

1 Sheaves

Definition 1. A (Grothendieck) topology on a category \mathcal{C} is the data of: for every object $U \in \mathcal{C}$, a collection of families of morphisms $\{U_i \rightarrow U\}_{i \in I}$. The families in these collections are called coverings of U . This data is required to satisfy the following axioms:

1. $\{U \xrightarrow{\text{id}} U\}$ is a covering, for every object U .
2. If $\{U_i \rightarrow U\}_{i \in I}$ is a covering of U , and $V \rightarrow U$ is a morphism, then each fibre product $U_i \times_U V$ exists, and $\{U_i \times_U V \rightarrow V\}_{i \in I}$ is a covering of V .
3. If $\{U_i \rightarrow U\}_{i \in I}$ is a covering of U , and for each $i \in I$ we have a covering $\{U_{ij} \rightarrow U_i\}_{j \in J_i}$ of U_i , then $\{U_{ij} \rightarrow U\}_{i \in I, j \in J_i}$ is a covering of U .

Exercise 1. Suppose that X is a topological space in the conventional sense.¹ Define $Op(X)$ to be the category whose objects are open sets of X , and morphisms are inclusions. For $U \in Op(X)$, define the coverings of U to be the families $\{U_i \rightarrow U\}_{i \in I}$ such that $\cup_{i \in I} U_i = U$. Show that this defines a Grothendieck topology on $Op(X)$. (Note that in this category $V \times_U W = V \cap W$.)

Exercise 2. Let X be a topological space, and define $LH(X)$ to be the category whose objects are local homeomorphisms² $Y \rightarrow X$ and morphisms are commutative triangles $\begin{array}{ccc} & Y' \rightarrow Y & \\ & \searrow & \swarrow \\ & X & \end{array}$. Show that this category has fibre products.

For $Y \in LH(X)$, define the coverings of Y to be the families $\{f_i : Y_i \rightarrow Y\}_{i \in I}$ such that $\cup_{i \in I} f_i(Y_i) = Y$. Show that this defines a Grothendieck topology on $LH(X)$.

Exercise 3. Recall that a morphism $f : Y \rightarrow X$ of schemes is étale if it is locally of finite presentation, and for every $y \in Y$, the ring morphism $\mathcal{O}_{X, f(x)} \rightarrow \mathcal{O}_{Y, y}$ is étale. Let $Et(X)$ denote the category whose objects are étale morphisms $Y \rightarrow X$, and morphisms are commutative triangles. Do Exercise 2 with $Et(X)$ instead of $LH(X)$.

Definition 2. A presheaf F on a category \mathcal{C} is just a functor $\mathcal{C}^{op} \rightarrow \text{Set}$. A morphism of presheaves $F \rightarrow G$ is just natural transformation of functors $F \rightarrow G$.

Definition 3. If \mathcal{C} is equipped with a Grothendieck topology, then a presheaf F is called a sheaf if for any object U and any covering $\{U_i \rightarrow U\}_{i \in I}$ we have

$$F(U) = \text{eq} \left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i, j \in I} F(U_i \times_U U_j) \right). \quad (\text{III})$$

¹I.e., a set equipped with a collection of subsets of X declared to be *open*, preserved by finite intersection, arbitrary union, and containing X and \emptyset .

²A morphism $f : Y \rightarrow X$ is a local homeomorphism if for every point $y \in Y$, there is an open neighbourhood $V \ni y$ such that $f : V \rightarrow f(V)$ is a homeomorphism.

A morphism of sheaves is just a morphism of presheaves. A sheaf on $Et(X)$ for some scheme X is called an étale sheaf on X .

Example 4. We have the following important examples of étale sheaves on $Et(k)$, cf. Exercise 10.

1. $\mathcal{O} : L \mapsto (L, +)$.
2. $\mathcal{O}^* : L \mapsto (L^*, *)$.
3. $\mu_n : L \mapsto \{a \in L^* : a^n = 1\}$.

