References.

1. Cisinski, Deglise "Local and stable homological algebra in Grothendieck abelian categories"

2. Weibel, "An introduction to homological algebra"

3. The Stacks Project (online)

1 Road Map

In this lecture we work with a Grothendieck abelian category $\mathscr{A}$. We have in mind the following examples:

1. $\mathscr{A} = R$-mod, the category of R-modules for some ring R (for example, $R = \mathbb{Z}$ or a field K),

2. $\mathscr{A} = \mathsf{Shv}(X)$ sheaves of abelian groups on a topological space,

3. $\mathscr{A} = \mathsf{Shv}_{et}(X)$, étale sheaves of abelian groups on a scheme X,

4. $\mathscr{A} = G$-mod, the category of discrete G-modules for some profinite group (such as $G = Gal(k^{sep}/k)$ for some field k).

We are interested in *left exact* functors. That is, functors $\Phi : \mathscr{A} \to \text{Ab}$ (to the category of abelian groups, say) which send short exact sequences

$$0 \to A \to B \to C \to 0$$

to left exact sequences

$$0 \to \Phi(A) \to \Phi(B) \to \Phi(C).$$

Examples of such functors are:

1. If $\mathscr{A} = R$-mod, the functors $F(M) = M \otimes_R N$ for some fixed $N \in R$-mod.

2. $\mathscr{A} = \mathsf{Shv}(X)$ the functor $\Phi(F) = F(X)$,

3. $\mathscr{A} = \mathsf{Shv}_{et}(X)$ the functor $\Phi(F) = F(X)$,

4. $\mathscr{A} = G$-mod, the functor $\Phi(M) = M^G = \{m \in M : gm = m \ \forall\ g \in G\}$.

Our problem is to understand what happens on the right side $\Phi(B) \to \Phi(C)$ of the left exact sequence $0 \to \Phi(A) \to \Phi(B) \to \Phi(C)$. In order to address this, we formally invert quasi-isomorphisms (this will be explained later, see Definitions 15 and 16) in the category of chain complexes. This produces the *derived* *categories* $D(\mathscr{A})$ and $D(\text{Ab})$. In general, there is no functor $D(\mathscr{A}) \to D(\text{Ab})$ making the square

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & D(\mathscr{A}) \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ \text{Ab} & \longrightarrow & D(\text{Ab}) \end{array}$$

commute. But there does exist a functor $R\Phi : D(\mathscr{A}) \to D(\text{Ab})$ which is a "best approximation" to Φ in a precise sense (see Definition 19). This functor has the wonderful property that: If

