In this lecture we assume that $k = \mathbb{C}$ is the field of complex numbers. Then to any smooth variety X is associated the complex manifold X_{an}, whose topology is induced by the usual topology on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We will work exclusively with the sheaf cohomology on X_{an} in this lecture

$$H^n(X, \Lambda) := H^n_{\mathsf{Betti}}(X_{an}, \Lambda).$$

If $\Lambda = \mathbb{C}$, then we have the canonical isomophism $H^n_{\mathsf{Betti}}(X_{an}, \mathbb{C}) \cong H^n_{\mathsf{dR}}(X_{an}, \mathbb{C})$, and (when X is projective) the Hodge decomposition on $H^n_{\mathsf{dR}}(X_{an}, \mathbb{C})$, leading to a decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{r+s=n} H^{r,s}(X).$$

1 The cycle map

Proposition 1. *Let X be a smooth, quasi-projective, irreducible variety defined over $\mathbb{C}$. Let $0 \leq p \leq \dim X$.Then there exists a homomorphism, the cycle map*

$$\mathcal{Z}^p(X) \to H^{2p}(X, \mathbb{Z})$$

which factors through $CH^p(X)$. If X is projective, the image lies in $\mathrm{Hdg}^p(X) = j^{-1}H^{p,p}(X)$, the preimage of $H^{p,p}(X)$:

$$\begin{array}{ccccc} H^{2p}(X,\mathbb{Z}) & \xrightarrow{j} & H^{2p}(X,\mathbb{C}) & \cong & \bigoplus_{r+s=2p} H^{r,s}(X) \\ \cup\uparrow & & \uparrow\cup & & \\ \mathrm{Hdg}^p(X) & \longrightarrow & H^{p,p}(X,\mathbb{C}) & & \end{array}$$

Remark 2. We shall only construct the map for $\mathcal{Z}^p(X)$ itself. The factorization exists because $\mathcal{Z}_{rat}(\cdot) \subseteq \mathcal{Z}_{hom}(\cdot)$ by our definition of a Weil cohomology theory as one which factors through the category of motives $\mathcal{M}$.

Outline of the construction of the $\gamma_{\mathbb{Z}}$. (For more details see [Voisin, 11.1.2]). Let $Z \subset X$ be a closed irreducible subvariety of (algebraic) codimension p. We want to define $\gamma(Z) \in H^{2p}(X, \mathbb{Z})$. Recall the Thom isomorphism theorem: If $Z \subseteq X$ is a smooth closed subvariety of codimension p, then

$$T : H^j(X, U; \mathbb{Z}) \xrightarrow{\cong} H^{j-2p}(Z; \mathbb{Z}).$$

In particular, we deduce from this that

$$H^j(X, \mathbb{Z}) \cong H^j(U, \mathbb{Z}); \qquad j < 2p.$$

Even if Z is not smooth, there exists a sequence of closed subvarieties

$$\varnothing = Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = Z$$

such that $Z_i \setminus Z_{i+1}$ is smooth. Then by induction, we discover that

$$H^{2p}(X, \mathbb{Z}) \cong H^{2p}(X \setminus Z_1, \mathbb{Z})$$

Since we want the cycle map to be functorial,

$$\begin{array}{ccc} \mathcal{Z}^p(X) & \longrightarrow & \mathcal{Z}^p(X \setminus Z_1) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ H^{2p}(X,\mathbb{Z}) & \xrightarrow[\cong]{} & H^{2p}(X \setminus Z_1,\mathbb{Z}) \end{array}$$

to define $\gamma(Z) \in H^{2p}(X,\mathbb{Z})$, it suffices to define $\gamma(Z \setminus Z_1) \in H^{2p}(X \setminus Z_1,\mathbb{Z})$. In other words, we can assume that Z is smooth. Now we use the following exact sequence with $U = X - Z$ and define $\gamma(Z) := \rho \circ T^{-1}(1_{\mathbb{Z}})$

$$1 \mapsto \gamma(Z)$$
$$\cdots \to H^{2p-1}(U;\mathbb{Z}) \to \underbrace{H^{2p}(X,U;\mathbb{Z})}_{\cong H^0(Z;\mathbb{Z}) = \mathbb{Z}} \xrightarrow{\rho} H^{2p}(X;\mathbb{Z}) \to H^{2p}(U,\mathbb{Z}) \to \cdots$$

