References are:

[BS] Bhatt, Scholze, "The pro-étale topology for schemes"

[SGA] Artin, Grothendieck, Verdier, et al. "Théorie des topos et cohomologie étale des schémas. Tome 3 (SGA4)"

[Stacks project] The Stacks Project, `https://stacks.math.columbia.edu/`

1 Motivation

Let $i : Z \to X$ be a closed immersion, and $j : U \to X$ the open complement. Defining

$$j^! := j^* \qquad \text{and} \qquad i_! := i_*,$$

we obtained, in addition to the adjunctions (j^*, j_*) and (i^*, i_*), adjunctions

$$(j_!, j^!) \qquad \text{and} \qquad (i_!, i^!). \tag{1}$$

These were compatible in various ways such as

$$j^* i_! = 0, \qquad i^* i_! = \mathrm{id}, \qquad i^* j_! = 0, \qquad j^* j_! = \mathrm{id}\,. \tag{2}$$

We defined cohomology with compact support as

$$H^r_c(U, F) = R^r\Gamma(X, j_! F) = R^r \pi_{X*} j_! F$$

where $j : U \to X$ is an open immersion of from a smooth curve to a projective smooth curve over an algebraically closed field k, and $\pi_X : X \to k$ is the structural morphism. We say that this fit into the long exact sequence

$$\cdots \to H^r_c(U, \mathbb{Z}/n) \to H^r_{\mathsf{et}}(X, \mathbb{Z}/n) \to H^r_{\mathsf{et}}(Z, \mathbb{Z}/n) \to H^{r+1}_c(U, \mathbb{Z}/n) \to \dots. \tag{3}$$

When F was a locally constant sheaf of $\mathbb{Z}/n$-modules with finite stalks, we also had Poincaré Duality

$$\hom(H^r_c(U, F), \mathbb{Z}/n) \cong H^{2-r}_{\mathsf{et}}(U, \underline{\hom}(F, \mu_n)). \tag{4}$$

Generalising $i_! := i_*$ to $f_! := Rf_*$ for arbitrary proper morphisms f, and writing

$$\pi_{U!} := R\pi_{X*} \circ j_! \tag{5}$$

for the quasi-projective morphism $\pi_U : U \to k$, the long exact sequence (3) and Poincaré Duality (4) become

$$\cdots \to H^r \pi_{U!}\mathbb{Z}/n \to H^r \pi_{X!}\mathbb{Z}/n \to H^r \pi_{Z!}\mathbb{Z}/n \to H^{r+1}\pi_{U!}\mathbb{Z}/n \to \dots$$

$$\hom(\pi_{U!}F, \Lambda) \cong \hom(F, \mu_n[2]) \tag{6}$$

Since $\mathsf{Shv}_{\mathsf{et}}(k, \mathbb{Z}/n) \cong \mathbb{Z}/n$-mod, (6) says that $\pi_{U!}$ has a right adjoint $\pi_U^! := \pi_U^*(- \otimes \mu_n[2])$, which is a nice picture considering the adjunctions (1).

These facts fit into a larger framework.

One can define an adjunction of exceptional functors $(f_!, f^!)$ for any finite type morphism between noetherian schemes $f : Y \to X$ generalising (1) and (5). Moreover, the identities 2 generalise to base change theorems: If

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[f]{} & Y \end{array}$$

is a cartesian square with f proper, or g smooth, then

$$Rg^* \circ f_! \cong f'_! \circ Rg'^*.$$

2 Notation and outline

In this lecture we write

$$D(X) = D(\mathsf{Shv}_{\mathsf{et}}(X)), \qquad D(\Lambda) = D(\Lambda\text{-mod}), \qquad D(X, \Lambda) = D(\mathsf{Shv}_{\mathsf{et}}(X, \Lambda))$$

where Λ is a ring. We have $\Lambda = \mathbb{Z}/n$ in mind here. To make the statements nicer we work with noetherian schemes. Since we are always working with derived categories and derived functors, we write $f_* = Rf_*$, $\otimes = \otimes^L$, etc.

