Noncommutative del Pezzo surfaces

Kazushi Ueda

Why noncommutative varieties?

- Noncommutative rings appear ‘in nature’,
  e.g., from integrable systems.
- Commutative varieties are closed under neither
  - deformations,
  - Fourier–Mukai transforms, nor
  - mirror symmetry.
- quest for ‘quantization of space’

Artin–Zhang (1994)

A *polarized noncommutative variety* is a triple $(\mathcal{A}, \mathcal{O}, (-)(1))$ consisting of

- a Noetherian abelian category $\mathcal{A}$
- an object $\mathcal{O} \in \mathcal{A}$, and
- an autoequivalence $(-)(1) \colon \mathcal{A} \to \mathcal{A}$

such that

- $H^0(\mathcal{O}) = \mathbf{k}$: a field,
- $\forall \mathcal{M} \in \mathcal{A}$, $\dim_{\mathbf{k}} H^0(\mathcal{M}) < \infty$, and
- the pair $(\mathcal{O}, (-)(1))$ is ample, i.e.,
  - $\forall \mathcal{M} \in \mathcal{A}$, $\exists \bigoplus_i \mathcal{O}(-l_i) \to \mathcal{M}$: epi,
  - $\forall \mathcal{M} \to \mathcal{N}$: epi, $\exists n_0$, $\forall n \geq n_0$, $H^0(\mathcal{M}(n)) \to H^0(\mathcal{N}(n))$: epi,

where $H^0(\mathcal{M}) := \operatorname{Hom}(\mathcal{O}, \mathcal{M})$. In this case, one has $\mathcal{A} \cong \operatorname{qgr} A := \operatorname{gr} A / \operatorname{tor} A$ for $A := \bigoplus_{i=0}^n H^0(\mathcal{O}(n))$.

Hochschild–Kostant–Rosenberg isomorphism

X: a smooth variety

- $\mathrm{HH}^2(X) \cong H^2(\mathcal{O}_X) \oplus H^1(\mathcal{T}_X) \oplus H^0\left(\bigwedge^2 \mathcal{T}_X\right)$
- $H^1(\mathcal{T}_X)$: the 'classical' direction
- $H^0\left(\bigwedge^2 \mathcal{T}_X\right)$: the 'strictly noncommutative' direction
- $H^2(\mathcal{O}_X)$: the 'gerby' direction

Artin's conjecture

Any noncommutative surface is birational to either

- a noncommutative projective plane,
- a noncommutative $\mathbb{P}^1$-bundle over a commutative curve, or
- a noncommutative surface which is finite over its center.

AS-regular algebras

- A finitely presented $\mathbb{N}$-graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ over a field $\mathbf{k}$ is *connected* if $A_0 = \mathbf{k}$.
- A connected algebra A is *AS-Gorenstein* of dimension d and parameter a if $\mathbb{R}\mathrm{Hom}_A(\mathbf{k}, A) \simeq \mathbf{k}(a)[-d]$.
- A connected algebra A is *AS-regular* of dimension d if
  - A is AS-Gorenstein of dimension d,
  - A has polynomial growth, and
  - A has global dimension d.
- d-dimensional AS-regular algebras are noncommutative generalizations of polynomial algebras in d variables.

Remark

$A := \mathbf{k}\langle x_1, \dots, x_n \rangle$: a free algebra
$\deg x_i = d_i$, $i = 1, \dots, n$

$$0 \to A(-d_1) \oplus \cdots \oplus A(-d_n) \to A \to \mathbf{k} \to 0 \qquad \text{(exact)}$$

- A is not AS-Gorenstein.
- A has exponential growth.

