Reference: [BS] Bhatt, Scholze, “The pro-étale topology for schemes”

In this lecture we consider replete topoi. This is a nice class of topoi that include the pro-étale topos, and whose derived categories are all complete.

Here are the main points:

$$\mathcal{X} = \mathsf{Shv}_{\mathsf{pro\acute{e}t}}(X)$$

$\Rightarrow$ $\mathcal{X}$ is *locally weakly contractible* (next lecture)

$\Rightarrow$ $\mathcal{X}$ is *replete*

$\Rightarrow$ $D(\mathcal{X})$ is *left complete*

$\Rightarrow$ $\forall\ K \in D(\mathcal{X})$ we have $R\varprojlim \tau^{\geq n} K \cong K$.

In the last section we consider $D(\mathcal{X}, R)$ the derived category of sheaves of R-modules for a dvr R with uniformiser π, and consider what it means for an object of $D(\mathcal{X}, R)$ to be π-adically complete.

1 Replete topoi

Definition 1. *A* topos *is a category equivalent to a category of the form* $\mathsf{Shv}_\tau(\mathcal{C})$ *for some category* $\mathcal{C}$ *and some Grothendieck topology* τ *on* $\mathcal{C}$*. Given an object* $X \in \mathcal{C}$*, we write* h_X *for the sheaf represented by* X*. I.e., the sheafification of the presheaf* $\hom_{\mathcal{C}}(-, X)$*.*

Example 2. Given a topos $\mathcal{X}$, the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences $(\dots, F_2, F_1, F_0)$ of objects in $\mathcal{X}$ is also a topos.[1] The category $\mathcal{X}^{\mathbb{N}}$ of sequences $(\dots \to F_2 \to F_1 \to F_0)$ of morphisms in $\mathcal{X}$ is also a topos.[2]

Definition 3. *Let* $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ *be a topos. A morphism* $F \to G$ *of objects of* $\mathcal{X}$ *is* surjective *if for every object* $X \in \mathcal{C}$ *and* $s \in G(X)$*, there exists a covering* $\{U_i \to X\}_{i \in I}$ *such that* $s|_{U_i}$ *is in the image of* $F(U_i) \to G(U_i)$ *for each* $i \in I$*.*

Remark 4. It can be shown that a morphism $F \to G$ of sheaves is surjective if and only if for every sheaf H the induced map $\hom(G, H) \to \hom(F, H)$ is injective. I.e., if and only if $F \to G$ is surjective in the categorical sense. Another equivalent condition for $F \to G$ to be surjective is asking that $im(F{\to}G) \to G$ become an isomorphism (of presheaves) after sheafification. Here, by $im(F{\to}G)$ we mean the *presheaf* image, i.e., $im(F{\to}G)(U) = im(F(U){\to}G(U))$ (this is not necessarily a sheaf).

`exer:epi` **Exercise 1.** Note that:

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[1] It is the category of sheaves on the disjoint union $\amalg_{n \in \mathbb{N}} \mathcal{C}$ equipped with the coarsest topology such that the inclusions $\mathcal{C} \to \amalg_{n \in \mathbb{N}} \mathcal{C}$ send covers to covers.

[2] It is the category of sheaves on the category $\mathbb{N} \times \mathcal{C}$ whose objects are pairs (n, X) consisting of an $n \in \mathbb{N}$ and an object $X \in \mathcal{C}$. Morphisms are $\hom((n, X), (m, Y)) = \varnothing$ if $n > m$ and $\hom(X, Y)$ otherwise. Again, the topology is the coarsest topology such that the inclusions $\mathcal{C} \to \mathbb{N} \times \mathcal{C}$ send covers to covers.

(∗) For any presheaf F with sheafification aF, object $X \in \mathcal{C}$, and section $s \in aF(X)$, there exists a covering $\{U_i \to X\}_{i \in I}$ such that each $s|_{U_i}$ is in the image of $F(U_i) \to aF(U_i)$.

Let $\{V_i \to Y\}_{i \in I}$ be a covering family in a site $\mathcal{C}$. Using the axioms of a Grothendieck topology, and (∗) show that $\amalg_{i \in I} h_{V_i} \to h_Y$ is a surjective morphism of sheaves.

Remark 5. In the SGA definition of a covering family, the converse is also true: a family $\{V_i \to Y\}_{i \in I}$ is a covering family if and only if the induced morphism of sheaves $\amalg_{i \in I} h_{V_i} \to h_Y$ is a surjective.

Definition 6 ([BS, Def.3.1.1])**.** *A topos is* replete *(充実した) if for every sequence of surjective morphisms* $\cdots \to F_2 \to F_1 \to F_0$ *the induced morphisms*

$$\begin{tikzcd}
\varprojlim_{i \geq 0} F_i \arrow[dr, dotted] \arrow[drr, two heads] \arrow[drrr, two heads] \arrow[drrrr, two heads] & & & & \\
& \dots \arrow[r, two heads] & F_2 \arrow[r, two heads] & F_1 \arrow[r, two heads] & F_0
\end{tikzcd}$$

are surjective for all n*.*

rema:limits **Remark 7.** Note: the inclusion $\mathsf{Shv}_\tau(\mathcal{C}) \subseteq \mathsf{PreShv}(\mathcal{C})$ preserves limits (but not all colimits). That is, $(\varprojlim_{i \in I} F_i)(X) = \varprojlim_{i \in I}(F_i(X))$ for any diagram of sheaves $I \to \mathsf{Shv}_\tau(\mathcal{C})$ and $X \in \mathcal{C}$ (to calculate colimits of sheaves, one takes the colimit in the category of presheaves and then sheafifies).