3 Constructible complexes

We are interested in $H_c^r(-, \mu_n^{\otimes r})$. However, in order to effectively use the six operations $f^*, f_*, f_!, f^!, \otimes, \underline{\hom}$ and the operations on complexes $\oplus$ and Cone, we end up having to deal with a larger class of objects.

Definition 1 ([Stacks Project, Tag 0657])**.** *An object $K \in D(\Lambda)$ is* perfect *if it is isomorphic to a bounded complex of finite projective Λ-modules.*

Exercise 1. Show that if K, L are perfect, then so is $K \oplus L$. Use the fact that $\hom_{D(\Lambda)}(K, L) = \hom_{K(\Lambda)}(K, L)$ if K, L are bounded complexes of projectives to show that for any map $K \to L$ between perfect complexes, $\mathrm{Cone}(K \to L)$ is also a perfect complex.

Lemma 2 ([Stacks Project, Tag 0ATI])**.** *The category $D_{\mathrm{perf}}(\Lambda) \subseteq D(\Lambda)$ of perfect objects is the smallest subcategory closed under cone, (de)shift, and direct summand containing Λ.*

Proposition 3 ([Stacks Project, Tag 07LT]). *A complex $K \in D(\Lambda)$ is perfect if and only if*

$$\hom_{D(\Lambda)}(K, \bigoplus L_i) \cong \bigoplus \hom_{D(\Lambda)}(K, L_i)$$

for any family $\{L_i\}_{i\in I}$.

Remark 4. An object in any additive category satisfying the above property is called *compact*. So the proposition says that the compact objects of $D(\Lambda)$ are exactly the perfect ones. Being compact means that an object is small in a certain sense.

Definition 5. *An object $K \in D(X, \Lambda)$ is* constant *(with perfect values) if it is the image under $D(\Lambda) \to D(X, \Lambda)$ of a (perfect) object in $D(\Lambda)$. It is* locally constant *(with perfect values) if there exist an étale cover $\{U_i \to X\}$ such that $K|_{U_i}$ is constant (with perfect values) for each i.*

Exercise 2. Suppose that $f : Y \to X$ is a finite étale Galois[1] morphism, and $p, q : Y \times_X Y \to Y$ the two projections. Use the fact that $f^* \circ f_* \cong p_* \circ q^*$ to show that f_*K is locally constant for any constant complex $K \in D(Y_{\mathsf{et}}, \Lambda)$.

Definition 6 ([BS, Def.6.3.1]). *A complex $K \in D(X_{\mathsf{et}}, \Lambda)$ is called* constructible *if there exists a finite sequence of closed subschemes $\varnothing = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_m = X$ such that $i^*_{Z_j \setminus Z_{j-1}} K$ is locally constant with perfect values.*

Exercise 3. Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$ show that $j_!\Lambda$ (resp. $i_*\Lambda$) is constructible. Note that $j_!$ and i_* are exact, so just like j^* and i^* they are their own derived functors. In particular, the relations $j^*j_! = \mathrm{id}$, $i^*i_* = \mathrm{id}$, $j^*i_* = 0$, $j_!i^* = 0$ continue to hold at the derived category level.

Lemma 7 ([BS, Lem.6.3.11]). *For any open immersion $j : U \to X$, the functor $j_! : D(U_{\mathsf{et}}) \to D(X_{\mathsf{et}})$ preserves constructibilty.*

Exercise 4. Let $j : U \to X$ be an open immersion. Using the adjunction $(j_!, j^*)$ and the fact that f^* preserves direct sums, show that $j_!$ preserves compact objects. That is, if $K \in D(U_{\mathsf{et}}, \Lambda)$ is compact, then for any family $L_i \in D(X_{\mathsf{et}}, \Lambda); i \in I$ we have

$$\hom(j_!K, \bigoplus L_i) \cong \bigoplus \hom(j_!K, L_i).$$

Proposition 8 ([BS, Prop.6.4.8]). *An object of $D(X_{\mathsf{et}}, \Lambda)$ is constructible if and only if it is compact.*

Remark 9. The categories $D_{cons}(X_{\mathsf{et}}, \mathbb{Z}/n)$ of constructible objects are the smallest collection of subcategories closed under (de)shift, cone, the operations $f^*, f_*, f_!, f^!$, and containing each $\mu_n^{\otimes r}$.

