In this lecture we present some main theorems from algebraic topology, differential topology, symplectic geometry, ... that we will use later. More specifically, we discuss Betti, singular, and de Rham cohomology, the relationship between them, and finish with pure Hodge structures.

We will get back to algebraic geometry next week.

References:

1. Bott&Tu, Differential forms in algebraic topology.
2. Griffiths&Harris, Principles of algebraic geometry.
3. Hatcher, Algebraic topology.
4. Voisin, Hodge theory and complex algebraic geometry I.

1 Topological spaces

Definition 1. *If M is topological space, and R a ring, we define the* Betti cohomology $H^n_{\mathsf{Betti}}(M, R)$ *as the cohomology of the constant sheaf $\underline{R}$. That is, we choose an exact complex of sheaves on M*

$$0 \to \underline{R} \to I^0 \to I^1 \to \dots$$

with each I^i injective and define

$$H^n_{\mathsf{Betti}}(M, R) = \frac{\ker(d : I^n(M) \to I^{n+1}(M))}{\operatorname{im}(d : I^{n-1}(M) \to I^n(M))}.$$

Remark 2. Its also possible to calculate cohomology using exact complexes of *flasque*[1] sheaves, (cf. Voisin, Prop.4.34), or *fine*[2] sheaves (cf. Voisin, Prop.4.36).

The singular cohomology is one concrete option of a resolution calculating Betti cohomology. Cf. Voisin, Section.4.3.2.

Definition 3. *Define $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i,$ and $\sum x_i = 1\}$. Then for a topological space U, the group of singular cochains is hom set*

$$C^n_{\mathsf{sing}}(U, R) = \hom_{set}(\hom_{cont}(\Delta^n, U), R)$$

Addition in the ring R makes this set a group.

Example 4. $C^0_{\mathsf{sing}}(U, R) = \hom_{set}(U, R)$.

[1] Flasque means: (cf. Voisin, Def.4.33) for every inclusion of open subsets $U \subseteq V$ the induced map $F(V) \to F(U)$ is surjective.

[2] Fine means:)cf. Voisin, Def.4.35) F is a sheaf of R-modules where R is a sheaf of rings such that for every open cover $\{U_i \subseteq X\}_{i \in I}$ there exists a partition of unity $f_i, i \in I, \sum f_i = 1$, subordinate to the covering.

The inclusions $\mathbb{R}^{n+1} \to \mathbb{R}^{n+2}; (x_0, \dots, x_i, 0, x_{i+1}, \dots, x_n)$ induce maps $\delta_i : \Delta^n \to \Delta^{n+1}$ and, by composition, these induce maps $\delta_i^* : \hom_{cont}(\Delta^{n+1}, U) \to \hom_{cont}(\Delta^n, U)$, and from there, maps

$$d = \sum (-1)^i d_i : C_n^{\mathsf{sing}}(U, R) \to C_{n+1}^{\mathsf{sing}}(U, R).$$

These form a chain complex of sheaves

$$0 \to \underline{R}(-) \to C^0_{\mathsf{sing}}(-, R) \xrightarrow{d} C^1_{\mathsf{sing}}(-, R) \xrightarrow{d} \cdots \xrightarrow{d} C^n_{\mathsf{sing}}(-, R) \xrightarrow{d} \dots \tag{1}$$

Definition 5. *The* singular cohomology *of M is the cohomology of* (1). *I.e.,*

$$H^n_{\mathsf{sing}}(M, R) = \frac{\ker(d : C^n_{\mathsf{sing}}(M, R) \to C^{n+1}_{\mathsf{sing}}(M, R))}{\operatorname{im}(d : C^{n-1}_{\mathsf{sing}}(M, R) \to C^n_{\mathsf{sing}}(M, R))}$$

Exercise 1. Show that each $C^n_{\mathsf{sing}}(-, R)$ is a flasque sheaf. That is, for every inclusion of open sets $U \subseteq V$ the map $C^n_{\mathsf{sing}}(V, R) \to C^n_{\mathsf{sing}}(U, R)$ is surjective.

