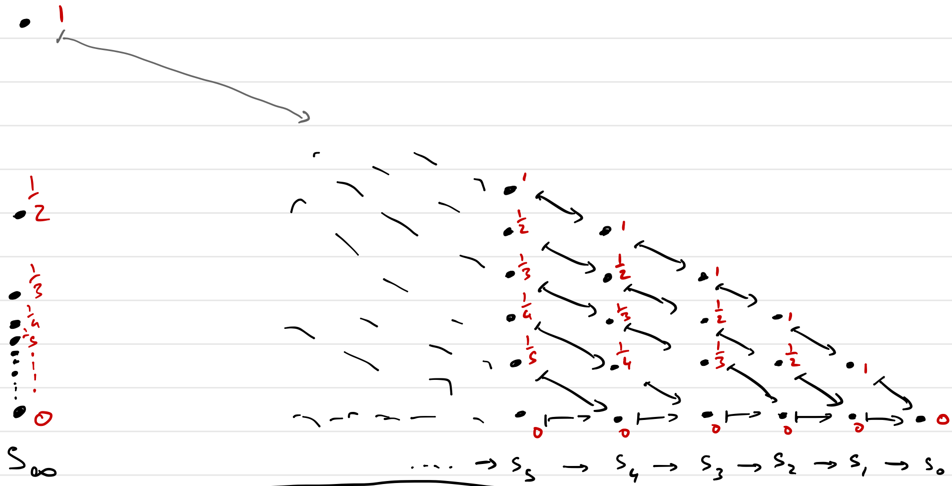

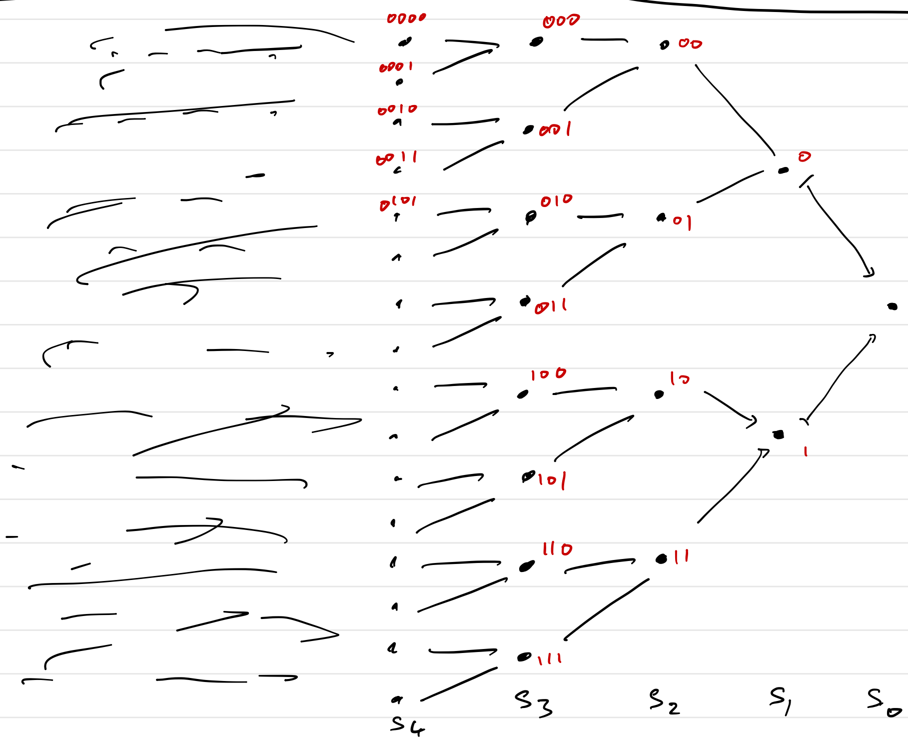

Cantor $\subset \mathbb{R}$

$$\mathrm{Spec}(\mathbb{C})_{\text{proét}} \cong \mathrm{ProFinSet}.$$

$$X \longmapsto X_{top}$$

$$\mathrm{Spec}\Big(\varinjlim \prod_{S_\lambda} \mathbb{C}\Big) \longleftarrow S = \varprojlim_{\lambda \in \Lambda} S_\lambda$$

Grothendieck topology on this category.

