All fields are of characteristic $\neq 2$ (just to make some statements nicer).

The reference for most of these notes is Dugger's "Notes on the Milnor Conjecture". We use Totaro's, "Milnor K-theory is the simplest part of algebraic K-theory" in the section on higher Chow groups.

1 The Witt ring

Definition 1. *A* quadratic space *over a field F is a pair (V,q) consisting of a finite dimensional vector space V, and a function $q : V \to F$ such that*

1. *for all $v \in V, a \in F$, we have $q(av) = a^2 q(v)$, and*
2. *$\beta_q(v,u) = q(u+v) - q(u) - q(v)$ is bilinear.*

A morphism *of quadratic spaces $(V,q) \to (V',q')$ is a linear morphism $\phi : V \to V'$ such that $q' \circ \phi = q$.*

Exercise 1.

1. Show that for any matrix $A \in M_{n\times n}(F)$, the map $F^n \to F$ defined by $q_A : v \mapsto v^t \cdot A \cdot v$ makes (F^n, q_A) a quadratic space.
2. Let (V,q) be a quadratic space. Show that (V,q) is isomorphic to a space of the form (F^n, q_A) for some n and some matrix $A \in M_{n\times n}(F)$. Hint: choose a basis $e_1,\ldots,e_n$ for V and consider the elements $\beta_q(e_i,e_j)$.

An isomorphism of quadratic spaces is called an *isometry*. If (V,q) is isometric to (V',q') we write $(V,q) \sim (V',q')$.

Question: Classify quadratic spaces over a field F up to isometry.

Proposition 2. *Suppose that (V,q) is a quadratic space. Then there exists a basis $e_1,\ldots,e_n$, and $a_1,\ldots,a_n \in F$ such that*

$$q(x_1e_1 + \ldots x_ne_1) = a_1x_1^2 + \cdots + a_nx_n^2.$$

Proof. Note that equivalently, there is a basis such that $\beta_q(e_i,e_j) = 0$ for all $i \neq j$. The proof is by induction on the dimension. It is true in dimension zero, and also if $q = 0$. So suppose there is $v \in V$ such that $q(v) \neq 0$. Consider the vector space $W = \{w : \beta_q(v,w) = 0\}$. For an arbitrary $u \in V$, we can write $u = \frac{\beta_q(u,v)}{\beta_q(v,v)}v + w$ where $w = u - \frac{\beta_q(u,v)}{\beta_q(v,v)}v$. Then one sees that $\beta_q(w,v) = 0$. In other words, $V = W \oplus \langle v\rangle$. Since $\dim W < \dim V$, by induction there is a basis $e_1,\ldots,e_{n-1}$ satisfying $\beta_q(e_i,e_j) = 0$ for all $i \neq j$. Then by construction, $e_1, dots, e_{n-1}, v$ is a basis of V satisfying the requirement. □

Definition 3. *The quadratic space in the above theorem will be denoted*

$$\langle a_1,\ldots,a_n\rangle = (F^n, a_1x_1^2 + \cdots + a_nx_n^2)$$

Exercise 2. 1. Show that for any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, we have $\langle a_1, \dots, a_n \rangle \sim \langle a_{\sigma 1}, \dots, a_{\sigma n} \rangle$.

2. Suppose that $b_1, \dots, b_n \in F$ are squares, i.e., there exist $c_1, \dots, c_n \in F$ with $c_i^2 = b_i$. Show that $\langle a_1, \dots, a_n \rangle \sim \langle b_1 a_1, \dots, b_n a_n \rangle$.

3. Deduce that if every element of F is a square (e.g., F is algebraically closed, e.g., $F = \mathbb{C}$) then every quadratic space is isometric to one of the form $\langle 1, \dots, 1, 0, \dots, 0 \rangle$.

4. Deduce that if $F = \mathbb{R}$, then every quadratic space is isometric to one of the form $\langle 1, \dots, 1, -1, \dots, -1, 0 \dots, 0 \rangle$.