Lemma 6. *When M is a manifold (or more generally a locally contractible topology space) the chain complex* (1) *is exact.*

Sketch of proof. Since M is a smooth manifold it admits an open cover consisting of contractible spaces. Then use the result mentioned further down, that the singular cohomology of contractible spaces vanishes. $\square$

Corollary 7. *If M is a smooth manifold, then*

$$H^*_{\mathsf{Betti}}(M, R) = H^*_{\mathsf{sing}}(M, R).$$

We have the following properties:

Theorem 8.

1. *(Homotopy invariance) If there is a continuous map $h : M \times [0, 1] \to N$ then the map on cohomology induced by $h(-, 0)$ is equal to the map induced by $h(-, 1)$. In particular, if M is contractible, i.e., there is a continuous map $h : M \times [0, 1] \to M$ such that $h(-, 0) = \mathrm{id}_M$ and $h(-, 1)$ is constant, then $H^n(M, R) = 0$ for $n > 0$ and $H^0(M, R) \cong R$.*

2. *(Mayer-Vietoris) If $U, V \subseteq M$ are two open subspaces then there is a long exact sequence*

$$\begin{aligned} \cdots \to H^{n-1}_{\mathsf{Betti}}(U \cap V, R) &\to H^n_{\mathsf{Betti}}(U \cup V, R) \\ &\to H^n_{\mathsf{Betti}}(U, R) \oplus H^n_{\mathsf{Betti}}(V, R) \to H^n_{\mathsf{Betti}}(U \cap V, R) \to \dots \end{aligned}$$

3. *(Finiteness) Cf. Bott&Tu, Prop.5.3.2. If M is a compact smooth manifold, then the $H^i(M, \mathbb{Z})$ are finitely generated.*

Sketch of proof (omitted from the lecture).

1. We only prove the "in particular". Any map $\sigma : \Delta^n \to X$ induces a map $\Delta^n \times [0,1] \to X \times [0,1] \xrightarrow{h} X$. Since $h(-,1)$ is constant, this map factors through the projection $\Delta^n \times [0,1] \to \Delta^n \times [0,1]/\Delta^n \times \{1\} \cong \Delta^{n+1}$. So we obtain a map $\sigma' : \Delta^{n+1} \to X$ whose composition with $\Delta^n \cong \Delta^n \times \{0\} \subset \Delta^n \times [0,1]/\Delta^n \times \{1\} \cong \Delta^{n+1}$ is σ (because $h(-,0) = \mathrm{id}_X$). It also has the property that $\sigma' \circ \delta_i = (\sigma \circ \delta_i)'$. Hence, $\mathrm{id} = (- \circ d)' - d \circ (-)'$. So the chain complex $C^*_{\mathsf{sing}}(X, R)$ is exact.

2. Consider the free sheaves $\mathbb{Z}h_{U \cap V}, \mathbb{Z}h_U, \mathbb{Z}h_V, \mathbb{Z}h_{U \cup V}$ represented by $U \cap V, U, V, U \cup V$ respectively. In the category of sheaves, the sequence

$$0 \to \mathbb{Z}h_{U \cap V} \to \mathbb{Z}h_U \oplus \mathbb{Z}h_V \to \mathbb{Z}h_{U \cup V} \to 0$$

is exact. For any injective sheaf I the functor $\hom(-, I)$ is exact (by definition). So applying $\hom(-, I^*)$ to this short exact sequence gives a short exact sequence of complexes of abelian groups. Then the long exact sequence in the statement is the long exact sequence associated to this short exact sequence of chain complexes.

3. This follows from the theorem that M admits a "good" covering, Cf. Bott&Tu, Thm.5.1. That is, covering $\{U_i \to M\}_{i \in I}$ such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are contractible. Since M is compact, we can assume I is finite. Then one can use an induction argument and Mayer-Vietoris. □

Example 9. For $m \geq 1$ we have

$$H^n_{\mathsf{Betti}}(S^m, R) = \begin{cases} R & n = 0, m \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We prove this by induction on m. Let $U_+, U_- \subseteq S^m$ be two open contractible subsets whose intersection is homotopic to S^{m-1}. For example, if we are using the model $S^m = \{x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \text{ s.t. } |x| = 1\}$, then we could take $U_\pm = \{(x_0, \dots, x_m) \in S^m \text{ s.t. } \pm x_0 \geq -1/2\}$. Then (2) follows by induction from the Mayer-Vietoris sequence, and the homotopies $U_+ \sim \{*\}$, $U_- \sim \{*\}$, $U_+ \cap U_- \sim S^{m-1}$.