Remark 5. If a presheaf takes values in the category of abelian groups, then the sheaf condition (III) is equivalent to asking that the sequence

$$0 \rightarrow F(U) \rightarrow \prod_{i \in I} F(U_i) \rightarrow \prod_{i, j \in I} F(U_i \times_U U_j)$$

be exact, where the last morphism is the difference of the two morphisms induced by the two projections $U_i \times_U U_j \rightrightarrows U_i, U_j$.

Exercise 4. Let X be a topological space in the conventional sense. Consider the Grothendieck topology defined on $Op(X)$ in Exercise 1. Show that a presheaf on X is the same thing as a presheaf on $Op(X)$, and a presheaf on X is a sheaf if and only if its associated presheaf on $Op(X)$ is a sheaf. That is, Definition 3 is an honest generalisation of the classical notion of a sheaf.

Exercise 5. Let $\text{Spec}(L) \rightarrow \text{Spec}(L')$ be a morphism in $Et(k)$ such that L/L' is Galois with Galois group $G = \text{Aut}(L/L')$. Recall that there is a canonical isomorphism

$$L \otimes_{L'} L \cong \prod_G L$$

where two morphisms $L \rightrightarrows L \otimes_{L'} L$; $a \mapsto 1 \otimes a, a \otimes 1$ are identified with $a \mapsto (a, a, \dots, a)$ and $a \mapsto (a^{g_1}, \dots, a^{g_n})$ where g_i are the elements of G . Show that if F is an étale sheaf on $\text{Spec}(k)$, then $F(\prod_G L) \cong \prod_G F(L)$, and

$$F(L') = F(L)^G$$

where $F(L)^G = \{s \in F(L) : g^*s = s \ \forall g \in G\}$. Deduce that if $F_1 \rightarrow F_2$ is a morphism of étale sheaves such that $F_1(L) \cong F_2(L)$ for every Galois extension L/k , then $F_1 \cong F_2$.

Remark 6. We will be able to show later on that a presheaf F on $Et(k)$ is a sheaf if and only if

1. $F(\coprod_{i \in I} U_i) \cong \prod_{i \in I} F(U_i)$ for any collection $U_i, i \in I$, and
2. $F(L') = F(L)^{\text{Aut}(L/L')}$ for every Galois extension L/L' .

Theorem 7 (cf. Milne, Thm. II.1.9). *Suppose that k is a field, k^{sep}/k is a separable closure, and $G = Gal(k^{sep}/k)$. Then there is a canonical equivalence between the category G -set of discrete³ G -sets⁴ and the category $\text{Shv}(Et(\text{Spec}(k)))$ of étale sheaves on k .*

Proof. For $F \in \text{Shv}(Et(\text{Spec}(k)))$ we define

$$X_F = \varinjlim_{k^{sep}/L/k} F(\text{Spec}(L)) \quad (1)$$

as the colimit over all subfields L of k^{sep} which are finite Galois extensions of k .

X_F is a discrete G -set. For any Galois L/k and any $\sigma \in G$ we have $\sigma(L) = L$ so σ restricts to a (finite) automorphism of L/k (and hence an automorphism of $F(L)$) via the canonical map $G \rightarrow Gal(L/k) \cong G/Fix(L)$ where $Fix(L) = \{g \in G : g(a) = a \forall a \in L\}$. These actions inclusions $L' \subseteq L$ (and hence, the morphisms $F(L') \rightarrow F(L)$), hence we get an action of G on X_F . Moreover, every $x \in X_F$ is the image of some $x' \in F(\text{Spec}(L))$, so X_F is a discrete G -set. The assignment $F \mapsto X_F$ is clearly natural in F , that is, it defines a functor.