Artin–Schelter (1987)

A 3-dimensional AS-regular algebra A generated in degree 1 is either *quadratic*, i.e.,

$$0 \to A(-3) \to A(-2)^{\oplus 3} \to A(-1)^{\oplus 3} \to A \to \mathbf{k} \to 0 \quad \text{(exact)},$$

or *cubic*, i.e.,

$$0 \to A(-4) \to A(-3)^{\oplus 2} \to A(-1)^{\oplus 2} \to A \to \mathbf{k} \to 0 \quad \text{(exact)}.$$

Artin–Tate–Van den Bergh (1990)

3-dimensional quadratic AS-regular algebras A such that $\operatorname{qgr} A \ncong \operatorname{coh} \mathbb{P}^2$ are classified by triples (E, L, σ) consisting of

- a genus one curve E,
- a very ample line bundle L of degree 3 on E, and
- $\sigma \in \operatorname{Aut} E$.

3-dimensional cubic AS-regular algebras A such that $\operatorname{qgr} A \ncong \operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$ are classified by triples (E, L, σ) consisting of

- a genus one curve E,
- a line bundle L of degree 2, and
- $\sigma \in \operatorname{Aut} E$.

$\mathbb{Z}$-algebra

- An algebra over a filed **k** is a **k**-linear category with one object.
- A $\mathbb{Z}$-*algebra* is a **k**-linear category A whose set of objects is identified with the set $\mathbb{Z}$ of integers.
- An A-module is a functor $A^{\mathrm{op}} \to \mathrm{Mod}\, \mathbf{k}$.

$\mathbb{Z}$-algebra (paraphrase)

- A $\mathbb{Z}$-algebra is an algebra $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ such that
  - $A_{ij} A_{jk} \subset A_{ik}$,
  - $\exists e_i \in A_{ii}$ satisfying $e_i a = a = a e_j$ for any $a \in A_{ij}$, and
  - $A_{ij} A_{kl} = 0$ if $j \neq k$.
- A can be regarded as a category by

$$A_{ij} = \mathrm{Hom}(j, i).$$

- An A-module is $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that
  - $M_i A_{ij} \subset M_j$
  - e_i acts as the identity on M_i, and
  - $M_i A_{jk} = 0$ if $i \neq j$.

Qgr of $\mathbb{Z}$-algebra

- A $\mathbb{Z}$-algebra is *non-negatively graded* if $A = \bigoplus_{i \geq j} A_{ij}$.
- A non-negatively graded $\mathbb{Z}$-algebra is *connected* if $A_{ii} = \mathbf{k} e_i$ for all $i \in \mathbb{Z}$.
- A module over a $\mathbb{Z}$-algebra is *torsion* if it is a colimit of modules which are finite over $\mathbf{k}$.
- $\operatorname{Qgr} A := \operatorname{Gr} A / \operatorname{Tor} A$

$\mathbb{Z}$-algebras and graded algebras

- A graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ produces a $\mathbb{Z}$-algebra $\check{A} = \bigoplus_{i,j \in \mathbb{Z}} \check{A}_{ij}$ by $\check{A}_{ij} = A_{i-j}$. One has $\operatorname{Qgr} A \cong \operatorname{Qgr} \check{A}$.
- A $\mathbb{Z}$-algebra comes from a graded algebra if and only if it is 1-periodic, i.e., there exists a collection $(A_{ij} \xrightarrow{\sim} A_{i+1,j+1})_{i,j \in \mathbb{Z}}$ of linear isomorphisms compatible with multiplication.
- For a pair (A, B) of graded algebras, one has $\check{A} \cong \check{B}$ if and only if A and B are related by the *Zhang twist*.

3-dimensional quadratic AS-regular $\mathbb{Z}$-algebra

- A: a connected $\mathbb{Z}$-algebra
- $P_i = e_i A$: the i-th projective module
- $S_i = e_i A e_i$: the i-th simple module
- A is a *3-dimensional quadratic AS-regular $\mathbb{Z}$-algebra* if

$$\forall i \in \mathbb{Z}, \quad 0 \to P_{i-3} \to P_{i-2}^{\oplus 3} \to P_{i-1}^{\oplus 3} \to P_i \to S_i \to 0 \quad \text{(exact)}.$$

Bondal–Polishchuk (1994)

- 3-dimensional quadratic AS-regular $\mathbb{Z}$-algebras A with $\operatorname{qgr} A \not\cong \operatorname{coh} \mathbb{P}^2$ are classified by triples consisting of
  - a genus one curve E and
  - very ample line bundles L_1 and L_2 of degree 3 on E such that
    - $L_1 \not\cong L_2$ and
    - $\deg L_i|_C = \deg L_i|_C$ for every irreducible component C of E.
- The map $(E, L, \sigma) \mapsto (E, L, \sigma^* L)$ from ATV triples to BP triples is generically $9 : 1$.
- Fibers are related by 3-torsion translations.

