In Galois theory I we defined the étale fundamental group $\pi_1^{\mathsf{et}}(X)$ of a connected scheme X, and saw an equivalence between R-local systems on X and R-representations of $\pi_1^{\mathsf{et}}(X)$, where $R = \mathbb{Z}/n$ for some n. In this lecture we discuss the pro-étale story.

1 Galois theory I: Review

Recall that the main theorem of classical Galois theory is: for any Galois[1] field extension L/k, there is an (inclusion reversing) isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \text{subextensions} \\ L/L'/k \end{array} \right\} \cong \left\{ \begin{array}{c} \text{subgroups} \\ H \subseteq \mathrm{Aut}(L/k) \end{array} \right\}.$$

Taking the limit over all Galois extensions turns this into an isomorphism

$$\left\{ \begin{array}{c} \text{finite subextensions} \\ k^{sep}/L'/k \end{array} \right\} \cong \left\{ \begin{array}{c} \text{finite index subgroups} \\ H \subseteq \mathrm{Aut}(k^{sep}/k) \end{array} \right\}$$

where finite index means the set of cosets $\mathrm{Aut}(k^{sep}/k)/H$ is finite.

Exercise 1. Suppose that G is a group, acting on a set S. Show that if the action is transitive, then there exists a (not necessarily unique) subgroup $H \subseteq G$ such that there is an isomorphism of G-sets $S \cong G/H$ where G acts on the set $G/H = \{gH : g \in G\}$ of cosets of H by multiplication on the left. (Hint: choose an element $s \in S$ and consider its stabiliser).

Using the above exercise, and the fact that every étale k-algebra is a product of finite separable field extensions, the above isomorphism of partially ordered sets becomes an equivalence of categories

$$\mathrm{FEt}_k \cong \mathrm{Aut}(k^{sep}/k)\text{-FinSet}$$

between finite étale k-algebras and finite sets equipped with a continuous action of $\mathrm{Aut}(k^{sep}/k)$. More generally, for a connected scheme X with geometric point $\overline{x} \to X$, we considered the functor $F : \mathrm{FEt}_X \to \mathrm{Set}$ sending a finite étale X-scheme Y to the set of points $F_{\overline{x}}(Y) = |\overline{x} \times_X Y|$. We defined

$$\pi_1^{\mathsf{et}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

and obtained the equivalence of categories

$$\mathrm{FEt}_X \cong \mathrm{Aut}(F_{\overline{x}})\text{-FinSet}.$$

[1] A field extension L/k is Galois if any of the following equivalent conditions are satisfied:

1. L/k is finite separable and normal.
2. $[L : k] = \mathrm{Aut}(L/k)$.
3. Every k-morphism $L \to k^{sep}$ has the same image.

On the other hand, there is a very similar equivalence associated to "nice"[2] topological spaces[3]

$$\left\{ \begin{array}{c} \text{finite covering spaces} \\ Y \to X \end{array} \right\} \cong \pi_1(X)\text{-FinSet}$$

between finite covering spaces[4] and finite sets with an action of the classical topological fundamental group. We saw these situations are axiomitised in the notion of a *Galois category*. A Galois category is a pair (C, F) consisting of a "nice"[5] category C, and a "nice"[6] functor $F : C \to \text{Set}$. The main theorem about Galois categories is that F induces an equivalence

$$C \cong \text{Aut}(F)\text{-FinSet}.$$

Finally, we saw linear versions of the above equivalences, where G-sets are replaced by G-modules.

Definition 1. *An R-*local system *of rank n is a sheaf of R-modules F such that for some covering $\{U_i \to X\}_{i \in I}$ there are isomorphisms $F|_{U_i} \cong R^n$ to the constant sheaf R^n. We write $Loc_X(R)$ for the category of R-local systems.*

Proposition 2. *If X is a connected scheme, and $R = \mathbb{Z}/\ell$ for some prime ℓ, there is an equivalence of categories*

$$Loc_X(R) \cong \left\{ \begin{array}{c} \textit{continuous finite dimensional} \\ R\textit{-linear representations of } \pi_1^{\mathsf{et}}(X) \end{array} \right\}$$

We would like this result for R a characteristic zero field, for example $R = \mathbb{Q}_l$ (the representation theory of characteristic zero fields is easier than positive characteristic, for example). However, as usual, getting to this field involves awkward limits. The pro-étale case on the other hand is better behaved. However, the theory of Galois categories must be generalised to allow the larger, more interesting category $X_{\mathsf{proét}}$.

[2] I.e., connected and locally simply connected.

[3] Indeed, this, and strong connection between étale morphisms of schemes and local homeomorphisms of topological spaces is the motivation for the notation π_1^{et}.

[4] $Y \to X$ is a finite covering space if for all $x \in X$ there is an open neighbourhood $x \in U$ such that $f^{-1}(U) \cong \amalg_{i=1}^n U$ for some n.

[5]

1. C has all finite limits and finite colimits.
2. Every object of X is a finite coproduct of connected objects.

[6]

3. $F(Y)$ is finite for all $Y \in X$.
4. (a) F preserves all finite limits and finite colimits.
   (b) A morphism f in C is an isomorphism if and only if $F(f)$ is an isomorphism.