A family $\mathcal{Y} = \{Y_i \to X\}_{i \in I}$ is a covering family if $\amalg Y_i \to X$ is surjective, and I is finite

(or $\mathcal{Y}$ is refineable by a jointly surjective finite family).] can ignore this

So if S is an infinite profinite set, $\{\{s\} \to S\}_{s \in S}$ is $\boxed{\text{not}}$ a covering family.

$\leadsto$ category of sheaves Shv(ProFinSet)

A sheaf is:
1) For every profinite set S, a set (or abelian gp, or ...) $F(S)$
2) For every morphism of profinite sets $T \xrightarrow{f} S$ a morphism $F(S) \xrightarrow{F(f)} F(T)$ such that
   i) $F(\mathrm{id}) = \mathrm{id}$
   ii) $F(f \circ g) = F(g) \circ F(f)$

contravariant functor (i.e., presheaf)

3) For every covering family $\{Y_i \to X\}$,
$$F(X) = \mathrm{eq}\Big(\prod F(Y_i) \rightrightarrows \prod F(Y_i \times_X Y_j)\Big).$$

Special case of 3: $X = Y_0 \amalg \dots \amalg Y_n$ then (3) says $F(X) = F(Y_0) \times \dots \times F(Y_n)$

⚠ only true for finite coproducts in general. Infinite disjoint unions don't exist in ProFinSet (because they're not compact).

Example If X is a topological space,
$$S \longmapsto \hom_{\text{continuous}}(S, X)$$
profinite set

is a sheaf in $\mathrm{Shv}(\mathrm{ProFinSet})$

Remark If $S = \{0\} \amalg \{\frac{1}{n} \mid n \in \mathbb{N}\}$ then
$$\hom_{\text{cont.}}(S, X) = \left\{ \begin{array}{c} \text{convergent sequences} \\ (x_0, x_1, x_2, \dots) \text{ in } X \end{array} \right\}$$
S has the limit topology (in this case, its the same as the topology coming from the embedding $S \subseteq \mathbb{R}$)

For more general profinite sets S, a continuous $S \to X$ is like a "generalised convergent sequence".

For a general sheaf F, can think of $F(S)$ as "S-indexed convergent sequences in $F(*)$"
$*$ = one point profinite set.

$$s \in S \rightsquigarrow \{*\} \xrightarrow{s} S$$
$$\mapsto F(S) \to F(*)$$
$$x \longmapsto \text{"}x_s\text{"} \quad s\text{th term of the "sequence"}$$

$(x_0, x_1, x_2, \dots)$ converges to $x_\infty \in X$ if every open neighborhood $U \ni x_\infty$ (1) contains almost all x_i.
(2) contains $\{x_i \mid i > N\}$ for some N.

[(1) $\Leftrightarrow$ (2)]

Equivalently, $(x_0, x_1, \dots)$ converges to $x_\infty \in X$ if $\{0\} \amalg \{\frac{1}{n} \mid n \in \mathbb{N}\} \to X$
$$\tfrac{1}{n} \mapsto x_n$$
is continuous. $(\frac{1}{\infty} = 0, \ \frac{1}{0} = \infty)$