Definition 4. *A quadratic space (V, q) is* degenerate *if there is a nonzero $v \in V$ such that $\beta(v, w) = 0$ for all $w \in V$. It is* anisotropic *if there is no subspace isometric to $\langle 1, -1 \rangle$.*

Theorem 5 (Witt)**.** *Every quadratic space is isometric to one of the form*

$$\langle b_1, \dots, b_i, 1, -1, \dots, 1, -1, 0, \dots, 0 \rangle$$

with $\langle b_1, \dots, b_i \rangle$ anisotropic. Moreover, $\langle b_1, \dots, b_i \rangle$ is uniquely determined up to isometry. That is,

$$\langle b_1, \dots, b_i, 1, -1, \dots, 1, -1, 0, \dots, 0 \rangle \sim \langle b_1, \dots, b_i, 1, -1, \dots, 1, -1, 0, \dots, 0 \rangle$$

if and only if

$$\langle b_1, \dots, b_i \rangle \sim \langle b_1', \dots, b_i' \rangle$$

Definition 6. *The sum of two quadratic spaces (V, q) and (V', q') is the space $(V, q) + (V', q') = (V \oplus V', q + q')$. The product is $(V, q) \cdot (V', q') = (V \otimes_F V', q \otimes_F q')$.*

In the multiplication we mean $V \otimes_F V' \stackrel{q \otimes q'}{\to} F \otimes_F F \cong F$.

Exercise 3. Let $\langle a_1, \dots, a_n \rangle$, $\langle b_1, \dots, b_m \rangle$ be quadratic spaces. Describe their sum and product in the form $\langle c_1, \dots, c_N \rangle$.

Using the fact than any isometry class of quadratic spaces can be written in the form $\langle a_1 \rangle + \dots + \langle a_n \rangle$ show that we have

$$\Big((V, q) \oplus (V', q') \Big) \otimes (V'', q'') \sim \Big((V, q) \otimes (V'', q'') \Big) \oplus \Big((V', q') \otimes (V'', q'') \Big)$$

Definition 7. *The* Witt ring *is*

$$W(F) = \frac{\left\{ \begin{array}{c} \textit{equivalence classes of nondegenerate} \\ \textit{quadratic spaces} \end{array} \right\}}{\langle a_1, \dots, a_n \rangle = \langle a_1, \dots, a_n, 1, -1 \rangle}$$

Remark 8. By Witt's theorem, every element of W is represented by a unique isometry class of anisotropic quadratic spaces. The problem is that addition and multiplication of anisotropic quadratic spaces are not necessarily anisotropic anymore (e.g., $\langle 1\rangle + \langle -1\rangle = \langle 1,-1\rangle$, and $\langle 1,a\rangle \otimes \langle 1,-a\rangle = \langle 1,a,-a,-a^2\rangle \sim \langle 1,-1\rangle \oplus \langle a,-a\rangle$, but its possible that neither $\langle 1,a\rangle$ or $\langle 1,-a\rangle$ are isometric to $\langle 1,-1\rangle$. For example, if $a=2$ in $\mathbb{F}_5$ then $\mathrm{disc}\langle 1,\pm 2\rangle = \mp 2 \not\equiv 1 = \mathrm{disc}\langle 1,-1\rangle$ because 2 and -2 are not squares in $\mathbb{F}_5$).

Exercise 4. Show that in $W(F)$ we have $-\langle a_1,\dots,a_n\rangle = \langle -a_1,\dots,-a_n\rangle$. Hint: consider multiplication by $\langle 1,-1\rangle$.

Exercise 5. Recall that over $\mathbb{R}$, every nondegenerate quadratic space is isometric to one of the form $\langle 1,\dots,1,-1,\dots,-1\rangle$. Show that $W(\mathbb{R}) \cong \mathbb{Z}$.

Exercise 6.

1. Show that the map

$$\dim : W(F) \to \mathbb{Z}/2; \qquad (V,q) \mapsto \dim V$$

is a well-defined group homomorphism.

2. Show that the map

$$\mathrm{disc} : W(F) \to F^*/(F^*)^2; \qquad \langle a_1,\dots,a_n\rangle \mapsto (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} a_i$$

restricted to the kernel $I(F) = \ker(W(F) \stackrel{\dim}{\to} \mathbb{Z}/2)$ of dim (i.e., elements such that n is even) is a group homomorphism with addition on the left and multiplication on the right.

3. Show that if $\langle a_1,\dots,a_n\rangle$ and $\langle b_1,\dots,b_m\rangle$ are in $I(F)$ (i.e., n and m are both even), then $\mathrm{disc}(\langle a_1,\dots,a_n\rangle \otimes \langle b_1,\dots,b_m\rangle)$ is in $(F^*)^2$, i.e., it is a square.