[1] I.e., $Y \times_X Y \cong \amalg_G Y$

4 Six operations

In this section we assume that all schemes are $\mathbb{Z}[\frac{1}{n}]$-schemes.

As in the curves case, we define $f_!$ as $\overline{f} \circ j_!$ for some factorisation into an open immersion $Y \xrightarrow{j} \overline{Y}$ and a proper morphism $\overline{f} : \overline{Y} \to X$. However, in dimension higher than one, such factorisations are highly non-cannonical, so we must check that this is well-defined. This uses the first Proper Base Change theorem. The functor $f^!$ is then defined as the right adjoint. In the case that f is smooth of constant relative dimension d, we can identify $f^!$ as $(f^* -) \otimes \mu_n^{\otimes d}[2d]$.

Theorem 10 (Grothendieck, [BS, Thm.6.7.1], [SGA§XIV])**.** *Let $f : X \to Y$ be a proper morphism of schemes. Then $f_* : D(X_{\mathsf{et}}, \mathbb{Z}/n) \to D(Y_{\mathsf{et}}, \mathbb{Z}/n)$ preserves constructibility.*

Theorem 11 (Proper Base Change I [SGA73, §XII.Thm.5.1])**.** *Suppose that*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{7}$$

is a cartesian square with f proper. Then for any object K of $D(X, \mathbb{Z}/n)$, the canonical morphism is an isomorphism.

$$g^* f_* K \xrightarrow{\sim} f'_* g'^* K.$$

Theorem 12 ([SGA73, §XVI])**.** *Suppose that* (7) *is a cartesian square with f finite type, g smooth. Then for any object K of $D(X, \mathbb{Z}/n)$, the canonical morphism is an isomorphism.*

$$g^* f_* F \xrightarrow{\sim} f'_* g'^* F.$$

Exercise 5. Prove the base change theorem(s) when f is a closed immersion and g is the open complement.

Definition 13. *Let $f : Y \to X$ be a separated finite type morphism. Choose a factorisation $f = \overline{f} \circ j$ where $j : Y \to \overline{Y}$ is a dense open immersion and $\overline{f} : \overline{Y} \to X$ is proper. Define*

$$f_! := \overline{f}_* \circ j_!.$$

Lemma 14. *The above definition is well-defined.*

Proof. Given two compactifications $Y \xrightarrow{j} \overline{Y}_1 \xrightarrow{\overline{f}} X$ and $Y \to \overline{Y}_2 \to X$, the closure $\overline{Y}_3$ of Y in fibre product $\overline{Y}_1 \times_X \overline{Y}_2$ induces a third compactification $Y \xrightarrow{k} \overline{Y}_3 \xrightarrow{\overline{g}} X$. So it suffices to show that $\overline{f}_* j_! = \overline{g}_* k_!$.

Note that $\overline{Y}_3 \to \overline{Y}_\epsilon$ is proper[2] (for $\epsilon = 1, 2$). But since j, k are dense, $\overline{Y}_3 \to \overline{Y}_\epsilon$ sends generic points isomorphically to generic points, so $Y \times_{\overline{Y}_\epsilon} \overline{Y}_3 \to Y$

[2]This follows from the basic fact that if $\xrightarrow{u}\xrightarrow{v}$ are composeable morphisms of schemes with v separated and vu proper, then u is also proper.

is a proper birational morphism of schemes with a section. This implies it is an isomorphism (cf. Exercise 6). So we have cartesian squares

$$\begin{array}{ccccc} Y & \xrightarrow{j} & \overline{Y}_3 & \xleftarrow{i} & \overline{Y}_3 \setminus Y \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \theta} \\ Y & \xrightarrow[k]{} & \overline{Y}_1 & \xleftarrow[h]{} & \overline{Y}_1 \setminus Y \end{array}$$