Remark 10. Note that the underlying topological space of $\mathbb{A}^n(\mathbb{C}) \setminus \{0\}$ is homeomorphic to $\mathbb{R}^{2n} \setminus \{0\}$, which is homotopic to S^{2n-1} (via $(x,t) \mapsto \frac{x}{(1-t)+t||x||}$).

Exercise 2. Add details to the argument in Example 9. I.e., write out the long exact sequence and use the induction hypothesis for $m-1$ to deduce the result for m. Note that $S^0 = \{-1, 1\} \subseteq \mathbb{R}$ so $H^0_{\mathsf{Betti}}(S^0, R) \cong R \oplus R$, and $H^n_{\mathsf{Betti}}(S^0, R) \cong 0$ for $n > 0$.

Theorem 11.

1. *(Künneth Formula) Cf. Voisin, Thm.11.38. For smooth manifolds M, M',*

and any field[3] *K (e.g., $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) there is a canonical isomorphism*

$$H^n_{\mathsf{Betti}}(M \times M', K) \cong \bigoplus_{i+j=n} H^i_{\mathsf{Betti}}(M, K) \otimes_K H^j_{\mathsf{Betti}}(M', K)$$

2. *(Poincaré duality) Cf. Hatcher, Prop.3.38, Voisin, Thm.5.30, Rem.5.31. For X a connected smooth projective complex variety of complex dimension n then there is canonical isomorphism*

$$H^{2n}_{\mathsf{Betti}}(X(\mathbb{C}), R) \cong R.$$

which, when combined with the Künneth isomorphism, induces isomorphisms

$$H^i_{\mathsf{Betti}}(X(\mathbb{C}), R) \cong H^{2n-i}_{\mathsf{Betti}}(X(\mathbb{C}), R)^*$$

when R is a field, or when torsion is factored out of $H^*_{\mathsf{Betti}}(X(\mathbb{C}), \mathbb{Z})$.

These isomorphisms are easier to describe for H_{dR} so we omit details here.

Example 12. The cohomology of a torus is:

$$H^n_{\mathsf{Betti}}(\underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}}, R) \cong \bigoplus_{i=1}^{\binom{m}{n}} R.$$

Remark 13. Note that the underlying topological space $E(\mathbb{C})$ of a complex projective elliptic curve E is homeomorphic to $S^1 \times S^1$. More generally, The underlying topological space of a complex abelian variety is homeomorphic to $S^1 \times \cdots \times S^1$.

Exercise 3. Prove the isomorphism of Example 12 by induction using the Künneth Formula starting with the base case $m = 1$, cf. Exercise 9.

Definition 14. *If $U \subseteq M$ is an open immersion of topological spaces, the relative cohomology is defined as*

$$H^n_{\mathsf{Betti}}(M, U; R) = H^{n-1}\Big(Cone(I^\bullet(M) \to I^\bullet(U))\Big)$$

For any injective resolution $\underline{R} \to I^0 \to I^1 \to \dots$ (e.g., $F^\bullet = C^\bullet_{\mathsf{sing}}$).

Remark 15. By definition, there is a long exact sequence

$$\begin{aligned} \cdots \to H^n_{\mathsf{Betti}}(X, U; R) &\to H^n_{\mathsf{Betti}}(X, R) \\ &\to H^n_{\mathsf{Betti}}(U, R) \to H^{n+1}_{\mathsf{Betti}}(X, U; R) \to \dots \end{aligned}$$

[3] If $H^*(M, \mathbb{Z}), H^*(M', \mathbb{Z})$ are finitely generated, e.g., if M, M' are compact (cf. Voisin, Rema.4.46) then the result is also true for $K = \mathbb{Z}$.

