, \phi)$  is none-other-than  $\phi$ , under these identifications. Hence,  $ts(F_1, F_2, \phi) = (F_1, F_2, \phi)$ .  $\square$

**Theorem 17** (Milne Prop.II.3.14). *Its possible to define six functors*

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j^!} & \\ \text{Shv}_{\text{et}}(Z) & \xrightarrow{i_*} & \text{Shv}_{\text{et}}(X) & \xrightarrow{j^*} & \text{Shv}_{\text{et}}(U) \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

such that under the identification  $\text{Shv}_{\text{et}}(X) \cong T(X)$ , they correspond to:

$$\begin{array}{llll} F_1 & \leftarrow & (F_1, F_2, \phi) & \quad (0, F_2, 0) \leftarrow F_2 \\ F_1 & \mapsto & (F_1, 0, 0) & \quad (F_1, F_2, \phi) \mapsto F_2 \\ \ker(\phi) & \leftarrow & (F_1, F_2, \phi) & \quad (i^*j_*F_2, F_2, \text{id}) \leftarrow F_2 \end{array}$$

1. Each functor is left adjoint to the one below it.
2. The functors  $i^*, i_*, j^*, j^!$  preserve exact sequences;  $j_*, i^!$  preserve monomorphisms.
3. The composites  $i^*j^!, i^!j^!, i^!j_*, j^*i_*$  are zero.
4. The functors  $i_*, j_*$  are fully faithful.
5. The functors  $j_*, j^*, i^!, i_*$  map injective objects to injective objects.

**Remark 18.** Heuristically,  $j_*$  “fills in the gaps” over  $Z$  in a canonical way, and  $i^!$  isolates the part of  $F$  which cannot be recovered from  $F|_{\text{Et}(U)}$  by this “filling in the gaps” process.

**Exercise 15** (Not advanced). Prove Theorem 17 using what we have seen so far. Note that if a functor has a left adjoint preserving monomorphisms then it preserves injectives.

**Exercise 16** (Not advanced). In the situation of Theorem 17 show that there are short exact sequence

$$\begin{aligned} 0 &\rightarrow j^!j^* \rightarrow \text{id} \rightarrow i_*i^* \rightarrow 0 \\ 0 &\rightarrow i_*i^! \rightarrow \text{id} \rightarrow j_*j^* \rightarrow 0 \end{aligned}$$

**Remark 19.** Sometimes one defines  $j^* := j^!$  and  $i_! := i_*$  so that the short exact sequences can be written as

$$\begin{aligned} 0 \rightarrow j_!j^! &\rightarrow \text{id} \rightarrow i_*i^* \rightarrow 0 \\ 0 \rightarrow i_!i^! &\rightarrow \text{id} \rightarrow j_*j^* \rightarrow 0 \end{aligned}$$

## 5 Curves

**Example 20** (Milne, Exam.II.3.12). Let  $A$  be a discrete valuation ring (e.g.,  $\mathbb{C}[[z]], \mathbb{F}_p[[z]], \mathbb{Z}_p, \dots$ ). Let

1.  $K = \text{Frac}(A)$ ,
2.  $k = A/\mathfrak{m}$ ,
3.  $G_K = \text{Gal}(K^{sep}/K)$ ,  $\rightarrow$  sheaves on the open point
4.  $G_k = \text{Gal}(k^{sep}/k)$ ,  $\rightarrow$  closed point

$$\text{Spec}(A) = \{m, (0)\}$$

$\uparrow$        $\uparrow$   
 closed    open

Since  $A$  is a discrete valuation ring,  $X = \text{Spec}(A)$  has one open point, and one closed point. Let  $U = \text{Spec}(K), Z = \text{Spec}(k)$  be the corresponding open and closed subschemes. Recall that the category of étale sheaves over a field is equivalent to the category of discrete Galois modules. That is,  $\text{Shv}_{et}(Z) \cong G_k\text{-mod}$  and  $\text{Shv}_{et}(U) \cong G_K\text{-mod}$ . We can give an analogous description of  $\text{Shv}_{et}(X)$  using a similar construction to  $\underline{T}(X)$ . It suffices to work out what functor  $G_K\text{-mod} \rightarrow G_k\text{-mod}$  corresponds to  $i^*j_*: \text{Shv}_{et}(U) \rightarrow \text{Shv}_{et}(Z)$ .

Let  $A^h$  be the henselisation of  $A$ , and  $A^{sh}$  a strict henselisation. Since  $K^{sep}$  is separable closed, there are factorisations  $A \rightarrow A^h \rightarrow A^{sh} \rightarrow K^{sep}$  which are actually inclusions. The choice of  $A^{sh}$  and the inclusion define subgroups  $I = \text{Gal}(K^{sep}/\text{Frac}(A^{sh}))$  and  $D = \text{Gal}(K^{sep}/\text{Frac}(A^h))$ , with  $I \subseteq D \subseteq G_K$ , and it turns out that  $D/I$  is canonically isomorphic to  $\text{Gal}(k^{sep}/k)$  where we identify  $k^{sep} = A^{sh}/\mathfrak{m}_{A^{sh}}$ . Then we claim that the functor  $i^*j_*: \text{Shv}_{et}(U) \rightarrow \text{Shv}_{et}(Z)$  corresponds to the functor of  $I$ -invariants.

$$(-)^I: G_K\text{-mod} \rightarrow G_k\text{-mod}.$$

Hence, the category  $\text{Shv}_{et}$  is equivalent to the category of triples  $(M_1, M_2, \phi)$  where  $M_1 \in G_k\text{-mod}$ ,  $M_2 \in G_K\text{-mod}$ , and  $\phi: M_1 \rightarrow M_2$  is compatible with the actions of  $G_k \cong D/I$  and  $G_K$ .

**Example 21.** Example 20 can be generalised to any normal curve, see Milne Exer.II.3.16 for details.

sheaf on  $X$

E.g., smooth complex curve  $X$

- $\text{Gal}(k(X)^{sep}/k(X))\text{-mod } M_\eta$
- Abelian group  $A_x$  for each closed point  $x \in X$
- $M_\eta \rightarrow A$  compatible with the group action.

8

$\text{Gal}(k(X)^{sep}/k(X))$  is related to  $\pi_1(U, x)$  for  $U \subset X$  open