□

Remark 3. If the variety X is defined over an algebraically closed field k but otherwise of arbitrary characteristic we have essentially the same construction (see [Milne, p.268]) working with $H_{et}(X,\mathbb{Z}_\ell)$, where $\ell \neq \mathrm{char}(k)$, and using instead of the above sequence the sequence

$$\cdots \to H^{2p}_Z(X,\mathbb{Z}_\ell) \to H^{2p}(X,\mathbb{Z}_\ell) \to H^{2p}(U,\mathbb{Z}_\ell) \to \cdots$$

Remark 4. If Z is codimension one, we have isomorphisms

$$CH^1(X) \cong Pic(X) \cong H^1_{\mathsf{Zar}}(X,\mathcal{O}^*_X) \cong H^1(X_{an},\mathcal{O}^*_{X_{an}}),$$

and the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_{an}} \stackrel{exp}{\to} \mathcal{O}^*_{X_{an}} \to 1$$

induces a map

$$CH^1(X) \cong H^1(X_{an},\mathcal{O}^*_{X_{an}}) \to H^2(X_{an},\mathbb{Z}).$$

It can be shown that this map agrees with the one defined above. Moreover, since $\gamma : CH^*(X) \to H^{2*}(X,\mathbb{Z})$ is a morphism of graded rings, the class of any cycle Z which is an intersection of codimension one cycles $Z_1,\ldots,Z_m \in CH^1(X)$ is the product of the classes

$$\gamma(Z_1 \cdot \cdots \cdot Z_m) = \gamma(Z_1) \cdot \ldots \gamma(Z_m)$$

Position of $\gamma_{\mathbb{Z}}(Z)$ *in the Hodge decomposition.* If X is (smooth) projective of dimension d, Poincaré Duality induces an isomorphism

$$H^i(X,\mathbb{C}) \cong H^{2d-i}(X,\mathbb{C})^*$$

(where $(-)^* = \hom(-, \mathbb{C})$ means the dual vector space) which is compatible with the Hodge decomposition in the sense that

$$H^{r,s}(X) \cong H^{d-r,d-s}(X, \mathbb{C})^*.$$

Instead of showing that $\gamma(Z) \in H^{p,p}(X)$ we will show that the induced map $\bigoplus_{r+s=2p} H^{d-r,d-s} \to \mathbb{C}$ is zero unless $r = s$.

We claim (for more details see [Voisin, 11.1.2]) that the cycle class map sends a the closed subvariety Z to the map

$$\gamma(Z) : H^{2d-2p}_{\mathsf{dR}}(X, \mathbb{C}) \to \mathbb{C}$$
$$\beta \mapsto \int_Z (\beta|_Z)$$

Notice that $\beta|_Z \in H^{2d-2p}_{\mathsf{dR}}(Z, \mathbb{C})$ and consider its Hodge decomposition. Since $H^{r,s}(Z)$ is the set of cohomology classes representable by a closed form of type (r, s), and since $d - c = \dim Z$, the Hodge decomposition of $H^{2d-2p}_{\mathsf{dR}}(Z, \mathbb{C})$ has a single term: $H^{d-c,d-c}(Z)$. So $\beta|_Z = 0$ if $\beta \notin H^{d-c,d-c}(X)$. Hence,

$$\text{either} \qquad \gamma(Z) = 0 \qquad \text{or} \qquad \gamma(Z) \in H^{d-c,d-c}(X)^* \cong H^{c,c}(X).$$

□

2 Hodge classes. Hodge conjecture

Question 5. What is the image of $\mathcal{Z}^p(X) \to \mathrm{Hdg}^p(X)$?