Exercise 2.

1. Show that the category of sets is replete. (Note, this is a topos: Set is the category of sheaves on the category $\overset{\overset{\mathrm{id}}{\curvearrowright}}{*}$ with only one object equipped with the trivial Grothendieck topology).
2. Let $\mathcal{C}$ be a category equipped with the trivial Grothendieck topology,[3] so every presheaf is a sheaf. Show that $\mathsf{PreShv}(\mathcal{C})$ is replete.
3. Let G be a (discrete) group. Deduce that the category of G-sets is replete. Note: G-sets is the category of presheaves on the category BG which has one object, one morphism for every element of G, and composition is defined by the multiplication in G.

exam:etNotReplete **Example 8.** Let k be a field such that k^{sep}/k is not finite. Then the category $\mathsf{Shv}_{et}(k)$ of étale sheaves on k is not replete: Since k^{sep}/k is not finite there exists a tower $\dots/L_2/L_1/L_0 = k$ of nontrivial finite separable field extensions. Since each $\mathrm{Spec}(L_n) \to \mathrm{Spec}(L_{n-1})$ is a covering, each morphism in the tower induces a surjective morphism of sheaves. However,

$$\varprojlim_i h_{\mathrm{Spec}(L_i)} \to h_{\mathrm{Spec}(k)} \qquad (1)$$ equa:idSurjEq

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[3] I.e., the only covering families are families of the form $\{X \xrightarrow{\mathrm{id}} X\}$.

cannot be surjective.

Exercise 3. By evaluating on $X = \mathrm{Spec}(k)$ and considering $s = \mathrm{id}_k$ prove the claim that Example 8(1) is not surjective. Hint: recall that the coverings of $\mathrm{Spec}(k)$ are of the form $\{\mathrm{Spec}(K_j) \to \mathrm{Spec}(k)\}_{j \in J}$ with K_j/k finite separable field extensions.

Example 9 ([BS, Example 3.1.7])**.** The category of affine schemes with the fpqc topology[4] is replete. Suppose $\dots \to F_2 \to F_1 \to F_0$ is a tower of surjective morphisms, and consider some affine scheme $X = \mathrm{Spec}(A)$ and some $s \in F_0(X)$. Since $F_1 \to F_0$ is surjective, there is a faithfully flat morphism $A \to B_0$ such that $s|_{B_0}$ is in the image of $F_1(B_0) \to F_0(B_0)$. That is, there is some $s_1 \in F_1(B_0)$ mapping to $s|_{B_0}$. Repeating the argument, we find a sequence of faithfully flat morphisms $A \to B_0 \to B_1 \to B_2 \to \dots$ and elements $s_i \in F_i(B_{i-1})$ such that s_i maps to $s_{i-1}|_{B_{i-1}}$. Set $B = \varinjlim B_i$. Now $A \to B$ is again a faithfully flat morphism, and the sequence $(s_n \in F_n(B_{n-1}))$ induces a sequence $(t_n \in F_n(B))$ such that $t_n \mapsto t_{n-1}$ for all n. In other words, it induces an element $t \in (\varprojlim F_i)(B)$. By construction, $s|_B = t$, and so we deduce that $\varprojlim F_i \to F_0$ is surjective. The same argument shows each $\varprojlim F_i \to F_n$ is surjective.

Remark 10. Note that the reason the fpqc site is replete and the étale site is not replete is precisely because limits of coverings exist in the category, and are still coverings.

Our first goal is to show that countable products are exact in replete topoi. This is Proposition 14. Knowing that products are exact, makes derived limits easy to calculate, cf. Remark 16. The first step is the following lemma.

`lem:3.1.8` **Lemma 11** ([BS, Lem.3.1.8])**.** *Let*

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d] \\
\dots \arrow[r] & G_2 \arrow[r] & G_1 \arrow[r] & G_0
\end{tikzcd}$$

be morphisms in a replete topos, and suppose $F_i \to G_i$ and $F_{i+1} \to F_i \times_{G_i} G_{i+1}$ are surjective for all i. Then $\varprojlim F_i \to \varprojlim G_i$ is surjective.

Exercise 4. Prove Lemma 11 when the topos is the category of sets.

Exercise 5. This exercise shows that limits do not preserves surjections in general. So the hypotheses of Lemma 11 are really necessary.

1. Show that $(\dots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}) \to (\dots \to \mathbb{Z}/l^3 \to \mathbb{Z}/l^2 \to \mathbb{Z}/l)$ does not satisfy the hypotheses of Lemma 11.

2. Show that the limit of the above sequence of morphisms is $\mathbb{Z} \to \mathbb{Z}_l$. Show that this is not surjective.

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[4] I.e., the topology whose coverings are families $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)\}_{i \in I}$ such that each $A \to B_i$ is flat, and $\amalg \mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ is surjective.

`exer:surjective` **Exercise 6.** Suppose that $(f_i : B_i \to C_i)_{i\in\mathbb{N}}$ is a sequence of surjections. Show that the conditions of Lemma 11 are satisfied for $F_n = \prod_{0\leq i\leq n} B_i$, and $G_n = \prod_{0\leq i\leq n} C_i$. Note: $X \times_Y (Y \times Y') \cong X \times Y'$ so

$$(B_0 \times \cdots \times B_n) \times_{(C_0\times\cdots\times C_n)} (C_0 \times \cdots \times C_{n+1}) \cong (B_0 \times \cdots \times B_n \times C_{n+1})$$

Deduce that $\prod_{i\in\mathbb{N}} f_i : \prod_{i\in\mathbb{N}} B_i \to \prod_{i\in\mathbb{N}} C_i$ is surjective.