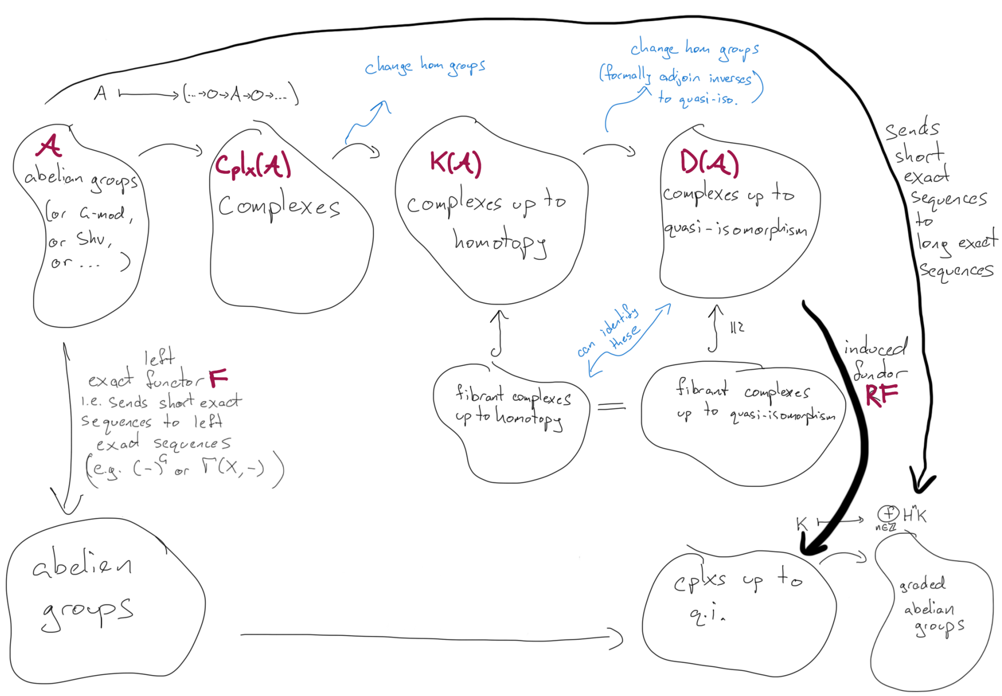
$$0 \to A \to B \to C \to 0$$

is a short exact sequence, then there is a long exact sequence

$$0 \to H^0 R\Phi(A) \to H^0 R\Phi(B) \to H^0 R\Phi(C) \to H^1 R\Phi(A) \to \dots .$$

The definition of $D(\mathcal{A})$ is quite abstract. Traditionally, the category $D(\mathcal{A})$ is accessed by identifying it with the category of fibrant complexes up to homotopy.

Here is the picture:



2 Chain complexes

Definition 1. *Let $\mathscr{A}$ be an abelian category. A* chain complex $C^\bullet$ *is a sequence of morphisms* $\cdots \xrightarrow{d} C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} \to \dots$ *such that* $d \circ d = 0$. *We define*

$$\text{the group of } n\text{-cycles as} \qquad Z^n = \ker(C^n \to C^{n+1}),$$

$$\text{the group of } n\text{-boundaries as} \qquad B^n = \operatorname{im}(C^{n-1} \to C^n), \text{ and}$$

$$\text{the } n\text{-th cohomology group as} \qquad H^n = Z^n / B^n.$$

A morphism *of chain complexes* $C^\bullet \to D^\bullet$ *is a sequence of morphisms* $f^n : C^n \to D^n$ *such that* $d \circ f^n = f^{n+1} \circ d$ *for all* n. *The category of chain complexes is denoted* $Ch(\mathscr{A})$.

If $C^n = 0$ *for* $n \ll 0$ *then* $C^\bullet$ *is said to be* bounded below. *The corresponding full subcategory is denoted* $Ch^+(A)$.

Exercise 1. Show that a morphism of chain complexes $C^\bullet \to D^\bullet$ induces a morphism on cohomology groups $H^n C \to H^n D$.

Definition 2. *A morphism of chain complexes* $f : C^\bullet \to D^\bullet$ *is called a* quasi-isomorphism *if the induced maps* $H^n f : H^n C \to H^n D$ *are isomorphisms* $\forall\, n$.

Exercise 2. Show that

$$\begin{array}{c} (\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \dots) \\ \downarrow \\ (\cdots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \dots) \end{array}$$

is a quasi-isomorphism.

Definition 3. *A* chain homotopy h *between two maps of chain complexes* $f, f' : C^\bullet \rightrightarrows D^\bullet$ *is a sequence of maps* $h^n : C^n \to D^{n-1}$ *such that*

$$d \circ h^n + h^{n-1} \circ d = f'^n - f^n.$$

If such an h *exists we say that* f *and* f' *are* homotopic *and write* $f \sim f'$. *If* $f \sim 0$ *we say that* f *is* null homotopic. *If we have two maps* $f : C^\bullet \rightleftarrows D^\bullet : g$ *such that* $g \circ f \sim \mathrm{id}_{C^\bullet}$ *and* $f \circ g \sim \mathrm{id}_{D^\bullet}$ *then we say that* f *(and* g*) is a* homotopy equivalence *and* $C^\bullet$ *and* $D^\bullet$ *are* homotopy equivalent.

Exercise 3. Show that if $f \sim g$ then $H^n f = H^n g$ for all n. Show that homotopy equivalences are quasi-isomorphisms. Show that the quasi-isomorphism from Exercise 2 is not a homotopy equivalence.

Exercise 4. Show that chain homotopy defines an equivalence relation on $\hom(C^\bullet, D^\bullet)$. Or in other words, show that the set of null homotopic morphisms is a subgroup of $\hom_{Ch(\mathscr{A})}(C^\bullet, D^\bullet)$.