For future reference, we note that since F is an étale sheaf, for each extension L'/L , the morphism $F(L) \rightarrow F(L')$ is injective, and moreover, for any two Galois extensions $L'/L/k$ of k , by Exercise 5 we have $F(L) = F(L')^{Aut(L'/L)} = F(L')^{Fix(L)}$. Since the action of G commutes with the colimit (1), we get

$$X_F^{Fix(L)} = \varinjlim_{k^{sep}/L'/L/k} F(L')^{Fix(L)} = \varinjlim_{k^{sep}/L'/L/k} F(L) = F(L). \quad (2)$$

Now suppose we have a discrete G -set X , we define a presheaf on $Et(\text{Spec}(k))$ as

$$F_X(U) = \text{hom}_G \left(\text{hom}_{\text{Spec}(k)}(\text{Spec}(k^{sep}), U), X \right),$$

where hom_G means G -equivariant morphisms, and $G = Gal(k^{sep}/k) = \text{hom}_k(k^{sep}, k^{sep})$ acts on $\text{hom}_{\text{Spec}(k)}(\text{Spec}(k^{sep}), U)$ by composition.

F_X is an étale sheaf. Cf. Milne, Lem. I.1.8. Recall that every étale morphism $Y \rightarrow \text{Spec}(k)$ is a of the form $\coprod_{i \in I} \text{Spec}(L_i) \rightarrow \text{Spec}(k)$ for some set of finite separable field extensions $L_i, i \in I$. Clearly, as $\text{Spec}(k^{sep})$ is connected, $\text{hom}_{\text{Spec}(k)}(\text{Spec}(k^{sep}), \coprod_I U_i) = \coprod_I \text{hom}_{\text{Spec}(k)}(\text{Spec}(k^{sep}), U_i)$ and so

$$F_X(\coprod \text{Spec}(L_i)) = \prod F_X(\text{Spec}(L_i)).$$

So by Remark 6, to show F_X is a sheaf, it suffices to check that

$$F_X(L') = F_X(L)^{Aut(L/L')}$$

for finite Galois extensions L/L' . Note that for any Galois extension L/k and any subextension $L'/L/k$ we have

$$\text{hom}_k(L, k^{sep})_{Aut(L/L')} \xrightarrow{\sim} \text{hom}_k(L', k^{sep}).$$

³Here *discrete* means that for every $x \in X$, there is a finite Galois extension L/k with fixed field $Fix(L) \subseteq G$ such that $x \in X^{Fix(L)}$.

⁴That is, a set X equipped with an action of G .

Using the identification $Aut(L/L') \cong Fix(L')/Fix(L) \subseteq G/Fix(L)$, it follows from this that $F_X(L') = F_X(L)^{Aut(L/L')}$. Note that for any finite Galois subextension $k^{sep}/L/k$ we have $hom_{Spec(k)}(Spec(k^{sep}), Spec(L)) = Gal(L/k)$. So

$$F_X(L) = hom_G(Gal(L/k), X) = X^{Fix(L)}. \quad (3)$$

Combining (1) and (3) we get

$$X_{F_X} = \varinjlim_L F_X(L) = \varinjlim_L X^{Fix(L)} = X.$$

On the other hand, combining (2) and (3) we get

$$F_{X_F}(L) = X_F^{Fix(L)} = F(L)$$

for Galois extensions L/k . Then by Exercise 5 we have $F_{X_F} = F$.

So the assignments $X \mapsto F_X$ and $F \mapsto X_F$ are inverse equivalences. \square

Exercise 6 (Omitted from lecture). Suppose that F is a presheaf on a category \mathcal{C} equipped with a Grothendieck topology. Suppose that $\{U_i \rightarrow U\}_{i \in I}$ and $\{U_{ij} \rightarrow U_i\}_{j \in J_i}$ are coverings. Using the diagram

$$\begin{array}{ccccc} F(U) & \longrightarrow & \prod F(U_i) & \rightrightarrows & \prod F(U_i \times_U U_{i'}) \\ \parallel & & \downarrow & & \downarrow \\ F(U) & \longrightarrow & \prod F(U_{ij}) & \rightrightarrows & \prod F(U_{ij} \times_U U_{i'j'}) \end{array}$$

show that if F satisfies the sheaf condition (III) for $\{U_{ij} \rightarrow U\}_{i \in I, j \in J_i}$ and each $F(U_i) \rightarrow \prod_{j \in J_i} F(U_{ij})$ is injective, then F satisfies the sheaf condition for $\{U_i \rightarrow U\}_{i \in I}$.