Theorem 16 (Thom isomorphism)**.** *Cf. Hatcher, Cor.4D.9, Thm.4D.10, Voisin, Proof of Lem.11.13. Let $Y \subseteq X$ be a closed complex submanifold of (complex) codimension c. Then*

$$H^j_{\mathsf{Betti}}(X, X \setminus Y; \mathbb{Z}) \cong H^{j-2c}_{\mathsf{Betti}}(Y, \mathbb{Z})$$

for $j \geq 2c$.

Some ideas of a proof (probably omitted from lecture, depending on time). The vague idea is to replace X with a small open neighbourhood of Y in X, then replace this open neighbourhood with the normal bundle to Y in X. In this way, we can assume $Y \to X$ is the zero section of a vector bundle (of real rank $2c$). If the vector bundle is trivial, i.e., $X \cong Y \times \mathbb{R}^{n-2c}$ then Künneth reduces the calculation to the case $Y = \{*\}$, in which case it is straightforward using the long exact sequence of Remark 15 and the homotopy $\mathbb{R}^{2c} \setminus \{0\} \sim S^{2c-1}$. □

2 Smooth manifolds

To an open subset $U \subseteq \mathbb{R}^n$ we associate $A^k(U)$ the $\mathbb{R}$-vector space of differential k-forms on X. This is a free module over the ring $C^\infty(U)$ of infinitely differentiable functions from U to $\mathbb{R}$ with basis

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Every differentiable map $\psi : U \to \mathbb{R}^m$ induces a map $A^k(\mathrm{im}(\psi)) \to A^k(U)$ defined by $dx_i \mapsto \sum(\frac{\partial}{\partial x_j}\psi_i)dx_j$, and in particular, the $A^k(U)$ define a sheaf on $\mathbb{R}^n$. If M is a smooth manifold of dimension n, then there is a unique sheaf A^k on M such that for any chart $U \subseteq M$, $\phi : U \to \mathbb{R}^n$ we have $A^k|_{\phi(U)} \xrightarrow{\sim} A^k|_U$, and for any two charts $\phi, \psi : U \to \mathbb{R}^n$ of the same open U, the isomorphism $A^k|_{\phi(U)} \xrightarrow{\sim} A^k|_U \xleftarrow{\sim} A^k|_{\psi(U)}$ is the canonical one associated to $\psi \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$.[4]

The exterior differential

$$d : f dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mapsto \sum_{j=1}^{n} (\tfrac{\partial}{\partial x_j} f) dx_j \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

induces maps of sheaves $d : A^k \to A^{k+1}$ producing a chain complex of sheaves

$$0 \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^n \to 0. \tag{3}$$

Definition 17. *The* de Rham cohomology *of a smooth manifold M is the cohomology of the above chain complex*

$$H^k_{dR}(M, \mathbb{R}) = \frac{\ker(d : A^k(M) \to A^{k+1}(M))}{\mathrm{im}(d : A^{k-1}(M) \to A^k(M))}$$

[4]For example, if $\{\iota_i : U_i \to M; \phi_i : U_i \to \mathbb{R}^n\}_{i \in I}$ is a system of charts of M, define A^k as the kernel of $\prod_{i \in I} \iota_{i*}\phi_i^* A^k \to \prod_{i,j \in I} \iota_{ij*}\phi_{ij}^* A^k$ where $\iota_{ij} : U_i \cap U_j \to M$ is the inclusion, $\phi_{ij} : U_i \cap U_j \to \mathbb{R}^n$ the chart induced by ϕ_i, and the morphism is the difference of the canonical maps $\iota_{i_*}\phi_i^* A^k \to \iota_{ij*}\phi_{ij}^* A^k$, and the compositions $\iota_{i_*}\phi_i^* A^k \to \iota_{ij*}\phi_{ij}^* A^k \cong \iota_{ji*}\phi_{ji}^* A^k$.