Exercise 5. Show that homotopy is preserved by pre-composition, and post-composition. That is, if $f \sim f'$ then $f \circ g \sim f' \circ g$ and $g' \circ f \sim g' \circ f'$ for any morphisms $g : B^\bullet \to C^\bullet$ and $g' : D^\bullet \to E^\bullet$.

Definition 4 (Weibel Def.1.4.4)**.** *The homotopy category $K(\mathscr{A})$ of an abelian category $\mathscr{A}$, is the category whose objects are chain complexes, and*

$$\hom_{K(\mathscr{A})}(C^\bullet, D^\bullet) = \hom_{Ch(\mathscr{A})}(C^\bullet, D^\bullet)/\sim .$$

The full subcategory of $K(\mathscr{A})$ consisting of bounded below chain complexes is denoted $K^+(\mathscr{A})$.

3 Resolutions

Definition 5. *Recall that an object I of an abelian category $\mathscr{A}$ is* injective *if $\hom_{\mathscr{A}}(-, I)$ sends monomorphisms to surjections.*

Exercise 6 (Omitted from lecture)**.**

1. Show that I is injective if and only if $\hom(-, I)$ sends exact sequences to exact sequences.

2. Show that if I is an injective abelian group then I is divisible.[1]

3. Advanced: show that any divisible abelian group is injective.

4. Show that if I is an injective abelian group, X is a topological space, and $x \in X$ a point, then the skyscraper sheaf[2] $sk_x I$ at x with value I is an injective object of $\mathsf{Shv}(X)$.

5. Show that a product of injective objects is injective.

Exercise 7. Recall that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group.

1. Show that for every abelian group A and nonzero $a \in A$, there exists a morphism $\phi : A \to \mathbb{Q}/\mathbb{Z}$ that $\phi(a) \neq 0$. (Hint: consider the inclusion of the subgroup $\langle a \rangle \subseteq A$ generated by a; note that either $\langle a \rangle \cong \mathbb{Z}$ or $\langle a \rangle \cong \mathbb{Z}/n$ for some n).

2. Show that $A \to \prod_{\hom(A,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ is a monomorphism from A to an injective abelian group. Note that it is functorial in A.

3. Let X be a topological space and F a sheaf. Show that $F \to \prod_{x \in X} sk_x I_{F_x}$ is a monomorphism of sheaves to an injective sheaf, where F_x is the stalk of F at x, and $sk_x I_{F_x}$ is the skyscraper sheaf at x with value the injective abelian group $\prod_{\hom_{\mathrm{Ab}}(F_x,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$.

[1] That is, multiplication by any nonzero integer $n : I \to I; a \mapsto na$ is surjective.

[2] This is the sheaf which sends an open $U \subseteq X$ to I if $x \in U$ and zero otherwise. It is characterised by the property that $\hom_{\mathsf{Shv}(X)}(F, sk_x I) = \hom_{\mathrm{Ab}}(F_x, I)$.

Theorem 6 (Grothendieck 1957, 東北数学雑誌)**.** *If $\mathscr{A}$ is a Grothendieck abelian category, then $\mathscr{A}$* has enough injectives*. That is, for any object $C \in \mathscr{A}$, there is a monomorphism $C \hookrightarrow I$ with I injective.*

Exercise 8. Suppose that $C^\bullet$ is a chain complex, that $f^i : C^i \to I^i$ for $i \leq n$ is a sequence of morphisms with I^i injective, that $\cdots \xrightarrow{d} I^{n-1} \xrightarrow{d} I^n$ is a sequence of morphisms such that

1. $d \circ d = 0$,
2. $df^i = f^{i-1}d$ for $i \leq n$,
3. $H^iC \cong H^iI$ for $i < n$,
4. $C^{n-1} \to C^n \oplus I^{n-1} \to I^n$ is exact in the middle.[3]

Define D to be the cokernel of $C^n \oplus I^{n-1} \to C^{n+1} \oplus I^n; (c, i) \mapsto (dc, di - fc)$, and let $D \hookrightarrow I^{n+1}$ be an inclusion into an injective object. Show that the above four points are still satisfied after we add I^{n+1} to the above data with the canonical morphisms $I^n \to I^{n+1}$ and $C^{n+1} \to I^{n+1}$ (now with n replaced by $n+1$).

We deduce the following from the above exercise.