Deduce that a presheaf F on $LH(X)$ from Exercise 2 is a presheaf if and only if $F|_{Op(Y)}$ is a sheaf on $Op(Y)$ from Exercise 1 for every $Y \in LH(X)$.

Exercise 7 (Omitted from lecture). Suppose that F is a presheaf on a category \mathcal{C} equipped with a Grothendieck topology. Suppose that $\{V \rightarrow U\}$ and $\{U \rightarrow X\}$ are coverings consisting of single morphisms. Using the diagram

$$\begin{array}{ccc} F(V \times_X V) & \longleftarrow & \prod F(U \times_X U) \\ \downarrow & \swarrow & \uparrow \\ \prod F(V \times_U V) & \longleftarrow & F(U) \\ & \swarrow & \uparrow \\ & & F(X) \end{array}$$

show that if F satisfies the sheaf condition (III) for $\{V \rightarrow U\}$ (cf. middle row) and $\{U \rightarrow X\}$ (cf. right column), and each $F(U \times_X U) \rightarrow F(V \times_X V)$ is injective (cf. top row), then F satisfies the sheaf condition for $\{V \rightarrow X\}$ (cf. diagonal).

Exercise 8 (Advanced. Omitted from lecture). Do Exercise 7 for coverings $\{U_i \rightarrow X\}_{i \in I}$ and $\{V_{ij} \rightarrow U_i\}_{j \in J_i}$ containing more than one element.

Exercise 9 (Advanced). Let X be a scheme. Deduce from Exercises 6 and Exercise 8 that a presheaf F on $Et(X)$ is a sheaf if and only if $F|_{\mathcal{O}_p(Y)}$ is a sheaf for every $Y \in Et(X)$, and F satisfies the sheaf condition (III) for every covering $\{Y_i \rightarrow Y\}_{i \in I}$ such that Y and each Y_i are affine schemes.

Exercise 10. Recall that for any faithfully flat ring morphism $A \rightarrow B$ the sequence $0 \rightarrow A \rightarrow B \rightarrow B \otimes_A B$ is exact. Deduce from this and Exercise 9 that for any scheme X and any affine scheme T , the presheaf $\text{hom}(-, T)$ is a sheaf on $Et(X)$. (Actually, its also true without the affine hypothesis, and for the category $Fppf(X)$).

Corollary 8. *The following representable presheaves are étale sheaves.*

1. $\text{hom}(-, \mathbb{A}^1); X \mapsto \Gamma(X, \mathcal{O}_X)$,
2. $\text{hom}(-, \mathbb{G}_m); X \mapsto \Gamma(X, \mathcal{O}_X^*)$,
3. $\mu_n = \text{hom}(-, \text{Spec}(\frac{\mathbb{Z}[T]}{T^n - 1})); X \mapsto \{a \in \Gamma(X, \mathcal{O}_X^*) : a^n = 1\}$,
4. $GL_n = \text{hom}(-, \text{Spec}(\frac{\mathbb{Z}[U, T_{ij} : 1 \leq i, j \leq n]}{U \cdot \det T_{ij} - 1})); X \mapsto GL_n(\Gamma(X, \mathcal{O}_X))$,

2 Sheafification

Definition 9. *A presheaf F on a category equipped with a Grothendieck topology is called separated if the morphism $F(U) \rightarrow \prod_{i \in I} F(U_i)$ is injective for every covering $\{U_i \rightarrow U\}_{i \in I}$.*

Remark 10. Every sheaf is separated.

Exercise 11. Suppose that \mathcal{C} is a category equipped with a Grothendieck topology, and let F be a presheaf. For $U \in \mathcal{C}$ define $F^s(U)$ as the quotient group

$$F^s(U) = F(U) / \bigcup \ker \left(F(U) \rightarrow \prod_{i \in I} F(U_i) \right)$$

where the union is over all covering families $\{U_i \rightarrow U\}_{i \in I}$. Show that for any morphism $V \rightarrow U$ in \mathcal{C} , there is an induced morphism $F^s(U) \rightarrow F^s(V)$, that is, F^s is a presheaf. Show that F^s is separated. Show that if $F \rightarrow G$ is any morphism from F to a separated presheaf G , there exists a unique factorisation $F \rightarrow F^s \rightarrow G$. In particular, this is true for every sheaf G .

