In this talk we compare the pro-étale site with the étale site. We see how the pro-étale site offers a technically simpler way to left complete the étale site. We also show how the pro-étale site can be used to recover the classical derived category of l-adic sheaves.

1 From étale to pro-étale

Since every étale morphism is weakly étale, we have a canonical functor

$$\nu : X_{\mathsf{et}} \to X_{\mathsf{pro\acute{e}t}}.$$

Moreover, this functor sends covering families to covering families and therefore induces an adjunction

$$\begin{aligned} \nu^* : \mathsf{Shv}(X_{\mathsf{et}}) &\rightleftarrows \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}) : \nu_* \\ (F|_{X_{et}}) &\leftarrow\!\shortmid F \end{aligned}$$

The left adjoint sends $F \in \mathsf{Shv}(X_{\mathsf{et}})$ to the *sheafification* of the presheaf

$$U \mapsto \varinjlim_{U \to V \to X} F(V) \tag{1}$$

where the colimit is over those factorisations with $V \in X_{et}$.

Lemma 1 ([Lem.5.1.1])**.** *For $F \in \mathsf{Shv}(X_{\mathsf{et}})$ and $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ with presentation $U = \varprojlim_i U_i$, we have $(\nu^* F)(U) = \varinjlim_i F(U_i)$. In other words, the presheaf* (1) *already satisfies the sheaf condition on $X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ before sheafification, and the colimit can be calculated using any presentation for U.*

Sketch of proof. It suffices to treat the case X is affine. In this case we have $\mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}) \cong \mathsf{Shv}(X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}})$, [Lem.4.2.4]. Now we use the lemma that we mentioned last time, that a presheaf is a sheaf if and only if it satisfies the sheaf condition for Zariski covers, and affine pro-étale morphisms. Both of these kind of covers (up to refinement) descend through filtered colimits. For example, if $B = \varinjlim B_i$ is an ind-étale algebra and $B \to C$ is an étale morphism, then there is some i, and étale algebra $B_i \to C_i$ such that $C = B \otimes_{B_i} C_i$. Then the sheaf condition for $B \to C$ is the filtered colimit of the sheaf conditions for $B_i \to C_i$

$$\begin{array}{ccccc} F(B) & \longrightarrow & F(C) & \rightrightarrows & F(C \otimes_B C) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \varinjlim_i F(B_i) & \longrightarrow & \varinjlim_j F(C_i) & \rightrightarrows & \varinjlim_j F(C_i \otimes_{B_i} C_i) \end{array}$$

Since filtered colimits preserve exact sequences, exactness of the top line follows from exactness of the lower line. □

Exercise 1. Prove the claim that filtered colimits preserve exact sequences. That is, suppose that Λ is a filtered category, and $A, B, C : \Lambda \to Ab$ are functors from Λ to the category of abelian groups, and $A \to B \to C$ are natural transformations such that for each $\lambda \in \Lambda$, the sequence

$$0 \to A_\lambda \to B_\lambda \to C_\lambda \to 0$$

is exact. Then show that

$$0 \to \varinjlim_\lambda A_\lambda \to \varinjlim_\lambda B_\lambda \to \varinjlim_\lambda C_\lambda \to 0$$

is an exact sequence.

Example 2. Suppose k is a field with separable closure k^{sep} such that k^{sep}/k is not a finite extension. Then consider the sheaf $F(-) = \hom(-, \mathrm{Spec}(k^{sep}))$ on the category $\mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$. For any $\mathrm{Spec}(A) \in \mathrm{Spec}(k)_{\mathsf{pro\acute{e}t}}$ we have $F(\mathrm{Spec}(A)) = \varnothing$. However, $\mathrm{Spec}(k^{sep}) \in \mathrm{Spec}(k)^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ and we have $F(\mathrm{Spec}(k^{sep})) \neq \varnothing = \varinjlim_{k \subseteq L \subseteq k^{sep}} F(\mathrm{Spec}(L))$ where the limit is over finite subextensions of k^{sep}/k. So F is not in the image of ν^*.