Definition 9. *The kernel $I(F) = \ker(W(F) \stackrel{\dim}{\to} \mathbb{Z}/2)$ of the dimension homomorphism is called the* augmentation ideal. *The* graded Witt ring *is the associated graded ring of $W(F)$ with respect to the ideal I.*

$$Gr_I W(F) = W/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots.$$

Remark 10. The augmentation ideal $I(F)$ is the subset $W(F)$ of classes that are represented by even dimension quadratic spaces.

Exercise 7. Show that $Gr_I W(\mathbb{R}) \cong \mathbb{Z}/2[t]$ (the polynomial ring in one variable over $\mathbb{Z}/2$.

Proposition 11. *The maps* dim *and* disc *induce isomorphisms*

$$\dim : W/I \cong \mathbb{Z}/2,$$
$$\mathrm{disc} : I/I^2 \cong F^*/(F^*)^2$$

Moreover, [Merkurjez 1981], there is an isomorphism

$$C : I^2/I^3 \cong {}_2Br(F)$$

with target the 2-torsion elements in the Brauer group.

2 Etale cohomology

Recall, the following two results from étale cohomology.

> **Theorem. (Hilbert 90)** $H^1_{et}(\mathrm{Spec}(F), \mathbb{G}_m) \cong 0$.
>
> **Proposition.** $H^2_{et}(\mathrm{Spec}(F), \mathbb{G}_m) \cong Br(F)$.

Putting these two isomorphisms into the long exact sequence associated to the short exact sequence of étale sheaves

$$0 \to \mathbb{Z}/2 \to \mathbb{G}_m \overset{(-)^2}{\to} \mathbb{G}_m \to 0$$

we get

$$\begin{aligned}
0 \to & \underbrace{H^0_{et}(F, \mathbb{Z}/2)}_{\cong \mathbb{Z}/2} \to \underbrace{H^0_{et}(F, \mathbb{G}_m)}_{\cong F^*} \overset{2}{\to} \underbrace{H^0_{et}(F, \mathbb{G}_m)}_{\cong F^*} \to \\
\to & \underbrace{H^1_{et}(F, \mathbb{Z}/2)}_{\cong F^*/(F^*)^2} \to \underbrace{H^1_{et}(F, \mathbb{G}_m)}_{\cong 0} \to \underbrace{H^1_{et}(F, \mathbb{G}_m)}_{\cong 0} \to \\
\to & \underbrace{H^2_{et}(F, \mathbb{Z}/2)}_{\cong {}_2Br(F)} \to \underbrace{H^2(F, \mathbb{G}_m)}_{\cong Br(F)} \overset{2}{\to} \underbrace{H^2(F, \mathbb{G}_m)}_{\cong Br(F)} \to
\end{aligned}$$

That is, our invariants take the form

$$\begin{aligned}
e_0 :& W/I \cong H^0_{et}(F, \mathbb{Z}/2) \\
e_1 :& I/I^2 \cong H^1_{et}(F, \mathbb{Z}/2) \\
e_2 :& I^2/I^3 \cong H^2_{et}(F, \mathbb{Z}/2)
\end{aligned}$$

Question 12. Does this sequence of isomorphisms continue?

3 Milnor K-theory

Exercise 8. Note that the discriminant $\mathrm{disc} = e_1 : I/I^2 \overset{\sim}{\to} F^*/(F^*)^2$ has inverse $\nu : a \mapsto \nu(a) = \langle a, 1\rangle$. For $a \in F^*$ show that if $a \neq 1$ then

$$\nu(a)\nu(1-a) = \Big(\langle a\rangle - \langle 1\rangle\Big) \cdot \Big(\langle 1-a\rangle - \langle 1\rangle\Big) = 0$$

Hint: show that for $a, b \in F^*$ we have $\langle a, b\rangle \sim \langle ab(a+b), a+b\rangle$.

Proposition 13 (Tate, Milne)**.** *For $a \in F^*, a \neq 1$ we have $\eta(a) \cup \eta(1-a) = 0$ in $H^2_{et}(F, \mathbb{Z}/2)$.*

So, the canonical ring homomorphisms

$$\bigoplus_{n\in\mathbb{N}}(F^*)^{\otimes n} \to Gr_I W(F)$$

$$\bigoplus_{n\in\mathbb{N}}(F^*)^{\otimes n} \to H^*(G,\mathbb{Z}/2)$$

both factor through:

Definition 14. *The Milnor K-theory ring*

$$K_*^M(F) = \frac{\bigoplus_{n\in\mathbb{N}}(F^*)^{\otimes n}}{a\otimes(1-a) : a\neq 1}$$

and we have seen that

Theorem 15. *The two graded ring homomorphisms*

$$Gr_I W(F) \stackrel{\nu}{\leftarrow} K_*^M(F)/2 \stackrel{\eta}{\to} H^*(G,\mathbb{Z}/2)$$

are isomorphisms in degrees 0, 1, 2.