Proposition 11. *Let \mathcal{C} be a category equipped with a Grothendieck topology. For every presheaf F on \mathcal{C} , there exists a universal morphism $F \rightarrow F^a$ to a sheaf. That is, a morphism towards a sheaf such that for any other morphism $F \rightarrow G$ towards a sheaf, there is a unique factorisation $F \rightarrow F^a \rightarrow G$.*

In other words, the (fully faithful) inclusion $\text{Shv}(\mathcal{C}) \rightarrow \text{PreShv}(\mathcal{C})$ admits a left adjoint $(-)^a : \text{PreShv}(\mathcal{C}) \rightarrow \text{Shv}(\mathcal{C})$.

Proof. By Exercise 11 it suffices to consider the case that F is separated. For $U \in \mathcal{C}$ define

$$\check{H}^0(U, F) = \varinjlim \text{eq} \left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i, j \in I} F(U_i \times_U U_j) \right).$$

Omitted from lecture: Note that this is functorial in F , and if F is a sheaf we have $\check{H}^0(U, F) = F(U)$ by the sheaf condition. It follows from this (with a little bit of work) that we get a unique factorisation $F \rightarrow \check{H}^0 F \rightarrow G$ for any sheaf G . So it suffices to show that $\check{H}^0 F$ is a sheaf. For simplicity we assume that F is a sheaf of abelian groups, and all covers have a single element. The general case is the same proof, just more confusing chasing indices around.

So suppose that $\{V \rightarrow U\}$ is a covering of U . We want to show that

$$0 \rightarrow \check{H}^0(U, F) \rightarrow \check{H}^0(V, F) \rightarrow \check{H}^0(V \times_U V, F)$$

is exact. Let $(U', s \in F(U'))$ represent an element of $\check{H}^0(U, F)$ and suppose that it gets sent to zero in V . Putting in the definitions, we see that this means that there is a refinement $V' \rightarrow V \times_U U' \rightarrow V$ of the covering $V \times_U U' \rightarrow V$ such that $s|_{V'} = 0$. But this is also a refinement of $\{U' \rightarrow U\}$, so $(U', s \in F(U'))$ and $(V', 0 \in F(V'))$ represent the same element of $\check{H}^0(U, F)$.

Showing exactness in the middle is fiddly and not very informative, so we omit it. It can be found in [Artin, Grothendieck topologies, 1962, Lemma.2.1.2(ii)]. \square

Definition 12. The sheaf F^a in Proposition 11 is called the sheafification or associated sheaf of F .

Corollary 13. Let \mathcal{C} be a category equipped with a Grothendieck topology. Then the category $\text{Shv}(\mathcal{C}, \text{Ab})$ of sheaves of abelian groups is an abelian category.

Sketch of proof. Limits (i.e., products and kernels) can be calculated sectionwise. E.g., $\ker(F \rightarrow G)(U) = \ker(F(U) \rightarrow G(U))$. Colimits (i.e., sums and cokernels) are calculated sectionwise, and then sheafified. E.g., the sheaf cokernel of $F \rightarrow G$ is the sheafification of the presheaf $U \mapsto \text{coker}(F(U) \rightarrow G(U))$. \square

3 Stalks

Definition 14. A geometric point of a scheme X is a morphism $\bar{x} \rightarrow X$ such that $\bar{x} = \text{Spec}(\Omega)$ for some separably closed field Ω .

Definition 15. Let F be a presheaf on $\text{Et}(X)$. For a geometric point $\bar{x} \rightarrow X$ we define the stalk at \bar{x} as

$$F_{\bar{x}} = \varinjlim_{\bar{x} \rightarrow Y \rightarrow X} F(Y)$$

where the colimit is over factorisations of $\bar{x} \rightarrow X$ via some $Y \in \text{Et}(X)$.