Consider the canonical morphism

$$k_! \to k_! j^* j_! = k_! \operatorname{id}_* j^* j_! \overset{PBC}{\cong} k_! k^* \pi_* j_! \to \pi_* j_!.$$

It suffices to show that this is an isomorphism. A morphism in $D(\overline{Y}_{1\mathsf{et}}, \Lambda)$ is an isomorphism if and only if it is an isomorphism after applying k^* and h^*. Applying k^* gives

$$\operatorname{id} \cong k^* k_! \to k^* \pi_* j_! \overset{PBC}{\cong} j^* j_! \cong \operatorname{id}$$

and applying h^* gives

$$0 = h^* k_! \to h^* \pi_* j_! \overset{PBC}{\cong} \theta_* i^* j_! = 0. \qquad \square$$

Exercise 6. Suppose that $f : W \to V$ is a proper morphism which maps generic points of W isomorphically to generic points of V, and such that there is a morphism $s : V \to W$ with $f \circ s = \operatorname{id}_V$. Show that f is an isomorphism. (Hint: s is automatically proper, and any proper immersions is a closed immersion.)

Theorem 15 (Proper Base Change II. [BS, Prop.6.7.10])**.** *Suppose that* (7) *is a cartesian square with f separated and finite type. Then for any object K of $D(X_{\mathsf{et}}, \mathbb{Z}/n)$, the canonical morphism is an isomorphism.*

$$g^* f_! K \xrightarrow{\sim} f'_! g'^* K.$$

Proof. From the definition of $f_!, f'_!$, Proper Base Change I, and the case that f, f' are open immersions. For the open immersion case, it suffices to check the isomorphism on the (underived) sheaf categories. Note all functors in question are left adjoints, so it suffices to check that the adjoints are isomorphic $f_* g^* \cong g'^* f'_*$. But by definition, these are composition with the functors

$$\begin{array}{ccc} \operatorname{Et}(X') & \longrightarrow & \operatorname{Et}(X) \\ {\scriptstyle -\times_{Y'} X'} \uparrow & & \uparrow {\scriptstyle -\times_Y X} \\ \operatorname{Et}(Y') & \longrightarrow & \operatorname{Et}(Y) \end{array}$$

So the result follows from the identity $- \times_{Y'} X' = - \times_{Y'} Y' \times_Y X \cong - \times_Y X$. $\square$

Proposition 16 (SGA, §XVIII, Thm.3.1.4)**.** *Let f be a finite type morphism. The functor $f_!$ admits a right adjoint, $f^!$.*

Theorem 17 (SGA, §XVIII, Thm.3.2.5)**.** *If f is smooth of relative dimension d, then $f^! \cong (f^* -) \otimes \mu_n^{\otimes d}[2d]$.*

Remark 18. Note that any quasi-projective morphism $f : Y \to X$ factors (by definition) as a closed immersion $Y \xrightarrow{i} P$ and a smooth morphism $P \xrightarrow{p} X$. The above theorem allows us to calculate $f^!$ as $i^! \circ (p^*(-) \otimes \mu_n^{\otimes d}[2d])$ where in the derived world, $i^! = i^* Cone(\mathrm{id} \to j_* j^*)[-1]$ for j the open complement.

5 $\mathbb{Q}_l$-coefficients

Recall that the original motivation for étale cohomology was the Weil conjectures. We wanted a collection of vector spaces over a characteristic zero field with certain nice properties. In particular, for any curve C, the first cohomology group $H^1(C, \mathbb{Q})$ should be a vector space of dimension $2g$, the genus of the curve.

Theorem 19 (Serre)**.** *There is no Weil cohomology theory for smooth projective varieties over $\mathbb{F}_{p^2}$ such that $\dim_{\mathbb{Q}} H^1(C, \mathbb{Q}) = 2$ for every curve of genus 1.*

Proof. Recall that an elliptic curve E is a curve of genus 1, and an elliptic curve over a field of positive characteristic p is called *supersingular* if it has no p-torsion points. In this case, $End(E)_{\mathbb{Q}}$ is a quaternion algebra, and in particular, $\dim_{\mathbb{Q}} End(E)_{\mathbb{Q}} = 4$, and all non-zero elements are invertible. Supersingular elliptic curves exist over every field $\mathbb{F}_{p^2}$.