Theorem 18. *Let M be a smooth manifold. Then*

$$H^i_{\mathsf{Betti}}(M,\mathbb{R}) \cong H^i_{\mathsf{dR}}(M,\mathbb{R}).$$

Sketch of proof. Poincaré's Lemma (Bott&Tu,§4) says that $H^i_{\mathsf{dR}}(\mathbb{R}^n,\mathbb{R}) = 0$ for $i > 0$ so we deduce that the sequence of sheaves (3) is exact. Then one shows that each A^k is a "fine sheaf" (cf. Voisin, Def.4.35), and that resolutions of fine sheaves calculate cohomology (cf. Voisin, Prop.4.36, Cor.4.37). □

We deduce from Theorem 18 that de Rham cohomology satisfies Homotopy Invariance, Mayer-Vietoris, Poncaré Duality, Künneth Formula, and the Thom Isomorphism. However, de Rham cohomology makes Poincaré duality and Künneth a little easier to state. Recall that an *orientation* on a smooth n-dimensional manifold M is a choice of $\omega \in A^n(M)$ which is nowhere zero.

Theorem 19 (Poincaré Duality, Cf. Voisin, Thm.5.30)**.** *Let M be an n-dimensional smooth connected compact orientable manifold. Then $A^n(M) \to \mathbb{R}; \omega \mapsto \int_M \omega$ defines an isomorphism*

$$H^n_{dR}(M,\mathbb{R}) \xrightarrow{\sim} \mathbb{R}.$$

Moreover, the map $A^p(M) \otimes_{\mathbb{R}} A^{n-p}(M) \to A^n(M); (\alpha,\beta) \mapsto \int_M \alpha \wedge \beta$ induces a perfect pairing on cohomology. That is, it induces an isomorphism

$$H^p(M,\mathbb{R}) \cong H^{n-p}(M,\mathbb{R})^*$$

(which depends on the isomorphism $H^n_{dR}(M,\mathbb{R}) \cong \mathbb{R}$ above).

Theorem 20 (Künneth Formula, Cf. Voinsin, §11.3.3, Griffiths&Harris, pg.103,104)**.** *Let M, M' be smooth manifolds. Then the map*

$$A^k(M) \times A^{k'}(M') \to A^{k+k'}(M \times M'); \qquad \alpha \otimes \beta \mapsto pr_1^*\alpha \wedge pr_2^*\beta$$

induces the Künneth Formula

$$H^k_{\mathsf{dR}}(M,\mathbb{R}) \otimes H^k_{\mathsf{dR}}(M',\mathbb{R}) \xrightarrow{\sim} H^{k+k'}_{\mathsf{dR}}(M \times M',\mathbb{R}).$$

Here, $pr_1 : M \times M' \to M, pr_2 : M \times M' \to M'$ are the canonical projections.

3 Complex manifolds

Consider the canonical identification

$$\mathbb{C}^\nu \cong \mathbb{R}^{2\nu} : (x_1 + iy_1, \ldots, x_\nu + iy_\nu) \leftrightarrow (x_1, y_1, x_2, y_2, \ldots, x_\nu, y_\nu).$$

For an open subset $U \subseteq \mathbb{R}^{2\nu}$ we define

$$I : A^1(U) \to A^1(U); \qquad \begin{cases} dx_j \mapsto dy_j \\ dy_j \mapsto -dx_j \end{cases}$$

For any $U \subseteq \mathbb{R}^{2n}$, a smooth function $\psi : U \to \mathbb{R}^{2m}$ is called *holomorphic* if $I \circ d\psi = d\psi \circ I$.

Exercise 4. Show that ψ is holomorphic if and only if the Cauchy-Riemann equations hold. That is, if and only if

$$\partial_{y_i} u_j = -\partial_{x_i} v_j, \qquad \text{and} \qquad \partial_{y_i} v_j = \partial_{x_i} u_j.$$

for each i, j, where $\psi(z) = (u_1(z), v_1(z), \ldots, u_m(z), v_m(z))$ and $z = (x_1, y_1, \ldots, x_n, y_n)$.