2 Noohi groups

One of the consequences of the classical theory of Galois categories is that for any profinite group G, there is a canonical isomorphism of profinite groups $G \cong \mathrm{Aut}(F)$ where $F : G\text{-FinSet} \to \mathrm{Set}$ is the functor forgetting the action. This is not true more generally, but many groups do still satisfy this. These are called Noohi groups in [BS].

Definition 3. *Let S be a set, and consider the set $\mathrm{Aut}(S)$ of automorphisms of S. For any finite subset $S' \subseteq S$ and morphism $\tau : S' \to S$ we define $U(s, \tau) = \{\phi \in \mathrm{Aut}(S) : \phi|_{S'} = \tau\}$. Then $\mathrm{Aut}(S)$ is given the topology generated by $U(S', \tau)$.*

Exercise 2.

1. Show that if S is a finite set, then the topology defined above on $\mathrm{Aut}(S)$ is the discrete topology.
2. Show that for any finite family $\{U(S'_i, \tau_i)\}_{i=1}^n$ the intersection $\bigcap_{i=1}^n U(S'_i, \tau_i)$ is either empty, or of the form $U(\cup_{i=1}^n S'_i, \tau')$ for some τ'. Deduce that every open subset of $\mathrm{Aut}(S)$ is of the form $\bigcup_{i \in I} U(S_i, \tau_i)$ for some (possibly infinite, possibly empty) collection $\{U(S_i, \tau_i)\}_{i \in I}$.
3. Show that if S is not finite, then $\mathrm{Aut}(S)$ is not the discrete topology.

Definition 4. *Suppose that C is a small category, and $F, G : C \to Set$ two functors. Then the set $\hom(F, G)$ of natural transformations is canonically a subset of $\prod_{S \in Ob(C)} \hom(F(S), G(S))$, where the product is over all objects of C. We equip $\mathrm{Aut}(F)$ with the topology induced from the product topology on $\prod_{S \in Ob(C)} \mathrm{Aut}(F(S))$, where $\mathrm{Aut}(F(S))$ are given the topology from Def.3.*

Exercise 3. Suppose that C is a small category and $F : C \to \mathrm{FinSet}$ a functor taking values in finite sets. Show that $\mathrm{Aut}(F)$ is a profinite set.

Recall that a *topological group* is a topological space G equipped with a point $e \in G$ and continuous morphisms $m : G \times G \to G$, $i : G \to G$ satisfying the axioms of a group.

Exercise 4. Let S be any set. Show that $\mathrm{Aut}(S)$ is a topological group. That is, show that the morphisms of composition $\mathrm{Aut}(S) \times \mathrm{Aut}(S) \to \mathrm{Aut}(S)$, the inclusion of the identity $\{\mathrm{id}\} \to \mathrm{Aut}(S)$, and inverse $\mathrm{Aut}(S) \to \mathrm{Aut}(S); \phi \mapsto \phi^{-1}$ are all continuous for the topology on $\mathrm{Aut}(S)$ defined above.

Definition 5. *Suppose that G is a topological group G. Let G-Set be the category of discrete sets equipped with a continuous G-action, and let $F_G : G\text{-}Set \to Set$ be the forgetful functor. We say that G is a* Noohi group *if the natural map induces an isomorphism $G \cong \mathrm{Aut}(F_G)$ of topological groups, where the topology of $\mathrm{Aut}(F_G)$ is induced by the product topology, cf. Definition 4.*

Remark 6. In [BS, Def.7.1.1] the compact-open topology is used, but for discrete topological spaces X, Y, the compact-open topology on $\hom(X, Y)$ agrees with the product topology on $\prod_X Y$, so our definition is the same.

Many groups that we are interested in are Noohi groups.

Example 7. The following are Noohi groups.

1. $\mathrm{Aut}(S)$ for any set S, [Exam.7.1.2].
2. Any profinite group, [Exam.7.1.6].
3. The group $G(E)$ for any local field E (such as $\mathbb{Q}_l$) and any finite type E-group scheme (such as GL_n), [Exam.7.1.6].
4. Any (not necessarily finite) discrete group, [Exam.7.1.6].
5. Any topology group G which admits an open subgroup U such that U is a Noohi group, [Lem.7.1.8].

3 Infinite Galois theory

In the étale case, a central rôle is played by Galois categories. Here we consider their infinite generalisation.

Definition 8 ([Def.7.2.1, 7.2.3])**.** *An* infinite Galois category *is a pair* $(C, C \xrightarrow{F} Set)$ *satsifying:*

1. *C is a category admitting (all small) colimits and finite limits.*
2. *Each $X \in C$ is a (possibly infinite) disjoint union of connected objects.*
3. *C is generated under colimits by a set of connected objects.*
4. (a) *F commutes with colimits and finite limits.*
   (b) *A morphism f is an isomorphism if and only if $F(f)$ is an isomorphism.*

The fundamental group *of (C, F) is the topological group $\pi_1(C, F) = \mathrm{Aut}(F)$ (topologised as above, cf. Def.4). An infinite Galois category is* tame *if for any connected $X \in C$, the action of $\pi_1(C, F)$ on $F(X)$ is transitive.*

Remark 9. Bhatt-Scholze also ask that F is faithful but this is automatic, cf. Exercise 5.