Corollary 7. *If $\mathscr{A}$ has enough injectives, then every bounded below complex is quasi-isomorphic to a complex of injectives.*

Definition 8. *A chain complex $Q^\bullet$ is called* fibrant *if for any monomorphic quasi-isomorphism $f : C^\bullet \to D^\bullet$ in $Ch(\mathscr{A})$, any map $C^\bullet \to Q^\bullet$ factors through $D^\bullet$.*

Lemma 9. *If $I^\bullet$ is a bounded below complex, then $I^\bullet$ is fibrant if and only if its a complex of injectives.*

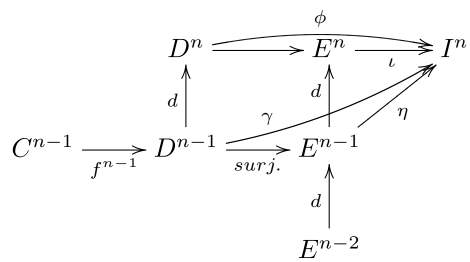
Proof. (Omitted from lecture). We will see in Exercise 9 that fibrant $\Rightarrow$ injectives, so we prove the converse.

Let $f : C^\bullet \to D^\bullet$ be a monomorphism which is a quasi-isomorphism, and suppose that $C^\bullet \to I^\bullet$ is a map. We will construct the factorisation by induction on I^n. It is easy to start the induction because $I^\bullet$ is bounded below. So suppose that we have a factorisation $C^i \xrightarrow{f^i} D^i \xrightarrow{g^i} I^i$ such that $dg^{i-1} = g^i d$ for all $i < n$ for some n. Since I^n is injective, and $C^n \to D^n$ injective, there exists a factorisation $C^n \to D^n \to I^n$, however the square

$$\begin{array}{ccc} D^n & \xrightarrow{g'} & I^n \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ D^{n-1} & \xrightarrow[g^{n-1}]{} & I^{n-1} \end{array}$$

[3] The first morphism does not have to be injective, and the second morphism does not have to be surjective.

may not commute. That is, $\gamma = dg^{n-1} - g'd$ may not be zero. Let $E^\bullet = \operatorname{coker}(C^\bullet \to D^\bullet)$ and note that since f is a quasi-isomorphism, $E^\bullet$ is an exact complex. Since $\gamma f^{n-1} = 0$, there is a factorisation $D^{n-1} \to E^{n-1} \xrightarrow{\eta} I^n$ of γ. Now since $E^\bullet$ is exact, I^n is injective, and $\eta d = 0$, there is a further factorisation $D^{n-1} \to E^{n-1} \xrightarrow{d} E^n \xrightarrow{\iota} I^n$. Composing ι with $D^n \to E^n$ gives a morphism $\phi : D^n \to I^n$. The diagram looks like this,



but we really only care that $\phi d = \gamma = dg^{n-1} - g'd$. Defining $g^n = g' + \phi$, one checks that $g^n d = dg^{n-1}$ as desired. □

Exercise 9. If $C^\bullet$ is fibrant, then $C^\bullet$ is a complex of injectives: Given an object M, let $D^n(M)$ denote the complex $(\cdots \to 0 \to M \xrightarrow{\text{id}} M \to 0 \to \dots)$ concentrated in degrees n and $n+1$.

1. Show that $\hom_{Ch(\mathscr{A})}(D^n(M), C^\bullet) = \hom_{\mathscr{A}}(M, C^n)$ for any complex $C^\bullet$.
2. Show that for any morphism $M \to M'$, the induced map $D^n(M) \to D^n(M')$ is a quasi-isomorphism.
3. Show that if $C^\bullet$ is fibrant, then each C^n is injective.

Remark 10. The converse to Exercise 9 is not necessarily true. While a bounded below complex is a complex of injectives if and only if it is fibrant, in general there are unbounded complexes of injectives which are not fibrant.

The real reason we are interested in fibrant objects is the following.