Lemma 3 ([Lem.5.1.2])**.** *The functor $\nu^* : \mathsf{Shv}(X_{\mathsf{et}}) \to \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ is fully faithful. Its essential image consists of those sheaves F such that $F(U) = \varinjlim_i F(U_i)$ for any $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ with presentation $U = \varprojlim_i U_i$.*

Proof. A left adjoint is fully faithful if and only if the unit $\mathrm{id} \to \nu_*\nu^*$ is an isomorphism.[1] Isomorphisms of sheaves can be detected locally, cf. Exercise 2, and in X_{et} every scheme is locally affine. For any affine étale $U \to X$, the constant diagram (U) is a presentation for U. So then by [Lem.5.1.1] we have $F(U) \cong \nu_*\nu^*F(U)$ for any $F \in \mathsf{Shv}(X_{\mathsf{et}})$.

For the second part, suppose $G \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ satisfies the conditions of the lemma. To show that G is in the image of ν^*, we will show that $\nu^*\nu_*G \to G$ is an isomorphism. Since $\mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}) \cong \mathsf{Shv}(X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}})$, [Lem.4.2.4], it suffices to show that $\nu^*\nu_*G(U) \to G(U)$ is an isomorphism for every $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$. But this follows from [Lem.5.1.1] and the hypothesis. $\square$

Exercise 2. Prove the claim in the above proof that a morphism of sheaves $\phi : F \to G$ on a site (C, τ) is an isomorphism if and only if for every $X \in C$, there is a τ-covering family $\{U_i \to X\}_{i \in I}$ such that $F(U_i) \to G(U_i)$ is an isomorphism for all i.

Hint: The hypothesis is for every $X \in C$, in particular, for any cover $\{U_i \to X\}$ with ϕ an isomorphism on each U_i, there are also covers $\{W_{ijk} \to U_i \times_X U_j\}_{k \in K_{ij}}$ with ϕ an isomorphism on each W_{ijk}.

Definition 4. *The sheaves in the image of $\mathsf{Shv}(X_{\mathsf{et}}) \subseteq \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$ is called* classical.

[1] This is becomes the composition $\hom(X, Y) \to \hom(LX, LY) \cong \hom(X, RLY)$ induced by the unit $Y \to RLY$.

Lemma 5 ([Lem.5.1.4])**.** *Suppose that $F \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}})$. If there is a pro-étale covering $\{Y_i \to X\}_{i \in I}$ such that $F|_{Y_i}$ is classical for all $i \in I$, then F is classical.*

Sketch of proof. We just treat the affine pro-étale case here. That is we assume $\{Y_i \to X\}_{i \in I}$ is of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ for some ind-étale morphism of rings $A \to B$. We need to check that for any other ind-étale A-algebra $A \to C$ (so $C = \varinjlim_j C_j$ for some filtered system of étale algebras $A \to C_j$), we have $F(C) = \varinjlim F(C_j)$. We have the following diagram

$$\begin{array}{ccccc} F(C) & \longrightarrow & F(C \otimes B) & \rightrightarrows & F(C \otimes B \otimes B) \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim_j F(C_j) & \longrightarrow & \varinjlim_j F(C_j \otimes B) & \rightrightarrows & \varinjlim_j F(C_j \otimes B \otimes B) \end{array}$$

The the left vertical morphism is an isomorphism since the middle and right one is, and F is a sheaf. □

Definition 6. *Suppose that R is a ring equipped with the discrete topology. We write $Loc_{X_{\mathsf{et}}}(X)$ (resp. $Loc_{X_{\mathsf{pro\acute{e}t}}}(X)$) for the category of sheaves of R-modules on X_{et} which are locally free of finite rank. That is, those sheaves F such that there exists a covering $\{U_i \to X\}_{i \in I}$ and isomorphisms $F|_{U_i} \cong R^n$ for each i and some n, where R^n is the constant sheaf associated to the R-module R^n.*

Exercise 3. Show that every sheaf in $Loc_{X_{\mathsf{pro\acute{e}t}}}(R)$ is classical.