$\boxed{\text{In Sets}}$

$$\begin{tikzcd}
(\dots, y_{i+1}, y_i, \dots, y_0) \arrow[d, mapsto] & & & \exists y_{i+1} & y_i \arrow[r, mapsto] & y_{i-1} \arrow[r, mapsto] & \dots & \exists y_0 \\
\varprojlim F_i \arrow[r] \arrow[d] & \dots \arrow[r] & \dots \arrow[r] & F_{i+1} \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d] & \dots & \arrow[r] & F_0 \arrow[d] \\
\varprojlim G_i \arrow[r] & \dots \arrow[r] & \dots \arrow[r] & G_{i+1} \arrow[r] & G_i \arrow[r] & \dots & \arrow[r] & G_0 \\
(\dots, x_{i+1}, x_i, \dots, x_0) & & \dots \arrow[r, mapsto] & x_{i+1} \arrow[r, mapsto] & x_i \arrow[r, mapsto] & \dots & \dots \arrow[r, mapsto] & x_0
\end{tikzcd}$$

$G_{i+1} \times_{G_i} F_i \ni (x_{i+1}, y_i)$

$(x_{i+1}, y_i) \in G_{i+1} \times_{G_i} F_i$

$$\dots \xrightarrow{t} F_2 \xrightarrow{t} F_1 \xrightarrow{t} F_0$$

t are surjective

$$\varprojlim F_i := \{(\dots, x_2, x_1, x_0) \mid \underbrace{t x_{i+1} = x_i}_{0 = t x_{i+1} - x_i}\}$$

$$\varprojlim F_i = \ker\Big(\prod_{\mathbb{N}} F_i \xrightarrow{t - \mathrm{Id}} \prod_{\mathbb{N}} F_i\Big)$$

$$(\dots, x_2, x_1, x_0) \longmapsto (\dots, t x_3 - x_2, t x_2 - x_1, t x_1 - x_0)$$

$$0 \to \varprojlim F_i \to \prod F_i \xrightarrow{t - \mathrm{Id}} \prod F_i \to 0 \overset{?}{=} \varprojlim{}^1 F_i$$

$\uparrow$

If zero then

$$\varprojlim F_i \cong \mathrm{Cone}\Big(\prod F_i \xrightarrow{t - \mathrm{Id}} \prod F_i\Big)[-1]$$

In general, $\mathrm{Cone}(\prod F_i \to \prod F_i)$ contains both $\varprojlim$ and $\varprojlim{}^1$

$R\lim$

$K^\bullet \xrightarrow{f^\bullet} L^\bullet$ morphism of chain complexes.

$$\begin{array}{lccc} \mathrm{Cone}(f)^n := & K^{n+1} \oplus L^n & & (dk, d\ell - k) \\ & \uparrow & & \uparrow \\ & K^n \oplus L^{n-1} & & (k, \ell) \end{array}$$

$$d := \begin{bmatrix} d_K & f \\ & d_L \end{bmatrix}$$

$\rightsquigarrow$ short exact sequence

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0$$

$\rightsquigarrow$ long exact sequence

$$\cdots \to H^n(K^\bullet) \xrightarrow{f} H^n(L^\bullet) \to H^n(\mathrm{Cone}) \to H^{n+1}(K^\bullet) \xrightarrow{f} H^{n+1}(L^\bullet) \to \cdots$$

<u>Example</u>

$$\begin{aligned} K^\bullet &= (\cdots \to 0 \to 0 \to A \to 0 \to \cdots) \\ L^\bullet &= (\cdots \to 0 \to 0 \to B \to 0 \to \cdots) \\ \mathrm{Cone} &= (\cdots \to 0 \to A \to B \to 0 \to \cdots) \end{aligned}$$

$$\begin{array}{ccccccccccccc} \cdots \to H^{-1}(L) & \to & H^{-1}(\mathrm{Cone}) & \to & H^0(K) & \to & H^0(L) & \to & H^0(\mathrm{Cone}) & \to & H^1(K) & \to & \cdots \\ \cong & & \cong & & \cong & & \cong & & \cong & & \cong & & \\ \cdots \to 0 & \to & \ker f & \to & A & \to & B & \to & \mathrm{coker}(f) & \to & 0 & \to & \end{array}$$

Cone = combination of kernel and cokernel.