For easy of exposition, all presheaves will be presheaves of abelian groups,[1] and all sites small.[2] All schemes are assumed to be quasi-compact quasi-separated (e.g., Noetherian), and we recall that étale morphisms are by definition locally of finite presentation (this means finite type if working only with Noetherian schemes).[3]

# 1 Presheaf adjunctions

**Definition 1.** *Suppose that $\pi : C' \to C$ is a functor. We denote the functor induced by composition as*

$$\pi_p : \mathsf{PreShv}(C) \to \mathsf{PreShv}(C'); \qquad F \mapsto F \circ \pi.$$

**Exercise 1.** If $\pi' : C'' \to C'$ and $\pi : C' \to C$ are two functors, show that $(\pi \circ \pi')_p = \pi_p \circ \pi'_p$.

**Definition 2.** *Give a presheaf $F \in \mathsf{PreShv}(C')$ and $X \in C$ define*

$$(\pi^p F)(X) = \varinjlim_{X \to \pi(Y)} F(Y)$$

*where the colimit is over the comma category $(X \downarrow F)$ whose objects are morphisms $X \to \pi(Y)$ in $C$, and $\hom(X{\to}\pi(Y), X{\to}\pi(Y')) = \{f : Y \to Y' \text{ s.t. the}$ triangle $\begin{smallmatrix} \pi(Y) \longrightarrow \pi(Y') \\ \nwarrow \qquad \nearrow \\ X \end{smallmatrix}$ commutes $\}$.*

**Remark 3.** There is also a right adjoint to $\pi_p$ defined in an analogous way, but we will not use it.

**Exercise 2.** Using the universal property of the colimit, show that a morphism $X \to X'$ in $C$ induces a morphism $(\pi^p F)(X') \to (\pi^p F)(X)$, and that this makes $\pi^p F$ into a presheaf on $C$.

**Exercise 3.** Using the universal property of the colimit, show that there is a canonical isomorphism $\hom_{\mathsf{PreShv}(C)}(\pi^p F, G) \cong \hom_{\mathsf{PreShv}(C')}(F, \pi_p G)$.

**Corollary 4.** *The pair $(\pi^p, \pi_p)$ is an adjunction $\mathsf{PreShv}(C) \rightleftarrows \mathsf{PreShv}(C')$.*

**Exercise 4.** Using Exercise 1, Corollary 4, and the uniqueness properties of adjunctions show that $\pi'^p \circ \pi^p = (\pi \circ \pi')^p$.

**Exercise 5.** Suppose that $f : Y \to X$ is a morphism of topological spaces, and let $\pi : \mathrm{Op}(X) \to \mathrm{Op}(Y); U \mapsto f^{-1}U$ be the induced functor between the categories of open subsets of $X, Y$. Show that $\pi_p$ is the usual push-forward $\mathsf{PreShv}(Y) \to \mathsf{PreShv}(X)$ and $\pi^p$ is the usual inverse image of presheaves functor $\mathsf{PreShv}(X) \to \mathsf{PreShv}(Y)$.

---

[1] Sheaves of sets work just as well.

[2] This is to ensure that the colimits defining $\pi^p$ are well-defined. In practice, there are many functors between large categories for which these colimits are still well-defined.

[3] These finiteness assumptions ensure that $\mathrm{Et}(X)$ are small (enough) categories (so that the left adjoints $\pi^p$ existence is guaranteed), but otherwise are basically only used in the proof of Proposition 12.

**Exercise 6.** Suppose that the category $C$ has a final object $X$, and let $\pi : * \to C$ be the functor from the category with one morphism which sends the unique object to $X$. Show that $\pi_p$ is the global sections functor $F \mapsto F(X)$, and $\pi^p$ is the constant presheaf functor $(\pi^p A)(U) = A$ for $A \in \mathrm{Ab} = \mathsf{PreShv}(*)$.

**Exercise 7.** Let $Y \to X$ be an étale morphism of schemes, and consider the functors

$$\pi : \mathrm{Et}(X) \to \mathrm{Et}(Y); \qquad U \mapsto Y \times_X U \tag{1}$$

and

$$\gamma : \mathrm{Et}(Y) \to \mathrm{Et}(X); \qquad (V \to Y) \mapsto (V \to Y \to X) \tag{2}$$

Show that $(\gamma, \pi)$ is an adjunction. Show that $\gamma_p = \pi^p$.