`exer:EpiLemma` **Exercise 7.** Suppose that $\cdots \to F_2 \to F_1 \to F_0$ is a sequence of surjections in a replete topos $\mathcal{X}$.

1. Show that each map $\prod_{i=0}^{n+1} F_i \overset{Shift-\mathrm{id}}{\longrightarrow} \prod_{i=0}^{n} F_i$ is surjective.
2. Using $B_n = \prod_{i=0}^{n+1} F_i$ and $C_n = \prod_{i=0}^{n} F_i$ and Exercise 6, show that

$$\prod_{\mathbb{N}} F_i \xrightarrow{t-\mathrm{id}} \prod_{\mathbb{N}} F_i$$

   is surjective.

Definition 12. *If $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ is a topos we write $D(\mathcal{X}) = D(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))$ for its derived category.*

Recall that if $\mathcal{X}$ is a topos, then the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences of objects and the category $\mathcal{X}^{\mathbb{N}}$ of sequences of morphisms are also topoi. We can this consider the right derived functors associated to product and limit

$$R\Pi : D(\prod_{\mathbb{N}} \mathcal{X}) \to D(\mathcal{X}),$$

$$R\varprojlim : D(\mathcal{X}^{\mathbb{N}}) \to D(\mathcal{X}).$$

We prove the following proposition in an appendix to this lecture.

`prop:rlimDef` **Proposition 13** (See Proposition 40 below)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category with products and $(\ldots\to C_2 \xrightarrow{t} C_1 \xrightarrow{t} C_0)$ a sequence of chain complexes (the t's are different, but we ommit the indices). Then there is a isomorphism in $D(\mathcal{A})$*

$$R\varprojlim C_n \cong \mathrm{Cone}\left(R\Pi C_n \xrightarrow{t-\mathrm{id}} R\Pi C_n\right)[-1]$$

where $t - \mathrm{id}$ is the morphism $(\ldots, c_2, c_1, c_0) \mapsto (\ldots, tc_3-c_2, tc_2-c_1, tc_1-c_0)$.

One of the reasons we are interested in replete topoi is that limits work very well.

`prop:derivedProd` **Proposition 14.** *Let $\mathcal{X}$ be a replete topos. Then the functor $\Pi : \prod_{\mathbb{N}} \mathcal{X} \to \mathcal{X}$ preserves injections and surjections. In particular, Π preserves quasi-isomorphism of chain complexes and so induces a well-defined functor*

$$\Pi : D(\prod_{\mathbb{N}} \mathcal{X}) \to D(\mathcal{X})$$

which is just Π on each object.

Proof. We want to show that if $(f_i : F_i \to G_i)_{i\in\mathbb{N}}$ is a sequence of morphisms in $\mathcal{X}$ which is injective (resp. surjective) then $\prod f_i$ is injective (resp. surjective). It is automatically injective because limits always preserve monomorphisms. The surjective case is exactly Exercise 6. ☐

Proposition 15 ([BS, 3.1.10])**.** *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a replete topos and suppose $\cdots \to F_2 \overset{t}{\to} F_1 \overset{t}{\to} F_0$ is a sequence of* surjective *morphisms in $\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})$. Then we have $\varprojlim F_i \cong R\varprojlim F_i$ in $D(\mathcal{X})$.*

Proof. Since each $F_{i+1} \to F_i$ is surjective, the morphism $t - \mathrm{id}$ is surjective by Exercise 7. So, we have a short exact sequence

$$0 \to \varprojlim F_i \to \prod F_i \xrightarrow{t-\mathrm{id}} \prod F_i \to 0.$$

Since products are automatically derived by Proposition 13 and Lemma 14 , we have

$$\varprojlim F_i \cong \mathrm{Cone}\left(\prod F_i \xrightarrow{\text{shift} - \mathrm{id}} \prod F_i\right)[-1].$$

☐

`rema:derivedLimCalc`

Remark 16. Proposition 14 shows that in a replete topos, we could *define* $R\varprojlim F_i$ as $\mathrm{Cone}(\prod F_i \to \prod F_i)[-1]$. We will work with this definition from now on.

2 Locally weakly contractible topoi

Definition 17. *Let $\mathcal{X}$ be a topos. An object $F \in \mathcal{X}$ is called* compact *if for every family $\{Y_i \to F\}_{i\in I}$ such that $\amalg_I Y_i \to F$ is surjective, there is a finite sequence $i_1, \dots, i_n$ such that $\amalg_{i=1}^n Y_{i_n} \to F$ is still surjective.*

Exercise 8. Show that a set is compact in the category of sets if and only if it is finite.

Definition 18. *Let $\mathcal{X}$ be a topos. An object $F \in \mathcal{X}$ is called* coherent *if it is compact, and for any pair of morphisms $Y, Y' \rightrightarrows F$ from compact objects Y, Y', the fibre product $Y \times_F Y'$ is again compact.*

Definition 19 ([Bs, Def.3.2.1])**.** *An object F of a topos $\mathcal{X}$ is* weakly contractible *if every surjective $G \to F$ has a section. We say that $\mathcal{X}$ is* locally weakly contractible *if every object $X \in \mathcal{X}$ admits a surjection $\amalg_{i\in I} Y_i \to X$ with Y_i weakly contractible coherent objects.*

Example 20. The pro-étale site that we define in the next lecture is locally weakly contractible.