Theorem 6 (Lefschetz $(1, 1)$, 1924)**.** *Let X be a smooth, irreducible, projective variety defined over $\mathbb{C}$. Then*

$$\gamma^1_{\mathbb{Z},X} : \mathrm{Div}(X) \longrightarrow \mathrm{Hdg}^1(X) \subset H^2(X_{an}, \mathbb{Z})$$

is onto. That is, every class of type $(1, 1)$ is "algebraic" (i.e. is the cohomology class of a divisor).

Indication of the proof. See [Griffiths and Harris, p.163] for details. It follows from the exponential sequence description of the cycle class map

$$\cdots \to H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \stackrel{\alpha}{\to} H^2(X_{an}, \mathbb{Z}) \stackrel{\beta}{\to} H^2(X_{an}, \mathcal{O}_{X_{an}}) \to \dots$$

that the image of $\gamma : CH^1(X) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \to H^2(X_{an}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ is the kernel of β. On the other hand, the pieces of the Hodge decomposition of $H^2(X, \mathbb{C})$ can be identified as

$$H^{r,s}(X) \cong H^s(X, \Omega^r)$$

and under this identification, the map β corresponds to the canonical projection

$$\beta : H^2(X_{an}, \mathbb{Z}) \stackrel{j}{\to} H^2(X_{an}, \mathbb{C}) \cong \bigoplus_{r=0}^{2} H^r(X_{an}, \Omega^{2-r}) \to H^2(X_{an}, \mathcal{O}_{X_{an}}).$$

Now since $H^{2,0} \cong H^{0,2}$, we see that a class lies in $\mathrm{Hdg}^1(X)$ if and only if it is in the kernel of β.

□

Motivated by the Lefschetz $(1,1)$ theorem for divisors Hodge conjectured that, or at least raised the question whether, $\gamma_{\mathbb{Z}}$ is onto always for all p ("integral Hodge conjecture"). However Atiyah-Hirzebruch discovered that this integral form is *not* true (1962), later other counterexamples were given by Kollár (1992) and Totaro (1997). Therefore the question has to be modified to rational coefficients.

Conjecture 7 (Hodge)**.** $\gamma_{\mathbb{Q}} : \mathcal{Z}^p(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Hdg}^p(X) \otimes \mathbb{Q}$ is onto.

This fundamental conjecture is wide open and only known for special cases (see for instance lectures by Murre and van Geemen in [Green al.]).

3 Intermediate Jacobian and Abel-Jacobi map

Let X be a smooth, irreducible, projective variety defined over $\mathbb{C}$. Recall the *Hodge decomposition*

$$H^i(X, \mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X), \quad H^{s,r}(X) = \overline{H^{r,s}(X)}$$

and the corresponding descending *Hodge filtration*

$$F^j H^i(X) = \bigoplus_{r \geq j} H^{r,i-r} = H^{i,0} + H^{i-1,1} + \cdots + H^{j,i-j}.$$

Definition 8. *The p-th intermediate Jacobian of X is*

$$J^p(X) = H^{2p-1}(X, \mathbb{C}) / \Big(F^p H^{2p-1}(X) + H^{2p-1}(X, \mathbb{Z}) \Big).$$

So writing

$$V = H^{0,2p-1} + \cdots + H^{p-1,p}$$

we have that $J^p(X) = V / H^{2p-1}(X, \mathbb{Z})$ (where -of course- we mean the image of $H^{2p-1}(X, \mathbb{Z})$ in V).

Exercise 1. Let Λ be a free abelian group of rank $2n$, and let $V \subset \Lambda \otimes \mathbb{C}$ be a sub-$\mathbb{C}$-vector space of dimension n such that $\Lambda \otimes \mathbb{C} \cong V \oplus \overline{V}$. Show that $\Lambda \otimes \mathbb{R} \to (\Lambda \otimes \mathbb{C}) / \overline{V}$ is an isomorphism of $\mathbb{R}$-vector spaces.