Exercise 5. Show that the morphism $I : A^1(U) \to A^1(U)$ induces a complex vector pace structure on $A^1(U)$ by

$$(a + ib)\omega = (a + bI)\omega.$$

Observe that a smooth function $\psi : U \to \mathbb{R}^{2m}$ is holomorphic if and only if the $\mathbb{R}$-linear morphism $d\psi : A^1(\mathbb{R}^{2m}) \to A^1(U)$ is actually $\mathbb{C}$-linear.

Exercise 6. Show that:

1. A composition of holomorphic functions is holomorphic.
2. A function $U \to \mathbb{R}^{2m}$ is holomorphic if and only if the composition with each projection $U \to \mathbb{R}^{2m} \to \mathbb{R}^2; (x_1, y_1, \ldots, x_m, y_m) \mapsto (x_i, y_i)$ is holomorphic.
3. A sum of holomorphic functions $U \to \mathbb{R}^2$ is holomorphic.
4. Via the multiplication induced by $\mathbb{C} \cong \mathbb{R}^2$, a product of holomorphic functions $U \to \mathbb{R}^2$ is holomorphic.
5. Any polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ defines a holomorphic function $\mathbb{R}^{2n} \cong \mathbb{C}^n \to \mathbb{C} \cong \mathbb{R}^2$.
6. If a function $\phi : U \to \mathbb{R}^{2m}$ admits a smooth inverse $\psi : \phi(U) \to U$, then ϕ is holomorphic if and only if ψ is holomorphic.

Now suppose that X is a smooth complex manifold of dimension n. That is, X is equipped with charts $\{U_i \subseteq X; \phi_i : U_i \to \mathbb{R}^{2n}\}_{i \in I}$ such that the transition maps $\phi_i \circ \phi_j^{-1}$ are holomorphic. Then we obtain an induced automorphism

$$I : A^1 \to A^1$$

of the sheaf A^1. We now define

$$A^1_{\mathbb{C}}(U) := A^1(U) \otimes_{\mathbb{R}} \mathbb{C}.$$

Since $I^2 = -1$, there are two eigenvalues $-i$ and i. Their eigenspaces are denoted $A^{1,0}(U)$, $A^{0,1}(U)$ respectively. Since the transition maps are holomorphic, they preserve the eigenspaces, and we get a decomposition of sheaves

$$A^1_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}.$$

Example 21. Consider $X = \mathbb{R}^{2n}$. Then $A^{1,0}(X)$ (resp. $A^{0,1}(X)$) has basis

$$dz_j := dx_j + idy_j, \qquad (\text{resp.} \quad d\overline{z}_j := dx_j - idy_j), \qquad j = 1, \ldots, n$$

as a free $C^\infty(\mathbb{R}^{2n})$-module

Exercise 7. Check that $Idz = -idz$ and $Id\overline{z} = id\overline{z}$.

The eigenspace decoposition $A^1_{\mathbb{C}} = A^{1,0}(X) \oplus A^{0,1}(X)$ induces a decomposition of $A^n_{\mathbb{C}}(X) := A^n(X) \otimes_{\mathbb{R}} \mathbb{C}$ as

$$A^n_{\mathbb{C}}(X) = \sum_{p+q=n} A^{p,q}(X); \qquad A^{p,q}(X) := \bigwedge^p A^{1,0} \otimes \bigwedge^q A^{0,1}$$

(the tensor and wedge products are over the sheaf C^∞).

Example 22. $A^{p,q}(\mathbb{R}^{2n})$ is spanned by the

$$dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{j_{p+1}} \wedge \cdots \wedge d\overline{z}_{p+q}$$

as a $C^\infty(\mathbb{R}^{2n})$-module

Exercise 8. Let V, W be finitely generated free R-modules for some ring R. Show that there is a canonical isomorphism

$$\bigwedge^n (V \oplus W) \cong \sum_{p+q=n} \left(\bigwedge^p V \otimes \bigwedge^q W \right).$$

In analogy with $dz, d\overline{z}$ we define

$$\partial_z := \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} := \tfrac{1}{2}(\partial_x + i\partial_y)$$

Exercise 9. For $U \subseteq \mathbb{R}^{2n}$, show that with respect to the basis $dz_j, d\overline{z}_j$ the differential $C^\infty(U) \otimes_{\mathbb{R}} \mathbb{C} \to A^1_{\mathbb{C}}(U)$ becomes $f \mapsto \sum_{j=1}^n (\partial_{z_j} f dz_j + \partial_{\overline{z}_j} f d\overline{z}_j)$.