Proposition 21 ([BS, Prop.3.2.3])**.** *] Let $\mathcal{X}$ be a locally weakly contractible topos. Then $\mathcal{X}$ is replete, and for any object $K \in D(\mathcal{X})$ we have $R\lim_n \tau^{\geq n} K \cong K$ where*

$$\tau^{\geq n} K = (\cdots \to 0 \to (K^n / dK^{n-1}) \to K^{n+1} \to K^{n+2} \to \dots).$$

Remark 22. The property $R\lim_n \tau^{\geq n}K \cong K$ means that all the information of K is contained in its truncations. This lets us deduce properties of unbounded complexes from bounded below complexes.

Sketch of proof. Since $\mathcal{X}$ is locally weakly contractible, a morphism f in $\mathcal{X}$ is an isomorphism (resp. surjection) if and only if evaluating on each weakly contractible object of $\mathcal{C}$ is an isomorphism (resp. surjection). It follows that $\mathcal{X}$ is replete.

Similarly, for any complex of sheaves K and weakly contractible object U we have $(H^iK)(U) = H^i(K(U))$. It follows that $R\lim_n \tau^{\geq n}K \cong K$. $\square$

3 Truncation completing derived categories

Recall that if $\mathcal{X}$ is a category, then $\mathcal{X}^{\mathbb{N}}$ is the category of sequences $(\ldots \to F_2 \to F_1 \to F_0)$ of morphisms in $\mathcal{X}$. If $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ is a topos, then $D(\mathcal{X}^{\mathbb{N}})$ is the derived category of the abelian category $\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})^{\mathbb{N}}$.

Now that we are working with sequences of chain complexes, we will have two indices: an upper index for the terms in the chain complex, and a lower index for the terms in the sequence.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
\cdots \arrow[r] & K_2^{i+1} \arrow[u, "d"'] \arrow[r, "t"] & K_1^{i+1} \arrow[u, "d"'] \arrow[r, "t"] & K_0^{i+1} \arrow[u, "d"'] \\
\cdots \arrow[r] & K_2^{i} \arrow[u, "d"'] \arrow[r, "t"] & K_1^{i} \arrow[u, "d"'] \arrow[r, "t"] & K_0^{i} \arrow[u, "d"'] \\
\cdots \arrow[r] & K_2^{i-1} \arrow[u, "d"'] \arrow[r, "t"] & K_1^{i-1} \arrow[u, "d"'] \arrow[r, "t"] & K_0^{i-1} \arrow[u, "d"'] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

Here, the d's and t's should have indices too, but we did not write them. Note that $Ch(\mathcal{A})^{\mathbb{N}} = Ch(\mathcal{A}^{\mathbb{N}})$. That is, we can think about objects in this category as sequences of chain complexes $\overset{t_2}{\to} \begin{pmatrix} \vdots \end{pmatrix} \overset{t_1}{\to} \begin{pmatrix} \vdots \end{pmatrix} \overset{t_0}{\to} \begin{pmatrix} \vdots \end{pmatrix}$ or chain complexes of sequences

$$\begin{tikzcd}
{} \\
(\ldots) \arrow[u, "d^{i+1}"'] \\
(\ldots) \arrow[u, "d^{i}"'] \\
{} \arrow[u]
\end{tikzcd}$$

Definition 23 ([BS, 3.3.1])**.** *Let $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ be a topos. We define the* left-completion $\widehat{D}(\mathcal{X})$ *of* $D(\mathcal{X})$ *as the full subcategory of* $D(\mathcal{X}^{\mathbb{N}})$ *spanned by the projection systems* $(\ldots\to K_2\to K_1\to K_0)$ *in* $Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab})^{\mathbb{N}})$ *such that*

1. $K_n \in D^{\geq -n}(\mathcal{X})$*. That is,* $H^i K_n = 0$ *for* $i < -n$*.*
2. *The canonical map* $\tau^{\geq -n} K_{n+1} \to K_n$ *is an equivalence. In other words, the map* $H^i K_{n+1} \to H^i K_n$ *is an isomorphism for all* $i \geq -n$*.*

We say that $D(\mathcal{X})$ *is* left-complete *if the map*

$$\tau : D(\mathcal{X}) \to \widehat{D}(\mathcal{X}); \qquad K \mapsto \{\tau^{\geq -n} K\}$$

is an equivalence.

Remark 24. The definition is equivalent to asking that when we take cohomology, we get the following picture:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
\ldots \arrow[r, "\cong"] & H^2K_2 \arrow[r, "\cong"] & H^2K_1 \arrow[r, "\cong"] & H^2K_0 \\
\ldots \arrow[r, "\cong"] & H^1K_2 \arrow[r, "\cong"] & H^1K_1 \arrow[r, "\cong"] & H^1K_0 \\
\ldots \arrow[r, "\cong"] & H^0K_2 \arrow[r, "\cong"] & H^0K_1 \arrow[r, "\cong"] & H^0K_0 \\
\ldots \arrow[r, "\cong"] & H^{-1}K_2 \arrow[r, "\cong"] & H^{-1}K_1 \arrow[r] & 0 \\
\ldots \arrow[r, "\cong"] & H^2K_2 \arrow[r] & 0 \arrow[r] & 0 \\
\ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
 & \vdots & \vdots & \vdots
\end{tikzcd}$$

Remark 25. The inclusion $\widehat{D}(\mathcal{X}) \subseteq D(\mathcal{X})$ is not an inclusion of triangulated categories (because $\widehat{D}(\mathcal{X})$) is not preserve by the deshift $[-1]$ from $D(\mathcal{X})$.