## 2 Sheaf adjunctions

**Definition 5.** *Suppose that $C', C$ are sites, i.e., categories equipped with Grothendieck topologies. A functor $\pi : C' \to C$ is called* continuous *if for every sheaf $F$ on $C$, the presheaf $\pi_p F$ is a sheaf.*

**Exercise 8.** Suppose $\pi : C' \to C$ sends fibre products to fibre products. Show that $\pi$ is continuous if it sends covers to covers.

**Example 6.** If $Y \to X$ is a morphism topological spaces then the induced morphism of sites $\mathrm{Op}(X) \to \mathrm{Op}(Y)$ is continuous.

**Example 7.** If $f : Y \to X$ is a morphism of schemes, then $\pi$ from Equation 1 is continuous. If $f$ is an étale morphism of schemes then $\gamma$ from Equation 2 is also continuous.

**Definition 8.** *Suppose $\pi : C' \to C$ is a continuous morphism of sites. The induced functor on sheaves is denoted*

$$\pi_* : \mathsf{Shv}(C) \to \mathsf{Shv}(C').$$

*The composition of $\pi^p$ with sheafification $a : \mathsf{PreShv}(C) \to \mathsf{Shv}(C)$ is denoted*

$$\pi^* = a \circ \pi^p : \mathsf{Shv}(C') \to \mathsf{Shv}(C).$$

**Exercise 9.** Suppose we are in the situation of Definition 8. Using the fact that sheafification $a : \mathsf{PreShv}(C) \to \mathsf{Shv}(C)$ is the left adjoint to the canonical inclusion $\iota : \mathsf{Shv}(C) \to \mathsf{PreShv}(C)$, show that

$$\pi^* : \mathsf{Shv}(C) \rightleftarrows \mathsf{Shv}(C') : \pi_*$$

is an adjunction.

**Exercise 10.** Using Exercise 1 and Exercise 4, show that if $C, C', C''$ are equipped with Grothendieck topologies, and $\pi, \pi'$ are continuous, then $(\pi \circ \pi')_* = \pi_* \circ \pi'_*$ and $\pi'^* \circ \pi^* = (\pi \circ \pi')^*$.

**Definition 9.** *If $f : Y \to X$ is a morphism of schemes, and $\pi$ the pullback functor from Equation (1), we write*

$$f^* := \pi^*, \qquad f_* := \pi_*.$$

*If $f$ is étale, so $\pi$ has a left adjoint $\gamma$ from Equation 2 then we write*

$$f_! := \gamma^*$$

*Note that since $\gamma_* = \pi^*$, cf. Exercise 7, in addition to the adjunction $(f^*, f_*)$, we have another adjunction $(f_!, f^*)$.*

**Lemma 10.** *Let $f : Y \to X$ be a morphism of schemes. Then $f^*$ preserves exact sequences.*

*Proof.* It automatically commutes with colimits because it is a left adjoint. On the other hand, $\pi^p$ commutes with finite limits because limits of presheaves are calculated object wise, and $\pi^p$ is defined using filtered colimits, which commute with finite limits. To deduce that $f^* = \pi^*$ commutes with finite limits from $\pi^p$ commuting, we just recall that sheafification is exact, so $\pi^* = a \circ \pi^p$ is a composition of two functors which commute with finite limits. $\square$

**Lemma 11.** *Let $f : Y \to X$ and $X' \to X$ be morphisms of schemes. Let $h'_X$ denote the étale sheaf of sets $h_{X'} = \hom_X(-, X') \in \mathsf{Shv}_{\mathsf{et}}(X)$, and similarly, $h_{Y \times_X X'} = \hom_Y(-, Y \times_X X') \in \mathsf{Shv}_{\mathsf{et}}(Y)$. We have*

$$f^* h_{X'} = h_{Y \times_X X'}.$$

*Proof.* By Yoneda, it suffices to produce isomorphisms

$$\hom_{\mathsf{Shv}}(f^* h_{X'}, F) \cong \hom_{\mathsf{Shv}}(h_{Y \times_X X'}, F)$$

for each $F \in \mathsf{Shv}_{\mathsf{et}}(Y)$, which are natural in $F$. But we have

$$
\begin{aligned}
\hom_{\mathsf{Shv}}(f^* h_{X'}, F) &\cong \hom_{\mathsf{Shv}}(h_{X'}, f_* F) && \text{adjunction} \\
&\cong (f_* F)(X') && \text{Yoneda} \\
&\cong F(Y \times_X X') && \text{definition} \\
&\cong \hom_{\mathsf{Shv}}(h_{Y \times_X X'}, F) && \text{Yoneda.}
\end{aligned}
$$

$\square$

**Exercise 11.** Using the same argument as in Lemma 11 show that if $f : Y \to X$ is an étale morphism of schemes, and $Y' \to Y$ any morphism then

$$f_! h_{Y'} = h_{Y'}$$

where the left $Y'$ is considered as a $Y$-scheme, and the right one as an $X$-scheme.