Exercise 4. Show that for any $U \in X_{\mathsf{et}}$ and $F \in \mathsf{Shv}(X_{\mathsf{et}})$ we have $\nu^*(F|_{U_{\mathsf{et}}}) \cong (\nu^* F)|_{U_{\mathsf{pro\acute{e}t}}}$. Deduce that for any $F \in Loc_{X_{\mathsf{et}}}(R)$, the sheaf $\nu^* F$ is in $Loc_{X_{\mathsf{pro\acute{e}t}}}(R)$.

Corollary 7 ([Cor.5.1.5])**.** *Suppose that R is a ring equipped with the discrete topology. Then ν^* defines an equivalence of categories $Loc_{X_{\mathsf{et}}}(R) \cong Loc_{X_{\mathsf{pro\acute{e}t}}}(R)$.*

Proof omitted.

Corollary 8 ([Cor.5.1.6])**.** *For any $K \in D^+(X_{\mathsf{et}})$, the map $K \to \nu_* \nu^* K$ is an equivalence (here ν_* and ν^* are derived now). Moreover, if $U \in X^{\mathsf{aff}}_{\mathsf{pro\acute{e}t}}$ has presentation $U = \varprojlim_i U_i$ then $R\Gamma(U, \nu^* K) = \varinjlim_i R\Gamma(U_i, K)$.*

The proof is omitted. It uses the Čech cohomology spectral sequence (hence the boundedness hypothesis).

Corollary 9 ([Cor.5.1.9])**.** *Consider a short exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}, \mathrm{Ab})$. Then F is in $\mathsf{Shv}(X_{\mathsf{et}}, \mathrm{Ab})$ if and only if F' and F'' are in $\mathsf{Shv}(X_{\mathsf{et}}, \mathrm{Ab})$.*

Proof omitted. It is not long, and it uses neat standard homological algebra tricks.

2 From pro-étale to étale

We now start working with the derived categories (cf. Remark 11). Functors between derived categories are always derived (for example $\nu^* : D(X_{\mathsf{et}}) \to D(X_{\mathsf{pro\acute{e}t}})$), even if we don't explicitly write it.

Definition 10 ([Def.5.2.1])**.** *A complex $L \in D(X_{\mathsf{pro\acute{e}t}})$ is called* parasitic *(寄生) if $R\Gamma(U, L) = 0$ for all $U \in X_{\mathsf{et}}$. We write $D_p(X_{\mathsf{pro\acute{e}t}}) \subseteq D(X_{\mathsf{pro\acute{e}t}})$ for the full subcategory of parasitic complexes.*

Remark 11. The category $D_p(X_{\mathsf{pro\acute{e}t}})$ is closed under shift, cone, and direct sum. However, if we try and define parasitic sheaves (outside of the derived category) we do not get a nice subcategory. For example, it is not closed under quotient.

Example 12 ([Rem.5.2.4])**.** Let $X = \mathrm{Spec}(\mathbb{Q})$, and $\widehat{\mathbb{Z}}_l(1) := \varprojlim \mu_{l^n} \in \mathsf{Shv}(X_{\mathsf{pro\acute{e}t}}, \mathrm{Ab})$. Then there is a short exact sequence

$$1 \to \widehat{\mathbb{Z}}_l(1) \xrightarrow{l} \widehat{\mathbb{Z}}_l(1) \to \mu_l \to 1.$$

For all $U \in X_{\mathsf{et}}$, we have $\widehat{\mathbb{Z}}_l(1)(U) = 0$ (because there is no finite separable field extension of $\mathbb{Q}$ which contains all l^nth roots of unity). However, $\mu_l \neq 0$. So the category of "parasitic" sheaves of abelian groups is not closed under quotients.