Noncommutative $\mathbb{P}^2$

- A noncommutative $\mathbb{P}^2$ is an abelian category of the form $\operatorname{qgr} A$ for a 3-dimensional quadratic AS-regular algebra.
- The set $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ of 'line bundles' on a noncommutative $\mathbb{P}^2$ is characterized categorically.
- The set of isomorphism classes of noncommutative $\mathbb{P}^2$ are in bijection with the set of isomorphism classes of 3-dimensional quadratic AS-regular $\mathbb{Z}$-algebras.

Van den Bergh (2011)

A *3-dimensional cubic AS-regular $\mathbb{Z}$-algebra* is a connected $\mathbb{Z}$-algebra A with

$$0 \to P_{i-4} \to P_{i-3}^{\oplus 2} \to P_{i-1}^{\oplus 2} \to P_i \to S_i \to 0 \quad \text{(exact)}.$$

They are classified by quadruples (E, L_1, L_2, L_3) consisting of

- a genus one curve E and
- three line bundles L_1, L_2, and L_3 such that
  - both (L_1, L_2) and (L_2, L_3) embed E as a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$,
  - $\deg L_1|_C = \deg L_3|_C$ for every irreducible component C of E, and
  - $L_1 \not\cong L_3$.

A *noncommutative* $\mathbb{P}^1 \times \mathbb{P}^1$ is an abelian category of the form $\operatorname{qgr} A$ for a 3-dimensional cubic AS-regular $\mathbb{Z}$-algebra A.

Acyclic helix

- An object E of a dg category $\mathcal{D}$ is *exceptional* if $\hom(E, E) \simeq \mathbf{k}\,\mathrm{id}_E$.
- A sequence $(E_1, \ldots, E_\ell)$ of exceptional objects is an *exceptional collection* if $\hom(E_i, E_j) \simeq 0$ for $i > j$.
- An exceptional collection is *full* if it generates $\mathcal{D}$.
- A *helix* of dimension d and period ℓ is a sequence $(E_i)_{i\in\mathbb{Z}}$ of objects such that $(E_1, \ldots, E_\ell)$ is a full exceptional collection and $E_{i+\ell} = \mathbb{S}(E_i)[-d]$ for any $i \in \mathbb{Z}$, where $\mathbb{S}$ is the Serre functor of $\mathcal{D}$.
- A helix is *acyclic* if $\mathrm{Hom}^k(E_i, E_j) = 0$ for $i < j$ and $k \neq 0$.
- An acyclic helix $(E_i)_{i\in\mathbb{Z}}$ produces a connected $\mathbb{Z}$-algebra.

Acyclic helix (continued)

Noncommutative $\mathbb{P}^2$ and noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ have acyclic helices which are noncommutative generalizations of

$$\ldots, \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(3), \ldots$$

and

$$\ldots, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2), \ldots,$$

respectively.

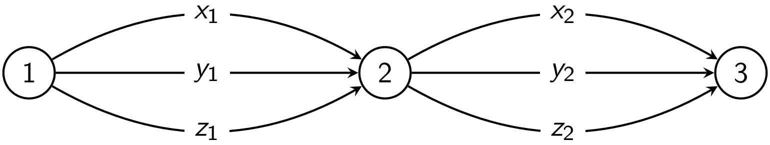
Abdelgadir–Okawa–U

- An acyclic helix $(E_i)_{i\in\mathbb{Z}}$ on a del Pezzo surface defines a *type* of an AS-regular $\mathbb{Z}$-algebra specified by a quiver.
- A *noncommutative weak del Pezzo surface* is qgr of an AS-regular $\mathbb{Z}$-algebra of that type.
- It is a *noncommutative del Pezzo surface* if the pair $(\mathcal{O}, (\mathbb{S}[-2])^{-k})$ of an appropriately defined 'structure sheaf' $\mathcal{O}$ and some power $k \geq 1$ of the shifted Serre functor is ample in the sense of Artin–Zhang.