We just state the main facts about completions without giving too many details.

Theorem 26. *Let* $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$ *be a topos.*

1. *[BS, Lem.3.3.2] The functor* $R\varprojlim : \widehat{D}(\mathcal{X}) \to D(\mathcal{X}^{\mathbb{N}}) \to D(\mathcal{X})$ *is the right adjoint of* τ*. In particular, if* $D(\mathcal{X})$ *is left-complete, then* $K \cong R\varprojlim \tau^{-n} K$ *for any* $K \in Ch(\mathsf{Shv}_\tau(\mathcal{C}, \mathrm{Ab}))$*.*
2. *[BS, Prop.3.3.3] If* $\mathcal{X}$ *is a replete topos then* $D(\mathcal{X})$ *is left-complete.*
3. *[BS, Exam.3.3.5] If* $k = \mathbb{C}(x_1, x_2, \ldots)$*, then* $D(\mathrm{Spec}(k)_{et})$ *is not left-complete.*

4. *[BS, Prop.3.3.7] If $U \in \mathcal{C}$ is an object such that $\Gamma(U, -)$ is exact then for any $K \in D(\mathcal{X})$ we have $R\Gamma(U, K) \cong R \varprojlim R\Gamma(U, \tau^{-n}K)$.*

5. *[BS, Prop.3.3.7] If for each $K \in D(\mathcal{X})$ and $U \in \mathcal{C}$ there exists some $d \in \mathbb{N}$ such that $H^p(U, \underline{H}^q K) = 0$ for $p > d$, then $D(\mathcal{X})$ is left-complete.*

Example 27. The finiteness condition of [BS, Prop.3.3.7] above is satisfied for the étale sites of $\mathrm{Spec}(\mathbb{F}_q)$, and X when X is a smooth affine variety over an algebraically closed field.

4 π-adically completing objects

Through-out this section we work with a discrete valuation ring R, with maximal ideal $\mathfrak{m}$ and uniformiser π (so $\mathfrak{m} = (\pi)$). We also fix a replete topos $\mathcal{X} = \mathsf{Shv}_\tau(\mathcal{C})$, and now our derived category will always be the derived category of sheaves of R-modules

$$D(\mathcal{X}, R) = D(\mathsf{Shv}_\tau(\mathcal{C}, R)).$$

Definition 28. *We say that $M \in \mathrm{Mod}_R$ is* classically complete *if $M \cong \varprojlim M/\pi^n M$. We write $\mathrm{Mod}_{R,\mathrm{comp}} \subseteq \mathrm{Mod}_R$ for the full subcategory of classically complete modules.*

We say that $K \in D(\mathcal{X}, R)$ is derived complete *if $T(K) \cong 0$ in $D(\mathcal{X}, R)$ where*

$$T(K) := R\varprojlim(\cdots \xrightarrow{\pi} K \xrightarrow{\pi} K \xrightarrow{\pi} K).$$

Here the transition maps are multiplication by the uniformiser π. We use the notation $D_{\mathrm{comp}}(\mathcal{X}, R) \subseteq D(\mathcal{X}, R)$ for the full subcategory of derived complete objects.

Remark 29. Since we are assuming that $\mathcal{X}$ is replete, by Proposition 14 we have

$$T(K) = \mathrm{Cone}\left(\prod_{\mathbb{N}} K \xrightarrow{\mathrm{id} - \pi} \prod_{\mathbb{N}} K\right)[-1].$$

Exercise 9. Show that $\mathbb{Z}_l$ is a complete $\mathbb{Z}_{(l)}$-module.

`exer:derivedCompleteCone` **Exercise 10.**

1. Show that if K is derived complete then so is $K[n]$ for any n.

2. Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of chain complexes in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, R))$. Using the fact that products in a replete topos are exact, show that $0 \to TA \to TB \to TC \to 0$ is also a short exact sequence. Deduce that if two of A, B, C are derived complete, then so is the third.

3. Consider a morphism $K \to L$ in $Ch(\mathsf{Shv}_\tau(\mathcal{C}, R))$ and define $C = Cone(K \to L)$. Use the second part above to show that if two of K, L, C are derived complete then the third is also derived complete.

exer:derivedCompleteLim

Exercise 11. Using the fact that for any *double* sequence of chain complexes $(K_{n,m})$ we have $R\varprojlim_n R\varprojlim_m K_{n,m} \cong R\varprojlim_m R\varprojlim_n K_{n,m}$, show that if $(\dots \to K_2 \to K_1 \to K_0)$ is a sequence of derived complete chain complexes then $R\varprojlim K_n$ is derived complete.

The relationship between classical complete and derived complete is the following.

Proposition 30 ([BS, Prop.3.4.2])**.** *An R-module $M \in \mathrm{Mod}_R$ is classically complete if and only if it is π-adically separated*[5] *and derived complete.*

In particular, for classical R-modules, classical completeness is strictly stronger than derived completeness.

We omit the proof of 3.4.2 as it is not used elsewhere.

Proposition 31 ([BS, Prop.3.4.4])**.** *An R-complex $K \in D(\mathcal{X}, R)$ is derived complete if and only if each $H^iK \in \mathsf{Shv}_\tau(\mathcal{C}, R)$ is derived complete.*

Recall that for a chain complex K we define

$$\tau^{\geq n}K = [\dots \to 0 \to 0 \to (K^n/dK^{n-1}) \to K^{n+1} \to K^{n+2} \to \dots]$$

$$\tau^{\leq n}K = [\dots \to K^{n-2} \to K^{n-1} \to (\ker\ d) \to 0 \to 0 \to \dots]$$

Exercise 12. Show that $H^i\tau^{\leq n}K = H^iK$ for $i \leq n$ and $H^i\tau^{\leq n}K = 0$ for $i > n$. Similarly, show that $H^i\tau^{\geq n}K = H^iK$ for $i \geq n$ and $H^i\tau^{\geq n}K = 0$ for $i < n$.