Lemma 13 ([Lem.5.2.3])**.** *We have:*

1. *A complex is in $D_p(X_{\mathsf{pro\acute{e}t}})$ if and only if it is sent to zero by the derived functor $\nu_* : D(X_{\mathsf{pro\acute{e}t}}) \to D(X_{\mathsf{et}})$.*
2. *The inclusion $i : D_p(X_{\mathsf{pro\acute{e}t}}) \to D(X_{\mathsf{pro\acute{e}t}})$ has a left adjoint L.*

Proposition 14 ([Prop.5.2.6])**.** *Consider the adjunctions (where ν^*, ν_* are derived functors).*

$$D_p^+(X_{\mathsf{pro\acute{e}t}}) \underset{i}{\overset{L}{\leftrightarrows}} D^+(X_{\mathsf{pro\acute{e}t}}) \underset{\nu_*}{\overset{\nu^*}{\leftrightarrows}} D^+(X_{\mathsf{et}})$$

1. *ν^* is fully faithful.*
2. *The essential image of ν^* are those complexes whose cohomology sheaves lie in $\mathsf{Shv}_{\mathsf{et}}(X, \mathrm{Ab}) \subseteq \mathsf{Shv}_{\mathsf{pro\acute{e}t}}(X, \mathrm{Ab})$.*
3. *For every $K \in D^+(X_{\mathsf{pro\acute{e}t}})$ we have*
$$\mathrm{Cone}(iLK \to K) \cong \nu^*\nu_*K.$$
4. *We have $\mathrm{hom}(i(K'), \nu^*(K'')) = 0$ for all $K' \in D_p^+(X_{\mathsf{pro\acute{e}t}}), K'' \in D^+(X_{\mathsf{et}})$.*

In other words, the above adjunctions define a semi-orthogonal decomposition of triangulated categories.

Remark 15 ([Rema.5.2.8])**.** In the case that $D(X_{\mathsf{et}})$ is left-complete (cf.[Prop.3.3.7]) then the above proposition extends to the unbounded categories.

Remark 16 ([Prop.5.2.9])**.** Another way to extend the above proposition to unbounded categories is to replace $D^+(X_{\mathsf{et}})$ with the smallest subcategory of $D(X_{\mathsf{pro\acute{e}t}})$ containing $\nu^*(D(X_{\mathsf{et}}))$, closed under cones, shift, and filtered colimits.

3 Left completion via the pro-étale site

Recall that the left completion $\widehat{D}(X_{\mathsf{et}})$ of $D(X_{\mathsf{et}})$ is the subcategory of $D(X_{\mathsf{et}}^{\mathbb{N}})$ consisting of those sequence of chain complexes $(\cdots \to K_2 \to K_1 \to K_0)$ in $Ch(\mathsf{Shv}(X_{\mathsf{et}})^{\mathbb{N}})$ such that

1. $K_n \in D^{\geq -n}(X_{et})$. That is, the sheaves $H^i K_n$ are zero for $i < -n$.
2. $\tau^{\geq -n} K_{n+1} \to K_n$ is an equivalence. That is the map $H^i K_{n+1} \to H^i K_n$ is an isomorphism of étale sheaves for $i \geq -n$.

The left completion $\widehat{D}(X_{\mathsf{proét}})$ is defined similarly, however, since $\mathsf{Shv}(X_{\mathsf{proét}})$ is replete [Prop.4.2.8], $D(X_{\mathsf{proét}})$ is left complete. That is, the canonical adjoints

$$\widehat{D}(X_{\mathsf{proét}}) \leftrightarrows D(X_{\mathsf{proét}})$$

are both equivalences of categories.

Left completion is functorial, so we get a commutative square of functors

$$\begin{array}{ccc} D(X_{\mathsf{et}}) & \xrightarrow{\nu^*} & D(X_{\mathsf{proét}}) \\ \downarrow & & \downarrow \cong \\ \widehat{D}(X_{\mathsf{et}}) & \xrightarrow[\nu^*]{} & \widehat{D}(X_{\mathsf{proét}}) \end{array}$$

Then, since $D(X_{\mathsf{proét}})$ is left complete, we end up with an adjunction

$$\nu^* : \widehat{D}(X_{\mathsf{et}}) \rightleftarrows D(X_{\mathsf{proét}})$$

This functor is full faithful, and its essential image admits the following simple description.