Theorem 16 (Orlov-Vishik-Voevodsky 1996, Morel 1999)**.** *For all fields F, the map ν is an isomorphism.*

Theorem 17 (Voevodsky 1996)**.** *For all fields F, the map η is an isomorphism.*

Example 18. Since we have $H^n_{et}(\mathbb{F}_q,\mathbb{Z}/2) = 0$ for $n > 1$, we see that any quadratic space over $\mathbb{F}_q$ is completely determined by its dimension and discriminant. Consequently, there are exactly four classes of anisotropic quadratic spaces over $\mathbb{F}_q$:

$$\langle\rangle, \qquad \langle 1\rangle, \qquad \langle\omega\rangle, \qquad \langle 1,-\omega\rangle$$

where $\omega\in\mathbb{F}_q^*$ is any nonsquare, e.g., a generator.

4 Higher Chow groups

The reference for this section is Totaro's, "Milnor K-theory is the simplest part of algebraic K-theory".

One of the most important ingredients in Voevodsky's proof of the Milnor conjecture was translating it into an isomorphism between motivic cohomology and étale cohomology. An earlier candidate for motivic cohomology was Bloch's higher Chow groups. Their definition is quite concrete, and Totaro gave an elementary proof that they specialise to Minor K-theory.

Let X be a quasi-projective variety over a field F. One defines $c^j(X,n)$ to be the free abelian group generated by closed subvarieties $Z\subseteq X\times\mathbb{A}^n$ of codimension j such that $Z\cap X\times\mathbb{A}^{n-i}\times\{\epsilon\}\times\mathbb{A}^i$ is a union of varieties of codimension $j+1$ for all $i=1,\ldots,n$ and $\epsilon=0,1$.

$- - - - - - - - - -$ draw picture $- - - - - - - - - -$

For example, $c^{\dim X+n}(X,n)$ is freely generated by the closed points of $X\times\mathbb{A}^n$ which do not lie on any of the "faces" (since the only variety of dimension -1 is $\varnothing$).

For each $i = 1,\ldots,n$ and $\epsilon = 0,1$, there are maps

$$\partial_i^\epsilon : c^j(X,n) \to c^j(X,n-1)$$

induced by intersection with the "faces" $X \times \mathbb{A}^{n-i} \times \{\epsilon\} \times \mathbb{A}^i$. We define

$$d = \sum_{i=1}^{n} (-1)^i (\partial_i^1 - \partial_i^0)$$

and obtain a complex

$$\cdots \to c^j(X,2) \to c^j(X,1) \to c^j(X,0) \to 0.$$

Inside each $c^j(X,n)$ is a canonical subgroup $d^j(X,n)$ of those cycles which are pulled back from cycles on $c^j(X,n-1)$ along a projection $X\times\mathbb{A}^n \to X\times\mathbb{A}^{n-1}$.

Definition 19. *Bloch's higher Chow groups are the homology groups of the complex*

$$CH^i(X,n) = c^i(X,*)/d^i(X,*).$$

Note that $CH^1(F,1)$ is generated by the points of $\mathbb{A}^1 \setminus \{0,1\}$. There is a canonical morphism $F^* \to CH^1(F,1)$ sending

$$a \in F^* \mapsto \begin{cases} 0 & a = 1 \\ [1/(1-a)] & a \neq 1 \end{cases}$$

Theorem 20 (Totaro)**.** *The morphism defined above induces an isomorphism*

$$K_n^M(F) \xrightarrow{\sim} CH^n(F,n).$$

The point is that now we have a theory defined for all varieties, not just fields. Moreover it is equipped with a lot of extra structure, for example, there are long exact sequences

$$\cdots \to CH^n(Z,i) \to CH^{n+c}(X,i) \to CH^{n+c}(X\setminus Z,i) \to CH^n(Z,i+1) \to$$

where $Z \subset X$ is a closed subvariety of codimension c.

5 Further remarks - Voevodsky's proof of the Milnor conjecture

1. (Reduction to *char* = 0). Using transfer maps, and discrete valuation rings, it is sufficient to prove that η is an isomorphism for all fields of charactersitic zero.