Proof. Suppose that each H^iK is derived complete. We will show that K is derived complete. For any $i \in \mathbb{N}, n \in \mathbb{Z}$ we have

$$\mathrm{Cone}\left(\tau^{\leq n+i}\tau^{\geq n}K \to \tau^{\leq n+i+1}\tau^{\geq n}K\right) \overset{q.i.}{\underset{\sim}{\to}} H^{i+1}K$$

so by induction on i, and Exercise 10, each $\tau^{\leq n+i}\tau^{\geq n}K$ is derived complete. Now we are assuming that $\mathcal{X}$ is replete, so in particular, we have

$$\tau^{\leq m}K \cong R\varprojlim_{n\in\mathbb{N}} \tau^{-n}\tau^{\leq m}K.$$

So by Exercise 11, we find that $\tau^{\leq m}K$ is derived complete. Now consider the short exact sequence of complexes

$$0 \to K \to \mathrm{Cone}\left(\tau^{\leq m}K \to K\right) \to \tau^{\leq m}K[1] \to 0$$

By Exercise 10 the functor T takes short exact sequence to short exact sequences. Since $\tau^{\leq m}K$ is derived complete, we deduce that

$$TK \overset{q.i.}{\to} T\,\mathrm{Cone}\left(\tau^{\leq m}K \to K\right)$$

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[5] π-adically separated means that $\cap_{n\in\mathbb{N}}\pi^nM = 0$.

But
$$\mathrm{Cone}\left(\tau^{\leq m}K \to K\right) \overset{q.i.}{\to} \tau^{\geq m+1}K$$
so
$$TK \overset{q.i.}{\to} T\tau^{\geq m+1}K$$
Finally, from the definition we see that $(T\tau^{\geq m+1}K)^i = 0$ for $i < m$. Since this is valid for any m, we deduce that $H^iTK = 0$ for all i. ☐

Definition 32. *Suppose that $K \in Ch(\mathsf{Shv}_\tau(\mathcal{C}, R))$ is a chain complex. Then we define*
$$K \overset{L}{\otimes}_R R/\mathfrak{m}^n := Cone(K \overset{\pi^n}{\to} K).$$

Remark 33. The functor $- \overset{L}{\otimes}_R R/\mathfrak{m}^n$ that we defined above actually calculates the left derived functor of $-\otimes_R R/\mathfrak{m}^n$ where here $\otimes_R$ is the usual tensor product. Since we only need the derived product in this case, we just take this as the definition.

`exer:derCompletion`

Exercise 13.

1. Show that there is a canonical morphism of sequences of chain complexes from $(\dots \overset{\pi}{\to} K \overset{\pi}{\to} K \overset{\pi}{\to} K)$ to $(\dots \overset{\mathrm{id}}{\to} K \overset{\mathrm{id}}{\to} K \overset{\mathrm{id}}{\to} K)$
2. Deduce that there is a canonical morphism from $(\dots \overset{\mathrm{id}}{\to} K \overset{\mathrm{id}}{\to} K \overset{\mathrm{id}}{\to} K)$ to $(\dots \to K \overset{L}{\otimes}_R R/\mathfrak{m}^2 \to K \overset{L}{\otimes}_R R/\mathfrak{m} \to K)$.
3. Show that there is a short exact sequence
$$0 \to K \to \widehat{K} \to TK \to 0$$
where
$$\widehat{K} := R\varprojlim(K \overset{L}{\otimes}_R R/\mathfrak{m}^n).$$
Deduce that K is derived complete if and only if the morphism $K \to \widehat{K}$ is a quasi-isomorphism.

Proposition 34 ([BS, Lem.3.4.9, Prop.3.5.1])**.** *The functor sending K to $\widehat{K}$ defines a left adjoint to the inclusion $D_{\mathrm{comp}}(\mathcal{X}, R) \subseteq D(\mathcal{X}, R)$.*

Sketch of proof. By Exercise 13 we see that $\widehat{K}$ is derived complete. Suppose that $L \in D(\mathcal{X}, R)$ is also derived complete. Then we want to show that
$$\hom_{D(\mathcal{X},R)}(\widehat{K}, L) \to \hom_{D(\mathcal{X},R)}(K, L)$$
is an isomorphism. By the short exact sequence in Exercise 13(3) it suffices to show that
$$\hom_{D(\mathcal{X},R)}(TK, L) = 0$$
(this uses some homological algebra that we have not covered, but it is not difficult homological algebra). Now we make two claims.

Claim 1. [BS, Lem.3.4.7] We have that $\hom(M, L) = 0$ for all $M \in D(\mathcal{X}, R[\frac{1}{\pi}])$.

Claim 2. [BS, Lem.3.4.8] We have that TK is in the essential image of the canonical functor $D(\mathcal{X}, R[\frac{1}{\pi}]) \to D(\mathcal{X}, R)$.

The proof of these claims is not difficult, but is omitted. □

Definition 35. *We define a tensor product on $D_{\mathrm{comp}}(\mathcal{X}, R)$ using the tensor product on $D(\mathcal{X}, R)$:*

$$K \widehat{\otimes}_R L := \widehat{K \overset{L}{\otimes}_R L}.$$

Here, $\overset{L}{\otimes}_R$ is the derived tensor product on $D(\mathcal{X}, R)$.