Theorem 23 (Cf. Voisin, Proof of Prop.6.11)**.** *Let X be a smooth projective variety. Then there is a canonical decomposition*

$$H^k_{\mathsf{dR}}(X(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is the set of cohomology classes representable by a closed form of type (p,q). Moreover, we have

$$\overline{H^{p,q}} = H^{q,p}$$

where complex conjugation acts via $H^{p+q}(X(\mathbb{C}), \mathbb{C}) \cong H^{p+q}(X(\mathbb{C}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Theorem 24 (Cf. Voisin, Lem.7.30). *Poincaré duality is compatible with the Hodge decomposition, in the sense that it induces isomorphisms*

$$H^{r,s} \cong (H^{d-r,d-s})^*$$

In particular, for $r' + s' = 2d - r - s$, and $(r', s') \neq (d - r, d - s)$, the morphism

$$H^{r,s} \otimes H^{r',s'} \to \mathbb{C}$$

is zero.

Theorem 25 (Künneth Formula, Cf. Voinsin, §11.3.3, Griffiths&Harris, pg.103,104). *Künneth is compatible with the Hodge decomposition in the sense that if X, Y are smooth complex projective varieties, then it induces isomorphisms*

$$H^{r,s}(X(\mathbb{C}) \times Y(\mathbb{C})) \cong \bigoplus_{\substack{p+p'=r,\\ q+q'=s}} H^{p,q}(X(\mathbb{C})) \otimes_{\mathbb{C}} H^{p',q'}(Y(\mathbb{C})).$$

4 Pure Hodge structures

Definition 26. *(Cf. Voisin, Def.7.4) A* pure Hodge structure *of weight k, is a free finitely generated abelian group V, equipped with a decomposition*

$$V \otimes \mathbb{C} \cong \bigoplus_{p+q=k} V^{p,q}$$

satisfying $V^{p,q} = \overline{V^{q,p}}$.

(Cf. Voisin, Def.7.22) A morphism *$(V, V^{p,q}) \to (W, W^{p,q})$ from a Hodge structure of weight n to a Hodge structure of weight m is a morphism $\phi : V \to W$ is of abelian groups such that $\phi \otimes \mathbb{C}(V^{p,q}) \subseteq W^{p+r,q+r}$ where $m = n + 2r$.*

(Cf. Voisin, Def.11.39) The tensor product *$(K, K^{r,s})$ of $(V, V^{p,q})$ and $(W, W^{p',q'})$ is defined to be the Hodge structure of weight $n + m$ whose abelian group is $K = V \otimes W$ and whose decomposition is*

$$K^{r,s} = \bigoplus_{\substack{p+p'=r,\\ q+q'=s}} V^{p,q} \otimes_{\mathbb{C}} W^{p',q'}.$$

Example 27. Putting together all of the above material, we get: If X is a smooth complex projective variety, then each

$$H^k_{\mathsf{Betti}}(X(\mathbb{C}), \mathbb{Z})/\text{torsion}$$

is canonically equipped with a pure Hodge structure of weight k. Moreover, if Y is another smooth complex projective variety, then Künneth induces an isomorphism of pure Hodge structures of weight $k + l$

$$\frac{H^k_{\mathsf{Betti}}(X(\mathbb{C}), \mathbb{Z})}{\text{torsion}} \otimes \frac{H^l_{\mathsf{Betti}}(Y(\mathbb{C}), \mathbb{Z})}{\text{torsion}} \xrightarrow{\sim} \frac{H^{k+l}_{\mathsf{Betti}}((X \times Y)(\mathbb{C}), \mathbb{Z})}{\text{torsion}}.$$