Exercise 5. We will show that F is automatically faithful. Suppose that $f, g : X \rightrightarrows Y$ are two morphisms such that $F(f) = F(g)$. Using property (4) in Def.8 above, show that $f = g$. Hint: Consider the equaliser of f and g.

Remark 10. Lets note the differences between a Galois category and an infinite Galois category:

1. Galois categories only have finite colimits.
2. Each $X \in C$ in a Galois category is a *finite* disjoint union of connected objects.
3. Instead of Axiom 3 above, the fibre functor F of a Galois category is required to take values in finite sets.

Theorem 11 ([Thm.7.2.5])**.** *There is an adjunction*

$$\begin{aligned} \{ \text{ Noohi groups } \} &\rightleftarrows \{\text{infinite Galois categories}\}^{op} \\ G &\mapsto G\text{-Set} \\ \pi_1(C,F) &\leftarrow\!\shortmid (C,F) \end{aligned}$$

and $C \cong \pi_1(C,F)$-Set for any tame (C,F). In particular, π_1 is fully faithful when restricted to tame infinite Galois categories.

4 Locally constant sheaves

Fix a scheme X which is connected and locally topologically noetherian. *That is, for every point $x \in X$ there is an open neighbourhood $x \in U \subseteq X$ such that U is topologically noetherian. Topologically noetherian means that for every decreasing family of closed subsets $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \dots$ there is some N such that $Z_n = Z_{n+1}$ for all $n \geq N$.*

Definition 12 ([7.3.1])**.** *We say that $F \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ is* locally constant *if there exists a cover $\{X_i \to X\}$ in $X_{\mathsf{pro\acute{e}t}}$ with $F|_{X_i}$ isomorphic to a constant sheaf on $(X_i)_{\mathsf{pro\acute{e}t}}$. We write Loc_X for the category of locally constant sheaves.*

Locally constant sheaves are particularly nice and have a number of characterisations (when X is locally topologically noetherian).

Definition 13. *We say that a morphism $Y \to X$ satisfies the* valuative criterion for properness *if for every valuation ring R, and every commutative square*

$$\begin{array}{ccc} \operatorname{Spec}(Frac\ R) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & X \end{array}$$

there exists *a* unique *diagonal morphism making the diagram commutative.*

Proposition 14 ([Lem.7.3.9])**.** *Let $F \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$. The following are equivalent.*

1. *F is locally constant.*
2. *There is an X-scheme $Y \to X$, locally etale (on Y) satisfying the valuative criterion for properness such that $F \cong \hom_X(-, Y)$.*

5 Fundamental groups

Let X be locally topological noetherian and connected, and $\overline{x} \to X$ is a geometric point. Write $\mathrm{ev}_x : \mathrm{Loc}_X \to \mathrm{Set}$ for the functor $F \mapsto F_x$.

Lemma 15 ([Lem.7.4.1])**.** *The pair (Loc_X, ev_x) is a tame infinite Galois category.*

Definition 16 ([Def.7.4.2])**.** *The* pro-étale fundamental group *is*

$$\pi_1^{\text{proét}}(X, \overline{x}) = \mathrm{Aut}(\mathrm{ev}_{\overline{x}}).$$

Lemma 17 ([Lem.7.4.5])**.** *Under the equivalence*

$$Loc_X \cong \pi_1^{\text{proét}}(X, \overline{x})\text{-}Set,$$

The full subcategory $Loc_{X_{\text{et}}} \subseteq Loc_X$ corresponds to the the full subcategory of those $S \in \pi_1^{\text{proét}}(X, \overline{x})$-Set where an open subgroup acts trivially.

Lemma 18 ([Lem.7.4.7])**.** *There is an equivalence of categories*

$$Loc_X(\mathbb{Q}_\ell) \cong Rep_{\mathbb{Q}_l, cont}(\pi_1^{\text{proét}}(X, \overline{x})).$$

Definition 19. *A local ring A is* geometrically unibranch *if A^{sh} has a unique minimal prime ideal (equivalently,* $\mathrm{Spec}(A^{sh})$ *has a unique irreducible component). A scheme X is geometrically unibranch if $\mathcal{O}_{X,x}$ is geometrically unibranch for all $x \in X$.*

Exercise 6. Show that $\mathrm{Spec}(k[x, y]/(xy))$ is not geometrically unibranch.

Lemma 20 ([Lem.7.4.10])**.** *If X is geometrically unibranch, then $\pi_1^{\text{proét}}(X, \overline{x}) \cong \pi_1^{\text{et}}(X, \overline{x})$.*

Example 21. $Y = \mathbb{P}^1/\{0 = \infty\}$. $\pi_1^{\text{et}}(Y) = \widehat{\mathbb{Z}}$, $\pi_1^{\text{proét}}(Y) = \mathbb{Z}$.