A Derived limits

In this subsection we consider a Grothendieck abelian category $\mathcal{A}$ that admits products (in other words, satisfies Grothendieck's axiom (AB3*)). We are concerned with the derived functors

$$R\Pi : D(\prod_{\mathbb{N}} \mathcal{A}) \to D(\mathcal{A})$$

$$R \varprojlim : D(\mathcal{A}^{\mathbb{N}}) \to D(\mathcal{A})$$

associated to product $\Pi : \prod_{\mathbb{N}} \mathcal{A} \to \mathcal{A}$ and limit $\varprojlim : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$. Note that $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ are again Grothendieck abelian categories (since they are functor categories from a small category to a Grothendieck abelian category).

Recall from the lecture Homological Algebra I that for a general left exact functor between Grothendieck abelian categories $F : \mathcal{B} \to \mathcal{B}'$, the derived functor $RF : D(\mathcal{B}) \to D(\mathcal{B}')$ can be calculated as follows. If $C \in Ch^+(\mathcal{B})$ is a bounded below chain complex, then there exists a quasi-isomorphism $C \to I$ with I a bounded below chain complex of injective objects,[6] and $RF(C) \cong F(I)$ in $D(\mathcal{B}')$. More generally, for any chain complex $C \in Ch(\mathcal{B})$, there exists a quasi-isomorphism $C \to Q$ to a fibrant chain complex,[7] and $RF(C) \cong F(Q)$ in $D(\mathcal{B}')$.

lemm:injProd **Lemma 36.** *An object $(I_i)_{i \in \mathbb{N}}$ in $\prod_{\mathbb{N}} \mathcal{A}$ is injective if and only if each I_i is injective in $\mathcal{A}$.*

Exercise 14. Prove Lemma 36.

lemm:injectiveSequences **Lemma 37.** *An object $(\cdots \to A_2 \to A_1 \to A_0)$ in $\mathcal{A}^{\mathbb{N}}$ is injective if and only if each A_i is injective and each $A_{i+1} \to A_i$ is a split surjection.*

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[6] Recall that an object $I \in \mathcal{B}$ is injective if for every monomorphism $A \to B$, every morphism $A \to I$ factors through $A \to B$.

[7] Recall that a chain complex $Q \in Ch(\mathcal{B})$ is fibrant if for every monomorphic quasi-isomorphism $A \to B$ of chain complexes, every morphism $A \to Q$ factors through $A \to B$.

Proof. Suppose $\mathcal{I}_\bullet = (\cdots \to I_2 \to I_1 \to I_0)$ is an injective object in $\mathcal{A}^{\mathbb{N}}$. Let $\lambda_n : \mathcal{A} \to \mathcal{A}^{\mathbb{N}}$ be the functor sending $A \in \mathcal{A}$ to $(\cdots \to 0 \to \underbrace{A \overset{id}{\to} \cdots \overset{id}{\to} A}_{n \text{ morphisms}})$. Then λ_n is exact and a left adjoint to the "evaluation at n" functor Ev_n (which sends $(\cdots \to B_2 \to B_1 \to B_0)$ to B_n). Since Ev_n has an exact left adjoint it sends injectives to injectives, and hence, each $I_n = Ev_n \mathcal{I}_\bullet$ is injective in $\mathcal{A}$. To see that each $I_{n+1} \to I_n$ is split surjective, consider the canonical monomorphism $\lambda_n I_n \to \lambda_{n+1} I_n$. Since $\mathcal{I}_\bullet$ is injective, the canonical morphism $\lambda_n I_n \to \mathcal{I}_\bullet$ factors as $\lambda_n I_n \to \lambda_{n+1} I_n \to \mathcal{I}_\bullet$. The degree $n+1, n, n-1$ piece of this factorisation is

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] \arrow[d] & I_n \arrow[r] \arrow[d, "id"'] & I_{n+1} \arrow[d] \\
I_n \arrow[r] \arrow[d, "id"'] & I_n \arrow[r, "id"] \arrow[d, "id"'] & I_n \arrow[d] \\
I_n \arrow[r] \arrow[d, "id"'] & I_n \arrow[r] \arrow[d, "id"'] & I_{n-1} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

So $I_{n+1} \to I_n$ is split surjective.

Conversely, suppose that $I_\bullet = (\cdots \to I_2 \to I_1 \to I_0)$ is an object of $\mathcal{A}^{\mathbb{N}}$ such that each I_n is injective in $\mathcal{A}$, and each $I_{n+1} \to I_n$ is split surjective. Suppose that $A_\bullet = (\ldots \to A_2 \to A_1 \to A_0) \to (\ldots \to B_2 \to B_1 \to B_0) = B_\bullet$ is a monomorphism in $\mathcal{A}^{\mathbb{N}}$, and that $A_\bullet \to I_\bullet$ is some morphism. We will show by induction that it factors through $A_\bullet \to B_\bullet$. In degree 0, this follows from the fact that I_0 is injective: $A_0 \to B_0$ is a monomorphism and I_0 injective so $A_0 \to I_0$ factors as $A_0 \to B_0 \to I_0$. Suppose that we have factorisations $A_i \to B_i \to I_i$ for all $0 \leq i < n$ which are compatible with the transition morphisms of $A_\bullet, B_\bullet, I_\bullet$ respectively. In particular, we have the following diagram