# 3 Immersions

**Exercise 12.** Suppose that $j : U \to X$ is an open immersion. Show that in this case, $\gamma : \mathrm{Et}(U) \to \mathrm{Et}(X)$ from Equation 2 is the inclusion of a *full* subcategory. Show that since this subcategory is full, the functor $j^* : \mathsf{Shv}(X) \to \mathsf{Shv}(U)$ is none-other-than the restriction functor

$$j^* F = F|_{\mathrm{Et}(U)}.$$

Show that $j_! : \mathsf{Shv}(U) \to \mathsf{Shv}(X)$ is "extension by zero" in the sense that for any $H \in \mathsf{Shv}(U)$,

$$(j_! H)(V) = \begin{cases} H(V) & V \in \mathrm{Et}(U) \\ 0 & V \notin \mathrm{Et}(U) \end{cases}$$

Show that

$$(j_* H)(V) = H(V) \text{ if } V \in \mathrm{Et}(U),$$

but give an example where $V \notin \mathrm{Et}(U)$, and $(j_* H)(V) \neq 0$.

Deduce that

$$j^* j_! = \mathrm{id} = j^* j_*.$$

**Exercise 13.** Let $X$ be a scheme, and $\iota : \overline{x} \to X$ a geometric point. Show that

$$\iota^* F = F_{\overline{x}}$$

is the stalk of $F$ at $\overline{x}$, where we implicitly use the identification $\mathsf{Shv}(\overline{x}) \cong \mathrm{Ab}$.

**Proposition 12** (Milne Cor.II.3.5). *Let $i : Z \to X$ be the inclusion of a closed immersion, $\overline{x} \to X$ a geometric point, and $G \in \mathsf{Shv}(Z)$. Then*

$$(i_* G)_{\overline{x}} = \begin{cases} G_{\overline{x}} & \mathrm{im}(\overline{x}) \in Z \\ 0 & \mathrm{im}(\overline{x}) \notin Z \end{cases}$$

*If $j : U \to X$ is the open complement of $Z$, then we have $j^* i_* = 0$.*

*Easy parts of the proof (Omitted from the lecture).* The second claim follows from the first claim, since a sheaf is zero if and only if all its stalks are zero, and the stalks of $j^* i_*$ are all zero by Exercises 10, 12, and 13.

Certainly, if $\mathrm{im}(\overline{x}) \notin Z$, then $(i_* G)_{\overline{x}} = 0$, since every $\overline{x} \to V \to X$ is refinable by some $\overline{x} \to U \times_X V \to X$, and $(i_* G)(U \times_X V) = G(Z \times_X U \times_X V) = G(\varnothing) = 0$.

The difficult part is to show that for any $\overline{x} \to V \to Z$, there is some $\overline{x} \to V' \to X$ and a factorisation $\overline{x} \to Z \times_X V' \to V \to Z$. If we know this, then the system defining $(i_* G)_{\overline{x}}$ is cofinal in the system defining $G_{\overline{x}}$.

The proof of this claim is omitted. See Milne Thm.II.3.2(b) for details. Really, check it out. Its a very neat argument using properties of limit schemes from EGA, in particular, EGA IV, Part 3, Cor.8.13.2. $\qquad\square$

**Exercise 14.** Suppose that $F \in \mathsf{Shv}_{\mathsf{et}}(U)$ is a constant sheaf (that is, $F(V) = A$ for connected $V$, for some abelian group $A$). If $i : Z \to X$ is a nowhere dense closed immersion with open complement $j : U \to X$, using the fact that étale morphisms send generic points to generic points, show that $j_* F$ and $i^* j_* F$ are constant sheaves on $X$ and $Z$ respectively.