Proposition 11. *Suppose $Q^\bullet$ is fibrant, and $f : C^\bullet \to D^\bullet$ is a quasi-isomorphism. Then*

$$f^* : \hom_{K(\mathcal{A})}(D^\bullet, Q^\bullet) \xrightarrow{\sim} \hom_{K(\mathcal{A})}(C^\bullet, Q^\bullet).$$

Proof. We can assume that f is a monomorphism: Consider the canonical factorisation $C^\bullet \to Cyl(f) \to D^\bullet$ via the mapping cylinder.[4] The second

[4][Weibel 1.5.5] The *cyclinder* is the complex

$$Cyl(f)^n = D^n \oplus C^{n+1} \oplus C^n; \qquad d : (b, a', a) \mapsto (db - fa', -da', a' + da).$$

It is equipped with canonical morphisms

$$C^\bullet \to Cyl(f); \qquad a \mapsto (0, 0, a),$$
$$D^\bullet \to Cyl(f); \qquad b \mapsto (b, 0, 0),$$
$$Cyl(f) \to D^\bullet; \qquad (b, a', a) \mapsto b + f(a)$$

morphism is a split quasi-isomorphism. Therefore $C^\bullet \to Cyl(f)$ is also a quasi-isomorphism. So if the result is true for the two monomorphic quasi-isomorphisms $C^\bullet \to Cyl(f)$ and $D^\bullet \to Cyl(f)$, then it will be true for $C^\bullet \to D^\bullet$. (Even though the triangle doesn't commute).

So now f is a monomorphic quasi-isomorphism, so f^* is surjective by definition of fibrant, and it suffices to show surjectivity. Recall that two morphisms $g, g' : D^\bullet \to Q^\bullet$ are homotopic if and only if they factor through the cyclinder $g + g' : D^\bullet \oplus D^\bullet \to Cyl(\mathrm{id}_D) \to Q^\bullet$. If we have two morphisms which become homotopic after composition with f, then $gf + g'f$ factors through some $Cyl(\mathrm{id}_C) \to Q^\bullet$. This induces a morphism on the pushout $(D^\bullet \oplus D^\bullet) \amalg_{Cyl(\mathrm{id}_C)} Cyl(\mathrm{id}_D) \to Q^\bullet$. But the canonical morphism $(D^\bullet \oplus D^\bullet) \amalg_{Cyl(\mathrm{id}_C)} Cyl(\mathrm{id}_D) \to Cyl(\mathrm{id}_D)$ is a monomorphic quasi-isomorphism (because f is), hence, by the definition of "fibrant", we get our desired factorisation $Cyl(\mathrm{id}_D) \to Q^\bullet$, and therefore a homotopy $g \sim g'$. □

Corollary 12. *A morphism between fibrant complexes is a quasi-isomorphism if and only if it is a homotopy equivalence.*

Proof. Let $K(Fib) \subseteq K(\mathscr{A})$ denote the full subcategory of fibrant objects. The corollary follows directly from Proposition 11 by applying Yoneda's lemma to $K(Fib)$. □

It is a theorem that there are an ample supply of fibrant complexes.

Theorem 13 (Beke, cf.Cisinski, Deglise, Thm.2.1)**.** *Suppose $\mathscr{A}$ is a Grothendieck abelian category. Then there exists a functor $Q : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ and a natural transformation $\mathrm{id} \to Q$ such that for each complex $C^\bullet$, the complex $QC^\bullet$ is fibrant, and the morphism $C^\bullet \to QC^\bullet$ is a quasi-isomorphism.*

Remark 14. A proof of the case where $\mathscr{A}$ is the category of modules over a ring is in Hovey, "Model categories", see Theorem 2.3.13.

4 Localisation

Definition 15. *If C is a category and S is a class of morphisms, the* localisation *$C \to C[S^{-1}]$ is the universal functor which sends every element of S to an isomorphism. In other words, every functor $C \to D$ sending elements of S to isomorphisms factors in a unique*[5] *way through D.*

Exercise 10. If R is a ring, let C_R denote the category with one object $*$, and $\hom(*, *) = R$. Composition is given by the multiplication of R. If $S \subseteq R$ is a multiplicatively closed set, show that any functor $C_R \to D$ inverting elements of S factors through $C_{R[S^{-1}]}$.

[5]More concretely, let $Func^S(C, D)$ denote the category of those functors which send elements of S to isomorphisms. Then $C \to C[S^{-1}]$ is a localisation if $Func(C[S^{-1}], D) \to Func^S(C, D)$ is an equivalence of categories.