Lemma 9. *The pth intermediate Jacobian $J^p(X)$ is a complex torus of dimension half the $(2p-1)$th Betti number $b_{2p-1} = \dim_{\mathbb{Q}} H^{2p-1}(X, \mathbb{Q})$ of X:*

$$\dim J^p(X) = \tfrac{1}{2} b_{2p-1}(X).$$

Proof. [Voisin, 12.1.1] Since $2p-1$ is odd, the Hodge filtration on $H^{2p-1}(X,\mathbb{C})$ gives the direct sum decomposition

$$H^{2p-1}(X,\mathbb{C}) = V \oplus \overline{V}.$$

Thus the composition map

$$H^{2k-1}(X,\mathbb{R}) \to H^{2k-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C})$$

is an isomorphism of $\mathbb{R}$-vector spaces. Therefore the lattice

$$H^{2p-1}(X,\mathbb{Z}) \subseteq H^{2p-1}(X,\mathbb{R})$$

gives a lattice in the $\mathbb{C}$-vector space $H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X)$. □

Remark 10. The complex torus $J^p(X)$ is *in general not* an *abelian variety*, i.e. can not be embedded in projective space. For a torus $T = V/L$ to be an abelian variety it is necessary and sufficient that there exists a so-called *Riemann form*. This is a $\mathbb{R}-$bilinear alternating form $E : V \times V \to \mathbb{R}$ satisfying

a. $E(iv, iw) = E(v, w)$.

b. $E(v, w) \in \mathbb{Z}$ whenever $v, w \in L$.

c. $E(v, iw)$ symmetric and *positive definite*

In our case there is a non-degenerate form on V given by $E(v, w) = v \cup w \cup h^{d+2-2p}$, where h is the hyperplane class in $H^2(X,\mathbb{Z})$. However this form is in general not positive definite because it changes sign on the different $H^{r,s}(X)$, but it is if only one $H^{r,s}$ occurs in V, for instance only $H^{p-1,p}$.

Special cases

a. $p = 1$. Then $J^1(X) = H^1(X,\mathbb{C})/H^{1,0} + H^1(X,\mathbb{Z})$. This is the *Picard variety* of X, which is an abelian variety.

b. $p = d$. Then $J^d(X) = H^{2d-1}(X,\mathbb{C})/H^{d,d-1} + H^{2d-1}(X,\mathbb{Z})$. This is the *Albanese variety* of X, which is an abelian variety.

c. If $X = C$ is a curve then $J^1(X)$ is the so-called *Jacobian variety* of C, an abelian variety which is at the same time the Picard variety and the Albanese variety of X.

Recall

$$\mathcal{Z}^p_{hom}(X) = \{Z \in \mathcal{Z}^p(X) : \ \gamma_{\mathbb{Z}}(Z) = 0\},$$

i.e. the algebraic cycles which are *homologically equivalent* to zero.

Theorem 11. *There exists a homomorphism*

$$\mathrm{AJ}^p : \mathcal{Z}^p_{hom} \to J^p(X)$$

which factors through $\mathrm{CH}^p_{hom}(X)$*. The map AJ is called the Abel-Jacobi map.*

Omitted. See [Voisin, 12.1.2]. The construction is a little complicated and leans heavily on Poincaré Duality. Using the commutative diagram

$$\begin{array}{ccc} H^{2p-1}(X,\mathbb{Z})/\text{torsion} & \cong & H_{2d-2p+1}(X,\mathbb{Z})/\text{torsion} \\ \downarrow & & \downarrow \\ H^{2p-1}(X,\mathbb{C}) & \cong & H^{2d-2p+1}(X,\mathbb{C})^* \end{array}$$

