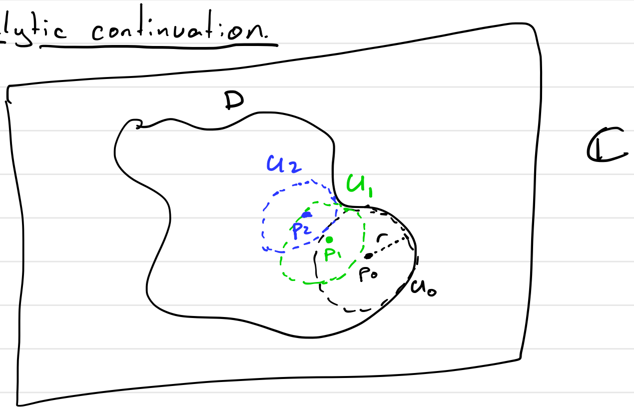
Analytic continuation.



$f : D \to \mathbb{C}$ analytic

On U_0, $\underbrace{f(z) = \sum_{i=0}^{\infty} a_i (z-p_0)^i}$ radius of convergence r

$\to$ not defined out side U_0

On U_1, $\underbrace{f(z) = \sum_{i=0}^{\infty} b_i (z-p_1)^i}$ $\to$ ~~~~~~~~ U_1

etc.

So an analytic function on D is

1) a collection of open balls U_i s.t. $\bigcup_{i \in I} U_i = D$

2) a convergent power series f_i on U_i

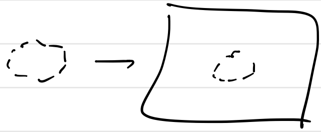
such that

3) $\forall\ i, j$. $f_i = f_j$ on $U_i \cap U_j$.

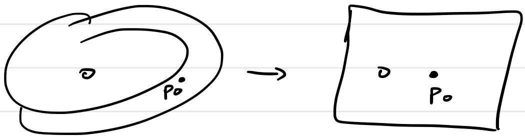
$\mapsto$ analytic functions form a <u>sheaf</u>

In algebraic geometry we don't have the open balls U:

Replace open balls



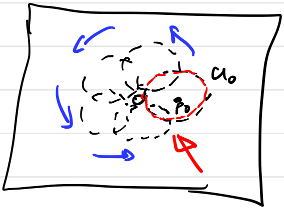
with étale morphisms



(analytically, a local homeomorphism)

$\mathbb{C} \setminus \{0\}$ $\qquad f = z^{\frac{1}{2}}$

analytically continue



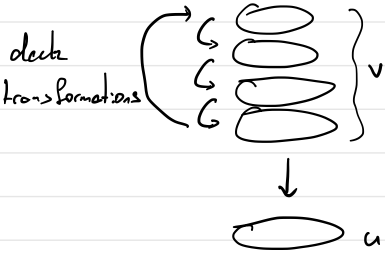
on U_0, there are two power series $g(z)$ s.t. $(g(z))^2 = z$

- Choose one.
- analytically continue around $0 \in \mathbb{C}$

$\mapsto$ the new power series on U_0 is $-g(z)$.

$\mapsto$ $\boxed{\text{Sheaves detect topological information.}}$

Replace open sets with local homeomorphisms.



$$\{\text{Functions on } U\} = \left\{\begin{array}{c}\text{deck transformation}\\ \text{invariant functions}\\ \text{on } V\end{array}\right\}$$

i.e., $\hom(U, \mathbb{C}) = \hom(V, \mathbb{C})^G$

$G = \{\text{deck transformation group}\}$

$U \subseteq \mathbb{C}$

Stalks: $F(U) = \{\text{analytic functions } U \to \mathbb{C}\}$

$p \in \mathbb{C}$, $\quad F_p := \varinjlim_{p \in U} F(U)$

$$\cong \left\{ \sum_{i=0}^{\infty} a_i (z-p)^i \right\} = \mathbb{C}[[z-p]]$$

$p \in U \qquad F(U) \xrightarrow{\exists} F_p$

$$f \longmapsto f(z) = \sum_{i=0}^{\infty} a_i (z-p)^i \in \mathbb{C}[[z-p]]$$

~~Open sets~~ $\rightsquigarrow$ etale morphisms (local homeomorphisms)

$$\varinjlim F(U)$$

$$\begin{tikzcd} & q \in U \arrow[d, mapsto] \arrow[d, shift left=6, "\text{etale}"] \\ \mathrm{Spec}(k(p)^{sep}) \arrow[r] \arrow[ur] & p \in X \end{tikzcd}$$

Note: $\varinjlim k(q) = k(p)^{sep}$

Valuative criterion for properness

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "a"] \arrow[d, "c"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "b"'] \arrow[ur, dashed, "d"] & X \end{tikzcd}$$

f is proper iff $\forall$ a, b, c exists a unique d

where R val. ring

$K = \operatorname{Frac} R$.

(separated $\iff$ uniqueness)

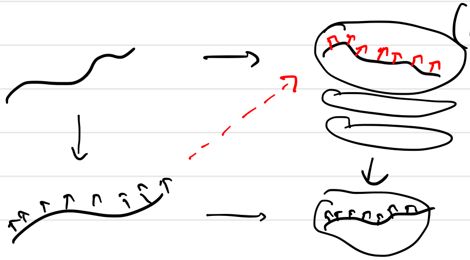
Formal criterion for étaleness

Suppose f is finite presentation.

f is étale iff $\forall$ ring A, ideal $I \subset A$ s.t. $I^2 = 0$, and $\forall$ a, b, c exists a unique d.

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r, "a"] \arrow[d, "c"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "b"'] \arrow[ur, dashed, "d"] & X \end{tikzcd}$$

(unramified $\iff$ uniqueness)



Corollary. $Y' \xrightarrow{g} Y \xrightarrow{f} X$ finite presentation

f unramified, fg étale $\Rightarrow$ g étale

f separated, fg proper $\Rightarrow$ g proper.