Definition 16. *The* derived category $D(\mathscr{A})$ *of an abelian category* $\mathscr{A}$ *is the localisation of the category of chain complexes* $Ch(\mathscr{A})[q.i.^{-1}]$ *at the class of quasi-isomorphisms.*

The cohomology *of a sheaf* F *is its image under the right derived global sections functor* $R\Gamma(X,-) : D(\mathsf{Shv}(X)) \to D(\mathrm{Ab})$.

Exercise 11 (Advanced)**.** For K a field, we will show that

$$D(K) \cong Gr(K)$$

where $D(K)$ is the derived category of K-vector spaces and $Gr(K)$ is the category of $\mathbb{Z}$-graded vector spaces. Using a decomposition $C^n = Z^n \oplus B^{n+1} \oplus H^n$ show that for any chain complex $C^\bullet$, there is a quasi-isomorphism $C^\bullet \overset{q.i.}{\to} \oplus_n H^n[-n]$ (where $H^n[-n] = (\dots \to 0 \to H^n \to 0 \to \dots)$ is the chain complex concentrated in degree n).

Let $Ch(K) \to Gr(K)$ be the functor $C^\bullet \mapsto \oplus_n H^n[-n]$. Show that this functor is a localisation of $Ch(K)$ at the class of quasi-isomorphisms.

Exercise 12. Using the fact that quotients of injective abelian groups are injective[6] we will show that for every complex $C^\bullet$ of abelian groups, there are quasi-isomorphisms

$$C^\bullet \overset{q.i.}{\to} K^\bullet \overset{q.i.}{\leftarrow} \oplus H^n[-n]$$

for some $K^\bullet$.

Let $\iota : C^n/B^n \hookrightarrow I$ be an immersion into an injective group, and define J as the cokernel of the canonical map $(C^n/B^n) \overset{(d,\iota)}{\longrightarrow} B^{n+1} \oplus I \twoheadrightarrow J$. Show that because $C^n \to B^{n+1}$ is surjective, the induced map $I \to J$ is surjective. Using the fact that J is again an injective abelian group, show that the canonical map $C^n \to J$ factors through C^{n+1}. Note that this produces a map of complexes $C^\bullet \to (\dots \to 0 \to I \to J \to 0 \to \dots)$. Show that this map induces an isomorphism on degree n cohomology, and that $(\dots \to 0 \to I \to J \to 0 \to \dots)$ is quasi-isomorphic to $H^n[-n]$. Deduce that there are quasi-isomorphisms $C^\bullet \to K^\bullet \leftarrow \oplus H^n[-n]$ for some appropriately chosen complex $K^\bullet$.

Remark 17. Even though every object of $D(\mathrm{Ab})$ is isomorphic to some $\oplus H^n[-n]$, we have $D(\mathrm{Ab}) \ncong Gr(\mathrm{Ab})$. In fact,

$$\hom_{D(\mathrm{Ab})}(\oplus H^n[-n], \oplus H'^n[-n])$$
$$\cong \left(\prod_n \hom_{\mathrm{Ab}}(H^n, H'^n)\right) \oplus \left(\prod_n \mathrm{Ext}^1(H^n, H'^{n-1})\right)$$

Theorem 18. *Let* $\mathscr{A}$ *be a Grothendieck abelian category. Then the derived category is equivalent to the subcategory of* $K(\mathscr{A})$ *whose objects are fibrant.*

$$D(\mathscr{A}) \cong K(Fib_{\mathscr{A}}).$$

[6] This is because an abelian group is injective if and only if its divisible. This is because $\mathbb{Z}$ is a principle ideal domain, and this fact does not generalise to more general abelian categories.

If we are working with bounded below chain complexes, we can instead use complexes of injectives.

$$D^+(\mathscr{A}) \cong K^+(Inj_{\mathscr{A}}).$$

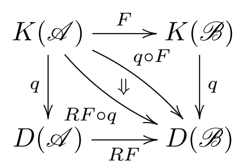
Proof. This follows from Lemma 9, Corollary 12, Theorem 13 ☐