# 4 The localistion sequences

**Definition 13.** *Let $i : Z \to X$ be a closed immersion of schemes, and $j : U \to X$ the open complement. Define $T(X)$ to be the category[4] whose objects are triples $(F_1, F_2, \phi)$ consisting of $F_1 \in \mathsf{Shv}(Z), F_2 \in \mathsf{Shv}(U)$, and a morphisms $\phi : F_1 \to i^* j_* F_2$. Morphisms $(F_1, F_2, \phi) \to (F_1', F_2', \phi')$ are pairs of morphisms $(F_1 \xrightarrow{\psi_1} F_1', F_2 \xrightarrow{\psi_2} F_2')$ such that the square commutes.*

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\phi} & i^* j_* F_2 \\
\psi_1 \downarrow & & \downarrow i^* j_* \psi_2 \\
F_1' & \xrightarrow{\phi'} & i^* j_* F_2'
\end{array}
$$

*We define a functor $t : \mathsf{Shv}(X) \to T(X)$ by*

$$
F \mapsto (i^* F, \qquad j^* F, \qquad i^*(F \xrightarrow{\eta} j_* j^* F))
$$

*where $\eta : \mathrm{id} \to j_* j^*$ is the unit of the adjunction $(j^*, j_*)$.*

**Theorem 14** (Milne Thm.II.3.10). *The functor $t : \mathsf{Shv}(X) \to T(X)$ is an equivalence of categories.*

*Proof.* Given a triple $(F_1, F_2, \phi)$ in $T(X)$ define

$$
s(F_1, F_2, \phi) := \ker\left( i_* F_1 \oplus j_* F_2 \xrightarrow{i_* \phi \,+\, \eta} i_* i^* j_* F_2 \right).
$$

Here, $\eta : \mathrm{id} \to i_* i^*$ is the unit of the adjunction $(i^*, i_*)$. Notice that this defines a functor

$$
s : T(X) \to \mathsf{Shv}(X).
$$

So it suffices to check that $st \cong \mathrm{id}$ and $ts \cong \mathrm{id}$. Consider $stF$. By definition, this is

$$
stF = \ker\left( i_* i^* F \oplus j_* j^* F \xrightarrow{i_* \phi \,+\, \eta} i_* i^* j_* j^* F \right).
$$

This comes equipped with a canonical morphism $F \to stF$, and so we are trying to show that the sequence

$$
0 \to F \to i_* i^* F \oplus j_* j^* F \xrightarrow{i_* \phi \,+\, \eta} i_* i^* j_* j^* F
$$

is exact. One can check exactness on stalks, so consider a geometric point $\overline{x} \to X$. If $\mathrm{im}(\overline{x}) \in U = X \setminus Z$ then our sequence becomes

$$
0 \to F_{\overline{x}} \to 0 \oplus F_{\overline{x}} \longrightarrow 0.
$$

---

[4]This is just the comma category $(\mathsf{Shv}(Z) \downarrow i^* j_*)$.

If $\mathrm{im}(\overline{x}) \in Z = X \setminus U$ then our sequence becomes

$$0 \to F_{\overline{x}} \to F_{\overline{x}} \oplus (j_* j^* F)_{\overline{x}} \longrightarrow (j_* j^* F)_{\overline{x}}.$$

Hence, we have confirmed exactness, and $F \overset{\sim}{\to} stF$.

Now consider $ts(F_1, F_2, \phi)$. We have

$$i^* s(F_1, F_2, \phi) = i^* \ker\left( i_* F_1 \oplus j_* F_2 \longrightarrow i_* i^* j_* F_2 \right)$$

$$= \ker\left( i^* i_* F_1 \oplus i^* j_* F_2 \longrightarrow i^* i_* i^* j_* F_2 \right)$$

$$= \ker\left( F_1 \oplus i^* j_* F_2 \longrightarrow i^* j_* F_2 \right)$$

$$= F_1$$

One similarly checks that $j^* s(F_1, F_2, \phi) \cong F_2$, and that the canonical morphism $i^* s(F_1, F_2, \phi) \to i^* j_* j^* s(F_1, F_2, \phi)$ is none-other-than $\phi$, under these identifications. Hence, $ts(F_1, F_2, \phi) = (F_1, F_2, \phi)$. $\qquad \square$

**Theorem 15** (Milne Prop.II.3.14). *Its possible to define six functors*

$$\mathsf{Shv}_{\mathsf{et}}(Z) \xrightarrow[\underset{i^!}{\overset{i_*}{\longrightarrow}}]{\overset{i^*}{\longleftarrow}} \mathsf{Shv}_{\mathsf{et}}(X) \xrightarrow[\underset{j_*}{\overset{j^*}{\longrightarrow}}]{\overset{j_!}{\longleftarrow}} \mathsf{Shv}_{\mathsf{et}}(U)$$