2. (Conversion to motivic cohomology). Instead of proving that η is an isomorphism, Voevodsky showed that it is isomorphic to the case $i = j$ of a "realisation map" between motivic cohomology and étale motivic cohomology

$$H^i_{\mathcal{M}}(F, \mathbb{Z}/2(j)) \to H^i_{\mathcal{M},et}(F, \mathbb{Z}/2(j))$$

   These motivic cohomology groups are the Zariski / étale cohomology groups of explicit complexes built using cycles (formal sums of irreducible closed subvarieties) in $\mathbb{A}^i \times (\mathbb{P}^1)^j$.

   This is useful because these groups have more structure. They are defined for all varieties, not just fields, they are bigraded (so there are more of them), there are various long exact sequences, etc.

3. (Hilbert 90). In this reformulation, it becomes sufficient to show that a Hilbert 90 type vanishing:

$$H^{n+1}_{\mathcal{M},et}(F, \mathbb{Z}_{(2)}(n)) = 0 \qquad \text{(H90(n))}$$

   for all characteristic zero fields, and all n.

4. (Splitting varieties). The statement H90(n) can be proven by induction on n. The statement H90(n) is true if the following holds:

   (a) H90(n -1) is true for all characteristic zero fields.

   (b) For every field F and tuple $\underline{a} = (a_1, \ldots, a_n)$ with $a_i \in F^*$ there is some smooth irreducible F-variety $X_{\underline{a}}$ such that

      i. $a_1 \otimes \cdots \otimes a_n$ vanishes in the function field $F(X_{\underline{a}})$ of $X_{\underline{a}}$. That is, it is killed by the map

$$K^M_n\big(F\big)/2 \to K^M_n\big(F(X_{\underline{a}})\big)/2$$

      (i.e., $X_{\underline{a}}$ is a splitting variety for $\underline{a}$).

      ii. the variety $X_{\underline{a}}$ becomes rational (i.e., birational to a projective space) over its function field $F(X_{\underline{a}})$.

      iii. A certain motivic cohomology group of the Čech simplicial scheme $\check{C}(X_{\underline{a}})$ of $X_{\underline{a}}$ vanishes: $H^{n+1}_{\mathcal{M}}(\check{C}(X_{\underline{a}}), \mathbb{Z}_{(2)}(n)) = 0$.

   Remarks:

   - The second two hypotheses imply that

$$H^{n+1}_{\mathcal{M},et}\big(F, \mathbb{Z}_{(2)}(n)\big) \to H^{n+1}_{\mathcal{M},et}\big(F(X_{\underline{a}}), \mathbb{Z}_{(2)}(n)\big)$$

     is injective.

   - By $\check{C}(X)$ we mean the simplicial scheme $X \leftleftarrows X \times X \Lleftarrow X \times X \times X \cdots$ . . . . There is a generalisation of motivic cohomology to simplicial schemes.

5. (Pfister quadrics). For $\underline{a}$ as above consider the symmetric bilinear form associated to $\langle 1, -a_1 \rangle \cdot \langle 1, -a_2 \rangle \cdot \dots \cdot \langle 1, -a_{n-1} \rangle + \langle a_n \rangle$. Then consider the variety $Q_{\underline{a}}$ in $\mathbb{P}_F^{2^n}$ defined by the quadratic form associated to this symmetric bilinear form. The theory of Pfister quadratic forms says that this is a splitting variety for $\underline{a}$. So now it suffices to show that condition (4(b)iii) is satisfied for this $Q_{\underline{a}}$. To begin with, one can show that

$$H_{\mathcal{M}}^{2^n-1}(\check{C}(Q_{\underline{a}}), \mathbb{Z}_{(2)}(2^{n-1})) = 0.$$

6. (Cohomology operations). The last piece is to show the following. Let $n \geq 2$ be an integer and F a field of characteristic 0. Then there exists a natural transformation

$$\Theta^n : H_{\mathcal{M}}^{n+1}(-, \mathbb{Z}/2(n))) \to H_{\mathcal{M}}^{2^n-1}(-, \mathbb{Z}/2(2^{n-1}))$$

on the category of simplicial smooth k-varieties which preserves the mod 2 reduction of integral classes, and such that H90(n - 1) implies that for all $\underline{a}$ the morphism $\Theta^n_{\check{C}(Q_{\underline{a}})}$ is an monomorphism.

This is where a careful study of the motivic Steenrod algebra (i.e., the algebra of certain nice natural transformations as above) is used.