5 Derived functors and cohomology

Definition 19. *Suppose that $F : \mathscr{A} \to \mathscr{B}$ is a left exact*[7] *functor between abelian categories. Its* right derived functor *is the universal functor*

$$RF : D(\mathscr{A}) \to D(\mathscr{B})$$

equipped with a natural transformation $qF \Rightarrow RFq$



Proposition 20. *If $\mathscr{A}$ is a category with enough injectives, then the composition*

$$D^+(\mathscr{A}) \cong K^+(Inj_{\mathscr{A}}) \to K^+(\mathscr{A}) \xrightarrow{F} K^+(\mathscr{B}) \to D^+(\mathscr{B})$$

is the derived functor of F. If $\mathscr{A}$ is a Grothendieck abelian category, and F has a left adjoint $E : \mathscr{B} \to \mathscr{A}$ which preserves monomorphisms and monomorphic quasi-isomorphisms, then the composition

$$D(\mathscr{A}) \cong K(Fib_{\mathscr{A}}) \to K(\mathscr{A}) \xrightarrow{F} K(\mathscr{B}) \to D(\mathscr{B})$$

is the derived functor of F.

Remark 21. It is quite common that all criteria of Proposition 20 are fulfilled.

Exercise 13 (Omitted from lecture.)**.** Consider the functor $F = \hom(\mathbb{Z}/2, -) : \mathrm{Ab} \to \mathrm{Ab}$ which sends a group to its subgroup of 2-torsion elements. Using the short exact sequence $0 \to (\frac{1}{2}\mathbb{Z})/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{2} \mathbb{Q}/\mathbb{Z} \to 0$ and the isomorphism $(\frac{1}{2}\mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}/2$, show that $RF(\mathbb{Z}/2) = (\cdots \to 0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0 \to \dots)$. (Note that $\mathbb{Q}/\mathbb{Z}$ is injective).

Definition 22. *Suppose that X is a small category equipped with a Grothendieck topology and admitting a terminal object X (for example, $Op(X)$ for a topological space X). Let $\Gamma(X, -) : \mathsf{Shv}(C) \to \mathrm{Ab}$ denote the global sections functor $F \mapsto F(X)$.*

[7] Left exact means it preserves kernels and products

The cohomology *of a sheaf of abelian groups* $F \in \mathsf{Shv}(C)$ *is the image under the right derived functor of* $\Gamma(-, F)$*.*

$$R\Gamma(X, F) \in D(\mathrm{Ab}).$$

The cohomology groups of F *are the cohomology groups of this complex*

$$H^n(X, F) = H^n(R\Gamma(X, F)).$$

Exercise 14. Show that if $0 \to F \to I^0 \to I^1 \to \dots$ is any exact complex of sheaves with each I^i injective, then

$$H^n(X, F) = \frac{\ker(I^n(X) \to I^{n+1}(X))}{\mathrm{im}(I^{n-1}(X) \to I^n(X))}$$

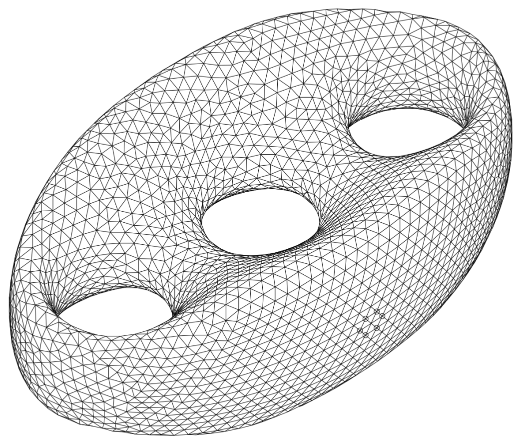
A Chain complexes as homotopy types

There is a canonical functor $Sing : \mathrm{Top} \to Ch(\mathrm{Ab})$ from the category Top of topological spaces to the category $Ch(\mathrm{Ab})$ of chain complexes of abelian groups (we will describe it below). In fact, originally, the whole purpose of considering chain complexes at all was to study topological spaces. As such, chain complexes in some ways behave a lot like topological spaces and it is often useful to consider them as a kind of topological space.

We do not want to give a lecture in algebraic topology, so we just give an idea about $Sing$.

0. $Sing_0(X)$ is the free abelian group generated by points of X.
1. $Sing_1(X)$ is the free abelian group generated by the "closed segments" of X.
2. $Sing_2(X)$ is the free abelian group generated by the "triangles" in X.
3. $Sing_3(X)$ is the free abelian group generated by the "tetrahedra" in X.
4. ...