Remark 16. If F is a presheaf defined on all schemes that commutes with fiwhere $\mathcal{O}_{X,x}^{sh}$ where $x = \text{im}(\bar{x}) \in X$ and $\mathcal{O}_{X,x}^{sh}$ is the strict henselisation of $\mathcal{O}_{X,x}$ defined by the separably closed extension $k(\bar{x})/k(x)$.

Remark 17. If k^{sep}/k is a separable closure, then $\bar{x} = \text{Spec}(k^{sep}) \rightarrow \text{Spec}(k)$ is a geometric point, and $F_{\bar{x}}$ is the G -set X_F defined above.

Proposition 18. *Suppose that F is a sheaf of abelian groups on $Et(X)$ and $Y \in Et(X)$. Then a section $s \in F(Y)$ is zero if and only if for any geometric point $\bar{x} \rightarrow Y$ its image in each $F_{\bar{x}}$ is zero.*

Proof. Since all sheaves are separated, it suffices to show that for every $s \in F(Y)$, there exists a covering $\{U_i \rightarrow Y\}_{i \in I}$ such that $s|_{U_i} = 0$ for all $i \in I$. For every point $x \in Y$, choose a separable closure $k(x)^s/k(x)$, and let $\bar{x} \rightarrow X$ be the corresponding geometric point. Since $F_{\bar{x}} = 0$, the image of s in $F_{\bar{x}}$ is zero, so there is some $\bar{x} \rightarrow V \rightarrow Y$ such that $s|_V = 0$. Since V is associated to x , let us write $V_x = V$. We do this for every point $x \in Y$, and obtain a family $\{V_x \rightarrow Y\}_{x \in Y}$ of étale morphisms indexed by points of Y . Since $x \in \text{im}(V_x \rightarrow Y)$ for each $x \in Y$, the family is surjective, and therefore is a covering. By construction $s|_{Y_x} = 0$ for each Y_x , so $s = 0$. \square

Corollary 19. *A sheaf of abelian groups F on $Et(X)$ is zero if and only if $F_{\bar{x}} = 0$ for each $x \in X$.*

Proof. (Omitted from lecture). We want to show that $s = 0$ for every $Y \in Et(X)$, $s \in F(Y)$. By Proposition 18, it suffices to show that $F_{\bar{x}} = 0$ for every geometry point $\bar{x} \rightarrow Y$. We claim that $F_{\bar{x} \rightarrow Y} = F_{\bar{x} \rightarrow Y \rightarrow X}$. Indeed, there is a canonical morphism

$$F_{\bar{x} \rightarrow Y \rightarrow X} = \varinjlim_{\bar{x} \rightarrow U \rightarrow X} F(U) \rightarrow \varinjlim_{\bar{x} \rightarrow V \rightarrow Y} F(V) = F_{\bar{x} \rightarrow Y}$$

defined by sending a representative $(U, s \in F(U))$ to $(U \times_Y V, s|_{U \times_Y V})$. Injectivity is straight-forward. For surjectivity, note that any representative $(V, s \in F(V))$ of $F_{\bar{x} \rightarrow Y}$ can be considered as a representative of an element $s' \in F_{\bar{x} \rightarrow Y \rightarrow X}$. Then due to the factorisation $V \rightarrow V \times_X Y \rightarrow Y$, the image of s' is precisely the element represented by $(V, s \in F(V))$. \square

Corollary 20. *A morphism of sheaves of abelian groups $\phi : F \rightarrow G$ is a monomorphism, (resp. epimorphism, resp. isomorphism) if and only if $\phi_{\bar{x}} : F_{\bar{x}} \rightarrow G_{\bar{x}}$ is for each geometric point $\bar{x} \rightarrow X$.*

Proof. (Omitted from lecture). Since the definition of $(-)_{\bar{x}}$ is defined by a filtered colimit, it commutes with kernels and cokernels. Applying Corollary 19 to $\ker \phi$ and $\text{coker } \phi$ gives the result. \square