Definition 17 ([Def.5.3.1])**.** *Let $D_{cc}(X_{\mathsf{proét}})$ be the full subcategory of $D(X_{\mathsf{proét}})$ consisting of complexes whose cohomology sheaves lie in $\mathsf{Shv}(X_{\mathsf{et}}, \mathrm{Ab}) \subseteq \mathsf{Shv}(X_{\mathsf{proét}}, \mathrm{Ab})$.*

Proposition 18 ([Prop.5.3.2])**.** *There is an adjunction*

$$D(X_{\mathsf{et}}) \rightleftarrows D_{cc}(X_{\mathsf{proét}})$$

induced by ν^*, ν_* which is isomorphic to the left-completion adjunction

$$\tau : D(X_{\mathsf{et}}) \rightleftarrows \widehat{D}(X_{\mathsf{et}}) : R\varprojlim .$$

In particular

$$\widehat{D}(X_{\mathsf{et}}) \cong D_{cc}(X_{\mathsf{proét}}).$$

4 l-adic sheaves via the pro-étale site

Suppose l is a prime, and X is a $\mathbb{Z}[1/l]$-scheme. The l-adic cohomology is classically defined as

$$H^i_{\mathsf{et}}(X, \mathbb{Z}_l) := \varprojlim_n H^i_{\mathsf{et}}(X, \mathbb{Z}/l^n).$$

On the other hand, it is useful to have a description of cohomology in terms of derived categories. We have

$$\hom_{D(X_{\mathsf{et}}, \mathbb{Z}/l^n)}(\mathbb{Z}/l^n, \mathbb{Z}/l^n[i]) = H^i_{\mathsf{et}}(X; \mathbb{Z}/l^n)$$

but to extend this to l-adic cohomology, we would need to consider something like

$$\varprojlim_n D(X_{\mathsf{et}}, \mathbb{Z}/l^n)$$

but categories are only well-defined up to equivalence, so limits of categories are technically complicated to define.

Exercise 5. In this exercise we show that naïve inverse limits of categories are not well-defined up to equivalence of categories. Let $\cdots \to C_2 \to C_1 \to C_0$ be a system of functors of small categories. Define $\varprojlim C_n$ to be the category with set of objects

$$Ob_{\varprojlim C_n} = \varprojlim Ob_{C_n}.$$

Given objects $x = (\dots, x_2, x_1, x_0)$ and $y = (\dots, y_2, y_1, y_0)$ in $\varprojlim C_n$ define

$$\hom_{\varprojlim C_n}(x, y) = \varprojlim \hom_{C_n}(x_n, y_n).$$

1. For an abelian group A, let BA be the category of one object, $*$, and $\hom_{BA}(*, *) = A$ with composition in BA given by addition in A. Note that any group homomorphism $A \to A'$ induces a functor $BA \to BA'$. Show that
$$\varprojlim_n B(\mathbb{Z}/l^n) = B\mathbb{Z}_l.$$

2. Now define C_n to be the category whose objects are $Ob\ C_n = \{i \in \mathbb{Z} : i \geq n\}$, morphisms are $\hom_{C_n}(i, j) = \mathbb{Z}/l^n$ for every i, j, and composition is given by addition in $\mathbb{Z}/l^n$. Note that there are canonical functors $C_{n+1} \to C_n$ induced by the group homomorphisms $\mathbb{Z}/l^{n+1} \to \mathbb{Z}/l^n$ and the inclusions $Ob\ C_{n+1} \subset Ob\ C_n$. Show that
$$\varprojlim_n C_n = \varnothing.$$

3. Show that for every n, the canonical functor $C_n \to B\mathbb{Z}/l^n$ is fully faithful, and essentially surjective. That is, it is an equivalence of categories. Deduce that $\varprojlim$, as defined above, does not preserve equivalences of categories.