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d] \arrow[rr, bend left] & B_n \arrow[r, dashed] \arrow[d] & I_n \arrow[d] \\
A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & I_{n-1}
\end{tikzcd}$$

and we are looking for the dashed morphism which makes the diagram commute. By hypothesis, $I_n \to I_{n-1}$ is split surjective. That is, $I_n \cong I_{n-1} \oplus J$ for some J, which is also injective as it is a direct summand of the injective object I_n. As J is injective, the induced morphism $A_n \to J$ factors as $A_n \to B_n \overset{a}{\to} J$. On the other hand, we have the morphism $b : B_n \to B_{n-1} \to I_{n-1}$ from the above diagram. Then we define the dashed morphism to be $(b, a) : B_n \to I_{n-1} \oplus J \cong I_n$. On checks that this makes the diagram commute. □

Now that we consider chain complexes in $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ we will have two indices, (an upper) one for the chain complex direction, and (a lower) one for the $\prod_{\mathbb{N}} \mathcal{A}$, $\mathcal{A}^{\mathbb{N}}$ direction. We will implicitly use the canonical equivalences of categories $Ch(\prod_{\mathbb{N}} \mathcal{A}) \cong \prod_{\mathbb{N}} Ch(\mathcal{A})$ and $Ch(\mathcal{A}^{\mathbb{N}}) \cong Ch(\mathcal{A})^{\mathbb{N}}$.

注意 Beware, however, that the canonical inclusions $Ch^+(\prod_{\mathbb{N}} \mathcal{A}) \subseteq \prod_{\mathbb{N}} Ch^+(\mathcal{A})$ and $Ch^+(\mathcal{A}^{\mathbb{N}}) \subseteq Ch^+(\mathcal{A})^{\mathbb{N}}$ are *not* essentially surjective.

Lemma 38. *A chain complex* $(Q_i^\bullet)_{i \in \mathbb{N}}$ *in* $Ch(\prod_{\mathbb{N}} \mathcal{A})$ *is fibrant if and only if each* $Q_i^\bullet$ *is fibrant in* $Ch(\mathcal{A})$.

Proof. It suffices to note that a morphism $(A_i^\bullet)_{i \in \mathbb{N}} \to (B_i^\bullet)_{i \in \mathbb{N}}$ is a monomorphic quasi-isomorphism if and only if each $A_i^\bullet \to B_i^\bullet$ is a monomorphic quasi-isomorphism. □

lemm:seqFibrant **Lemma 39.** *If a chain complex* $(Q_i^\bullet)_{i \in \mathbb{N}}$ *in* $Ch(\mathcal{A}^{\mathbb{N}})$ *is fibrant then each* $Q_i^\bullet$ *is fibrant in* $Ch(\mathcal{A})$.

Proof. As in the proof of Lemma 37, the “evaluation at n” functor $Ev_n : Ch(\mathcal{A}^{\mathbb{N}}) \to Ch(\mathcal{A})$ has a left adjoint $\lambda_n : Ch(\mathcal{A}) \to Ch(\mathcal{A}^{\mathbb{N}})$ which preserves monomorphisms and quasi-isomorphisms. Consequently, Ev_n sends fibrant objects to fibrant objects. □

prop:limTriangle **Proposition 40.** *Suppose that* $\mathcal{A}$ *is a Grothendieck abelian category with products. Then for any object* $(\dots \to C_2^\bullet \to C_1^\bullet \to C_0^\bullet)$ *in* $Ch(\mathcal{A}^{\mathbb{N}})$*, there is an isomorphism*

$$R \varprojlim C_n^\bullet \cong \operatorname{Cone}\left(R\Pi C_n^\bullet \overset{id-shift}{\longrightarrow} R\Pi C_n^\bullet \right)$$

in $D(\mathcal{A})$.

Proof. In order to calculate $R \varprojlim C_n^\bullet$, replace $(\dots \to C_2^\bullet \to C_1^\bullet \to C_0^\bullet)$ with a quasi-isomorphic fibrant complex $(\dots \to Q_2^\bullet \to Q_1^\bullet \to Q_0^\bullet)$ in $Ch(\mathcal{A}^{\mathbb{N}})$. Recall that every fibrant chain complex is a chain complex of injective objects (the converse is true if the complex is bounded below). In particular, for each i the sequence $(\dots \to Q_2^i \to Q_1^i \to Q_0^i)$ is injective in $\mathcal{A}^{\mathbb{N}}$, and therefore by Lemma 37, the morphisms $Q_{n+1}^i \to Q_n^i$ are split surjective. We will use this fact later.

Now by Lemma 39 each $Q_n^\bullet$ is fibrant. Hence, $(Q_\bullet^\bullet)$ can also be used to calculate the derived products as well. That is,

$$R \varprojlim C_n^\bullet \cong \varprojlim Q_n^\bullet, \qquad R\Pi C_n^\bullet \cong \Pi Q_n^\bullet$$

So it suffices to show that the canonical morphism

$$\varprojlim Q_n^\bullet \to \operatorname{Cone}\left(\Pi Q_n^\bullet \overset{id-\text{shift}}{\longrightarrow} \Pi Q_n^\bullet \right) [-1]$$

is a quasi-isomorphism. But since each $Q^i_{n+1} \to Q^i_n$ is split surjective, it follows that each $\Pi Q^i_n \overset{id-\text{shift}}{\longrightarrow} \Pi Q^i_n$ is surjective. So the sequence

$$0 \to \varprojlim Q^\bullet_n \to \Pi Q^\bullet_n \overset{id-\text{shift}}{\longrightarrow} \Pi Q^\bullet_n \to 0$$

is exact, and therefore the left term is quasi-isomorphic to the shifted cone of the right morphism. □