*such that under the identification $\mathsf{Shv}_{\mathsf{et}}(X) \cong T(X)$, they correspond to:*

$$
\begin{array}{rclcrcl}
F_1 & \leftarrow\!\shortmid & (F_1, F_2, \phi) & & (0, F_2, 0) & \shortmid\!\leftarrow & F_2 \\
F_1 & \mapsto & (F_1, 0, 0) & & (F_1, F_2, \phi) & \mapsto & F_2 \\
\ker(\phi) & \shortmid\!\leftarrow & (F_1, F_2, \phi) & & (i^* j_* F_2, F_2, \mathrm{id}) & \shortmid\!\leftarrow & F_2
\end{array}
$$

1. *Each functor is left adjoint to the one below it.*

2. *The functors $i^*, i_*, j^*, j_!$ preserve exact sequences; $j_*, i^!$ preserve monomorphisms.*

3. *The composites $i^* j_!, i^! j_!, i^! j_*, j^* i_*$ are zero.*

4. *The functors $i_*, j_*$ are fully faithful.*

5. *The functors $j_*, j^*, i^!, i_*$ map injective objects to injective objects.*

**Remark 16.** Heuristicaly, $j_*$ "fills in the gaps" over $Z$ in a canonical way, and $i^!$ isolates the part of $F$ which cannot be recovered from $F|_{\mathrm{Et}(U)}$ by this "filling in the gaps" process.

**Exercise 15** (Not advanced). Prove Theorem 15 using what we have seen so far. Note that if a functor has a left adjoint preserving monomorphisms then it preserves injectives.

**Exercise 16** (Not advanced)**.** In the situation of Theorem 15 show that there are short exact sequence

$$0 \to j_! j^* \to \mathrm{id} \to i_* i^* \to 0$$

$$0 \to i_* i^! \to \mathrm{id} \to j_* j^* \to 0$$

**Remark 17.** Sometimes one defines $j^* := j^!$ and $i_! := i_*$ so that the short exact sequences can be written as

$$0 \to j_! j^! \to \mathrm{id} \to i_* i^* \to 0$$

$$0 \to i_! i^! \to \mathrm{id} \to j_* j^* \to 0$$

# 5    Curves

**Example 18** (Milne, Exam.II.3.12)**.** Let $A$ be a discrete valuation ring (e.g., $\mathbb{C}[[z]], \mathbb{F}_p[[z]], \mathbb{Z}_p, \dots$). Let

1. $K = \mathrm{Frac}(A)$,

2. $k = A/\mathfrak{m}$,

3. $G_K = Gal(K^{sep}/K)$,

4. $G_k = Gal(k^{sep}/k)$,

Since $A$ is a discrete valuation ring, $X = \mathrm{Spec}(A)$ has one open point, and one closed point. Let $U = \mathrm{Spec}(K), Z = \mathrm{Spec}(k)$ be the corresponding open and closed subschemes. Recall that the category of étale sheaves over a field is equivalent to the category of discrete Galois modules. That is, $\mathsf{Shv}_{et}(Z) \cong G_k$-mod and $\mathsf{Shv}_{et}(U) \cong G_K$-mod. We can give an analogous description of $\mathsf{Shv}_{et}(X)$ using a similar construction to $T(X)$. It suffices to work out what functor $G_K$-mod $\to G_k$-mod corresponds to $i^* j_* : \mathsf{Shv}_{et}(U) \to \mathsf{Shv}_{et}(Z)$.

Let $A^h$ be the henselisation of $A$, and $A^{sh}$ a strict henselisation. Since $K^{sep}$ is separable closed, there are factorisations $A \to A^h \to A^{sh} \to K^{sep}$ which are actually inclusions. The choice of $A^{sh}$ and the inclusion define subgroups $I = \mathrm{Gal}(K^{sep}/\mathrm{Frac}(A^{sh}))$ and $D = \mathrm{Gal}(K^{sep}/\mathrm{Frac}(A^h))$, with $I \subseteq D \subseteq G_K$, and it turns out that $D/I$ is canonically isomorphic to $Gal(k^{sep}/k)$ where we identify $k^{sep} = A^{sh}/\mathfrak{m}_{A^{sh}}$. Then we claim that the functor $i^* j_* : \mathsf{Shv}_{et}(U) \to \mathsf{Shv}_{et}(Z)$ corresponds to the functor of $I$-invariants.

$$(-)^I : G_K\text{-mod} \to G_k\text{-mod}.$$

Hence, the category $\mathsf{Shv}_{et}$ is equivalent to the category of triples $(M_1, M_2, \phi)$ where $M_1 \in G_k$-mod, $M_2 \in G_K$-mod, and $\phi : M_1 \to M_2$ is compatible with the actions of $G_k \cong D/I$ and $G_K$.

**Example 19.** Example 18 can be generalised to any normal curve, see Milne Exer.II.3.16 for details.