It suffices to construct a map $H^{2d-2p+1}(X,\mathbb{C}) \to \mathbb{C}$ for each $Z \in \mathcal{Z}^p_{hom}$ (and show everything is well-defined). Using Poincaré Duality again, now between singular cohomology and singular homology, it suffices to construct a $\Gamma \in C^{\mathsf{sing}}_{2d-2p+1}(X,\mathbb{Z})$. In fact, the element $\gamma(Z) \in H^{2p}(X,\mathbb{Z}) \cong H_{2d-2p}(X,\mathbb{Z})$ is actually represented by a canonical cycle in $C^{\mathsf{sing}}_{2d-2p}(X,\mathbb{Z})$ (the pushforward of a triangulation of the smooth manifold associated to any desingularisation $\widetilde{Z} \to Z$). Saying $Z \in \mathcal{Z}^p_{hom}$ means precisely that $\gamma(Z) \in C^{\mathsf{sing}}_{2d-2p}(X,\mathbb{Z})$ is the image of some $\Gamma \in C^{\mathsf{sing}}_{2d-2p+1}(X,\mathbb{Z})$. This is the Γ that we choose. Then one must check that every thing is well-defined, and has the appropriate properties. □

Example 12. Let $X = C$ be a smooth projective curve defined over $\mathbb{C}$ of genus g. Now $p = 1$ and $\mathcal{Z}^1_{hom}(C) = \mathrm{Div}^{(0)}(C)$ are the divisors of degree zero, so we can write $D = \sum_i (P_i - Q_i)$ for some closed points $P_i, Q_i \in C$. Let Γ_i be a path from Q_i to P_i, so $D = \partial\Gamma$ with $\Gamma = \sum \Gamma_i$. Now

$$J(C) = H^1(C,\mathbb{C})/(F^1H^1 + H^1(C,\mathbb{Z})) = H^{0,1}(C)/H^1(C,\mathbb{Z})$$

$$\cong H^{1,0}(C)^*/H_1^{\mathsf{sing}}(C,\mathbb{Z}) \text{ (Poincaré Duality)}$$

and $H^{1,0}(\mathbb{C}) \cong H^0(C,\Omega^1_C)$, i.e. the space of holomorphic differentials, so $\dim H^{1,0} = \dim H^{0,1} = g$. Now $\Gamma \in C_1^{\mathsf{sing}}(C,\mathbb{Z})$ defines a map $H^0(C,\Omega^1_C) \to \mathbb{C}$, namely if $\omega \in H^0(C,\Omega^1_C)$ consider $\int_\Gamma \omega \in \mathbb{C}$, and hence, via the above isomorphisms, an element in $J(C)$.

If we choose other paths (or another ordering of the points Q_j) then we get a 1-chain Γ' and $\Gamma' - \Gamma \in H_1(C,\mathbb{Z})$ and the maps Γ and Γ' give the same element in $J(C)$.

4 Deligne cohomology. Deligne cycle map

In this section X is a smooth, irreducible, quasi-projective variety defined over $\mathbb{C}$. We denote the associated analytic space X_{an} by the same letter (X_{an} is a connected complex manifold, but not necessarily compact).

Definition 13. *Recall the decomposition* $A^n_{\mathbb{C}}(X) = \sum_{p+q=n} A^{p,q}(X)$ *of complex infinitely differentiable forms on the smooth (real) manifold* X_{an}*. A form* $\omega \in A^{p,0}(X)$ *is called* holomorphic *if its image* $d\omega \in \sum_{p+q=n+1} A^{p,q}(X)$ *lies* *entirely in $A^{p+1,0}$. We will write Ω_X^n for the sheaf of holomorphic n-forms. The differential $d: A^n_{\mathbb{C}}(X) \to A^{n+1}_{\mathbb{C}}(X)$ induces a complex*

$$\Omega_X^\bullet : 0 \to \Omega_X^0 \to \Omega_X^1 \to \cdots \to \Omega_X^d \to 0$$

Exercise 2. Show that Ω_X^0 is the sheaf of holomorphic functions.