1. The boundary map $d : Sing_1(X) \to Sing_0(X)$ takes a segment, to the difference of its end points.
2. The boundary map $d : Sing_2(X) \to Sing_1(X)$ takes a triangle, to the alternating sum of its edges.
3. The boundary map $d : Sing_3(X) \to Sing_2(X)$ takes a tetrahedron, to the alternating sum of its faces.
4. ...



Here, one should picture a smooth manifold equipped with a triangulation, however in order to make the construction functorial, we consider *all* triangules allowing folds, and overlaps, hence the name "singular".

Chain complex notions about such as cone, cyclinder of a morphism, homotopy, etc are the algebraic versions of such operations on topological spaces. Via the *Sing* functor the group of cycles are "hollow subspaces" and the group of boundaries are "boundaries of solid subspaces".

If we pick a randomly chosen triangulation of the a smooth manifold, it might not be fine enough to capture all topological features. The injective resolutions / fibrant replacements we used above are an algebraic version of taking an infinitely fine subtriangulation.

B (Pre)Triangulated categories

Triangulated categories are a formalism which isolate key properties of $D(\mathscr{A})$ (and $K(\mathscr{A})$), Using these properties, many things can be proven without having to touch chain complexes (any more).

Warning 23. The "triangle" in "triangulated category" is completely unrelated to the "triangle" in "manifold with triangulation". The topological version of the "triangle" in "triangulated category" is homotopy cofibre sequence, cf.Exercise 16 below. "Triangulated category" is something like a category equipped with "long exact sequences".

Definition 24 (Weibel Def.10.2.1)**.** *A* pretriangulated category *is an additive category T equipped with an autoequivalence* $[1] : T \to T$ *and a collection of* triangles*, i.e., diagrams of the form* $A \to B \to C \to A[1]$*, called* distinguished triangles *satisfying the following:*

TR1 (a) *The class of distinguished triangles is closed under isomorphism.*

(b) *For every object A, the triangle* $A \stackrel{\text{id}}{\to} A \to 0 \to A[1]$ *is distinguished.*

(c) *For every morphism $f : A \to B$ there exists a distinguished triangle $A \to B \to C \to A[1]$ containing f.*

TR2 If $A \to B \to C \to A[1]$ is a distinguished triangle, then so are

$$B \to C \to A[1] \to B[1] \text{ and } C[-1] \to A \to B \to C.$$

*TR3 For any two distinguished triangles, and morphisms f, g there exists a (*usually not unique!*) morphism h making the diagram commute:*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

Definition 25. *The* shift $C^\bullet[1]$ *of a chain complex* $C^\bullet$ *is the chain complex*

$$C^\bullet[1]^n = C^{n+1}, \qquad d_{C^\bullet[1]} = -d_{C^\bullet}.$$

Definition 26 (Weibel 1.5.1)**.** *Let $f : A^\bullet \to B^\bullet$ be a morphism of chain complexes. The* cone *of f is the complex*

$$Cone(f)^n = B^n \oplus A^{n+1}; \qquad d : (b, a) \mapsto (db - fa, -da).$$

Exercise 15. Show that the differentials defined above satisfy the condition $d \circ d = 0$.

Exercise 16. Show that there is a canonical morphism of complexes $Cone(f) \to A^\bullet[1]$, and that this induces a long exact sequence

$$\cdots \to H^n A \to H^n B \to H^n Cone(f) \to H^{n+1} A \to \ldots.$$

(Hint: it suffices to check that the map $H^{n+1}A \to H^{n+1}B$ defined by the Snake Lemma applied to the short exact sequence of complexes $0 \to B \to Cone(f) \to A[1] \to 0$ is the same as the map induced by f).

Proposition 27. *Let $\mathscr{A}$ be an abelian category. Define a triangle in $K(\mathscr{A})$ to be distinguished if it is isomorphic (in $K(\mathscr{A})$) to a triangle of the form $A \xrightarrow{f} B \to Cone(f) \to A[1]$. Then $K(\mathscr{A})$ equipped with $[1]$ and this class of triangles is a pretriangulated category.*

Remark 28. The category $K(\mathscr{A})$ also satisfies the octohedral axiom, making it a triangulated category, but we will not discuss this.

Exercise 17 (Advanced)**.** Prove Proposition 27.