There is a notion of 2-limit of categories defined by keeping track of isomorphisms, which does preserve equivalences, but the following is a better way of dealing with this problem.

Let R be a complete discrete valuation ring (e.g., $\mathbb{Z}_l$) with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$.

Definition 19 ([Def.5.5.2]). *Define $D^+_{Ek}(X_{\mathsf{et}}, R)$ as the full subcategory of $D^+(X^{\mathbb{N}}_{\mathsf{et}}, R)$ consisting of those sequences $(\cdots \to M_2 \to M_1 \to M_0)$ of complexes such that each M_n is a complex of sheaves of $R/\mathfrak{m}^n$-modules, and the induced maps*[2]

$$M_n \otimes_{R/\mathfrak{m}^n} R/\mathfrak{m}^{n-1} \to M_{n-1}$$

are quasi-isomorphisms for all n.

The category $D^+_{Ek}(X_{\mathsf{et}}, R)$ (and its unbounded version) is what was used classically to access l-adic cohomology in a derived category setting.

Recall that $K \in D(X_{\mathsf{pro\acute{e}t}}, R)$ is derived complete if $T(K)$ is quasi-isomorphic to zero, where $T(K) = R\varprojlim(\cdots \xrightarrow{\pi} K \xrightarrow{\pi} K \xrightarrow{\pi} K) \cong \mathrm{Cone}\left(\prod_{\mathbb{N}} K \xrightarrow{\mathrm{id}-\pi} \prod_{\mathbb{N}} K\right)[-1]$.

Definition 20 ([Def.5.5.3]). *Define $D^+_{Et}(X_{\mathsf{pro\acute{e}t}}, R) \subseteq D(X_{\mathsf{pro\acute{e}t}}, R)$ for the full subcategory of bounded below complexes K such that*

1. *K is derived complete, cf.[Def.3.4.1].*
2. *$K \otimes_R (R/\mathfrak{m}) \in D_{cc}(X_{\mathsf{pro\acute{e}t}})$.*

Proposition 21. *There is a natural equivalence*

$$D^+_{Ek}(X_{\mathsf{pro\acute{e}t}}, R) \cong D^+_{Ek}(X_{\mathsf{et}}, R).$$

Remark 22. If there is an integer N such that for all affine $Y \in X_{\mathsf{et}}$ and sheaves of κ-vector spaces F we have $H^n(Y, F) = 0$ for $n > N$, then the above proposition is true for unbounded complexes too.

Remark 23. Notice that $D^+_{Ek}(X_{\mathsf{et}}, R)$ is defined by adding structure to $D(X_{\mathsf{et}}, R)$, whereas $D^+_{Ek}(X_{\mathsf{pro\acute{e}t}}, R)$ is defined via properties of objects in $D^+(X_{\mathsf{pro\acute{e}t}}, R)$. So one would expect that the latter is easier to work with.

[2] Recall that all functors in the derived setting are derived, even if the notation does not expicitely say it. In particular, since $R/\mathfrak{m}^{n-1} \cong \mathfrak{m}R/\mathfrak{m}^n$ and $0 \to \mathfrak{m}R/\mathfrak{m}^n \to R/\mathfrak{m}^n \xrightarrow{\pi} R/\mathfrak{m}^n \to 0$ is a short exact sequence of R-modules for any $\pi \in R$ with $\mathfrak{m} = \langle \pi \rangle$, the functor $- \otimes_{R/\mathfrak{m}^n} R/\mathfrak{m}^{n-1}$ can be calculated as $\mathrm{Cone}(- \xrightarrow{\pi} -)[-1]$ for chain complexes of sheaves of $R/\mathfrak{m}^n$-modules. Similarly, $- \otimes_R (R/\mathfrak{m})$ can be calculated by $\mathrm{Cone}(- \xrightarrow{\pi} -)$ for complexes of sheaves of R-modules.