Proposition 14 (Holomorphic Poincaré lemma, [Griffiths and Harris, p.448])**.** *The canonical inclusion $\mathbb{C} \subseteq \Omega_X^0$ induces a quasi-isomorphism of $\mathbb{C} \to \Omega_X^\bullet$.*

Definition 15. *Let $A \subseteq \mathbb{C}$ be a subring (such as $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$).* Deligne-Beilinson cohomology *is the hypercohomology*

$$H^i_{\mathcal{D}}(X, A(n)) = \mathbb{H}^i(X, A(n)_{\mathcal{D}})$$

of the complex

$$A(n)^\bullet_{\mathcal{D}}: \qquad 0 \to A(n) \to \Omega_X^0 \to \Omega_X^1 \to \cdots \to \Omega_X^{n-1} \to 0 \to \dots,$$

where $A(n) = (2\pi i)^n A$.

Theorem 16 (Deligne)**.** *The cycle map induces a morphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Z}^p_{hom}(X) & \longrightarrow & \mathcal{Z}^p(X) & \longrightarrow & \mathcal{Z}^p(X)/\mathcal{Z}^p_{hom}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) & \longrightarrow & \mathrm{Hdg}^p(X) & \longrightarrow & 0 \end{array}$$

We don't say anything about the cycle map.

Sketch of exactness of the lower row. See [Voisin, p.304, 12.3]. Associated to

$$0 \to \Omega_X^{\bullet<p}[-1] \to \mathbb{Z}(p)_{\mathcal{D}} \to \mathbb{Z}(p) \to 0.$$

is the long exact sequence

$$\begin{aligned} H^{2p-1}(X, \mathbb{Z}(p)) &\xrightarrow{\alpha} \mathbb{H}^{2p-1}(X, \Omega_X^{\bullet<p}) \\ &\to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \\ &\to H^{2p}(X, \mathbb{Z}(p)) \xrightarrow{\beta} \mathbb{H}^{2p}(X, \Omega_X^{\bullet<p}). \end{aligned} \tag{1}$$

and it suffices to show that $\operatorname{coker} \alpha \cong J^p(X)$ and $\ker \beta \cong \mathrm{Hdg}^p(X)$.

The point essentially that $\Omega_X^{\bullet \geq p}$ calculates the Hodge filtration (this basically follows from the spectral sequence associated to the double complex $A^{p,q}(X)$)

$$\mathbb{H}^i(X, \Omega_X^{\bullet \geq p}) = F^p H^i(X),$$

so the short exact sequence $0 \to \Omega_X^{\bullet \geq p} \to \Omega_X^\bullet \to \Omega_X^{\bullet<p} \to 0$ implies

$$\mathbb{H}^i(X, \Omega_X^{\bullet<p}) \cong H^i(X, \mathbb{C})/F^p H(X) \tag{2}$$

The the claims about α and β follow from the definitions of $J^p(X)$ and $\mathrm{Hdg}^p(X)$. □

Exercise 3. Using the sequence (1) and the isomorphism (2) show that we have $\ker \beta = \mathrm{Hdg}^p(X)$. (Hint: Cf. the proof of Theorem 6).

Exercise 4.

1. Show that $H^i_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^i(X, \mathbb{Z})$.
2. Show that via the map $z \mapsto \exp(z)$, the complex $\mathbb{Z}(1)_{\mathcal{D}}$ is *quasi-isomorphic* to the complex $\mathcal{O}^*_X[-1]$ which has $\mathcal{O}^*_X$ placed in degree -1, and is trivial elsewhere.

Remark 17. We deduce from the exercise that $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}}) = \mathrm{Pic}(X)$, and that the lower short exact sequence of Theorem 16 is identified with the classical short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0 \tag{3}$$