Since all elements of $End(E)_{\mathbb{Q}}$ are invertible, every homomorphism to a nonzero ring must be injective. On the other hand, since $H^1(-, \mathbb{Q})$ is functorial, we have a canonical morphism $End(E)_{\mathbb{Q}} \to H^1(E, \mathbb{Q})$. But if $\dim_{\mathbb{Q}} H^1(E, \mathbb{Q}) = 2$, then $End(H^1(E, \mathbb{Q})) \cong M_{2\times 2}(\mathbb{Q})$. Since $\dim_{\mathbb{Q}} M_{2\times 2}(\mathbb{Q}) = 4 = \dim_{\mathbb{Q}} End(E)_{\mathbb{Q}}$, the morphism must be an isomorphism, which is a contradiction because there are nonzero noninvertible elements of $M_{2\times 2}(\mathbb{Q})$. □

Consequently, $H^r_{\mathsf{et}}(-, \mathbb{Q})$ is no good (for the Weil conjectures). However, we saw that $H^r_{\mathsf{et}}(C, \mathbb{Z}/n)$ had the "correct" ranks for smooth curves C over algebraically closed fields. It follows that for any l prime to the characteristic, $\varprojlim_m H^r_{\mathsf{et}}(C, \mathbb{Z}/l^m)$ also has the correct ranks. But these are $\mathbb{Z}_l = \varprojlim \mathbb{Z}/l^m$ modules, and so we can make them into $\mathbb{Q}_l$-vector spaces, and still have the correct ranks.

So this leads us to the following definition.

Definition 20. *For l a prime, and X a $\mathbb{Z}[\frac{1}{l}]$-scheme, define*

$$H^r_{\mathsf{et}}(X, \mathbb{Q}_l) := (\varprojlim H^r_{\mathsf{et}}(X, \mathbb{Z}/l^m)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

A problem with this definition is that since $H^r_{\mathsf{et}}(X, \mathbb{Q}_l)$ is not the cohomology of a sheaf, it is not the cohomology of Rf_*F for some sheaf F. Instead, to have access to the functors $f^*, f_*, f_!, f^!$ we must use limits of categories.

Definition 21 ([BS, Def.3.5.3]). *For a scheme X, define $\mathsf{Shv}_{\mathsf{et}}(X)^{\mathbb{N}}$ to be the category of $\mathbb{N}$-indexed projective systems in $\mathsf{Shv}_{\mathsf{et}}(X)$. So an object is a sequence of morphisms $F_\bullet = (\cdots \to F_2 \to F_1)$ and a morphism $F_\bullet \to F'_\bullet$ is a sequence of morphisms $F_m \to F'_m$ making the obvious squares commute. This is an abelian category, and we can consider its derived category $D(X_{\mathsf{et}}^{\mathbb{N}})$. Given a prime l, we write $D(X_{\mathsf{et}}, (\mathbb{Z}_l)_\bullet) \subseteq D(X_{\mathsf{et}}^{\mathbb{N}})$ for the full subcategory of those objects $(\cdots \to K_2 \to K_1)$ such that $K_m \in D(X_{\mathsf{et}}, \mathbb{Z}/l^m)$ and $K_m \otimes_{\mathbb{Z}/l^m} \mathbb{Z}/l^{m-1} \to K_{m-1}$ is a quasi-isomorphism. Here, $\otimes$ is the left derived tensor product.*

Theorem 22 (Ekedahl). *The functors $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{hom}}$ can be extended to the categories $D(X_{\mathsf{et}}, (\mathbb{Z}_l)_\bullet)$ in a sensible way.*

In the second half of this course we will see how the pro-étale topology allows us to avoid this mess, by "moving" the limits from the categories of sheaves into the underlying category of schemes.