Abdelgadir–Okawa–U (continued)

- A noncommutative weak del Pezzo surface has an acyclic helix $(E_i)_{i \in \mathbb{Z}}$.
- The algebra $\bigoplus_{i,j=1}^{\ell} \mathrm{Hom}(E_i, E_j)$ is described by a quiver with relations.
- The (rigidified) moduli stack of relations contains the moduli space of marked del Pezzo surfaces (the configuration space of points on $\mathbb{P}^2$) as a locally closed substack.
- A particularly nice (3-block) acyclic helix, known to exist except for $\mathbb{P}^2$ blown up at one or two points by Karpov–Nogin, allows one to define a compact moduli of relations as a GIT quotient with respect to a reductive group.

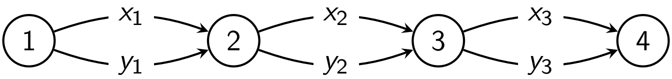
Noncommutative $\mathbb{P}^2$



$$V_1 := \mathbf{k}x_1 \oplus \mathbf{k}y_1 \oplus \mathbf{k}z_1$$
$$V_2 := \mathbf{k}x_2 \oplus \mathbf{k}y_2 \oplus \mathbf{k}z_2$$
$$V_3 := \mathbf{k}x_3 \oplus \mathbf{k}y_3 \oplus \mathbf{k}z_3$$

$$\begin{aligned}\overline{M}_{\mathrm{rel}} &= \mathrm{Gr}_3(V_1 \otimes V_2) /\!/ \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \\ &\cong V_1 \otimes V_2 \otimes V_3 /\!/ \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \\ &\cong \mathbb{P}(6, 9, 12)\end{aligned}$$

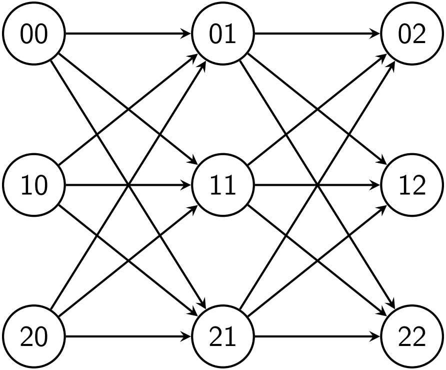
Noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$



$$V_i := \mathbf{k}x_i \oplus \mathbf{k}y_i, \qquad i = 1, 2, 3, 4$$

$$\begin{aligned}\overline{M}_{\mathrm{rel}} &= \mathrm{Gr}_2(V_1 \otimes V_2 \otimes V_3) /\!/ \, \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3) \\ &\cong V_1 \otimes V_2 \otimes V_3 \otimes V_4 /\!/ \, \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \times \mathrm{GL}(V_4) \\ &\cong \mathbb{P}(2, 4, 4, 6)\end{aligned}$$

Noncommutative cubic surfaces



$\overline{M}_{\mathrm{rel}} = \mathbb{A}^{27} // (\mathbb{G}_m)^{27}$ is an 8-dimensional toric variety containing the 4-dimensional configuration space $X(3,6)$ of 6 points in general position on $\mathbb{P}^2$.

Remark

- The quiver on the previous slide for noncommutative cubic surfaces is 3-block complete bipartite of block length $(3, 3, 3)$.
- Similarly, the 3-block complete bipartite quiver of block length $(2, 4, 4)$ gives noncommutative del Pezzo surfaces of degree 2, and
- that of block length $(2, 3, 6)$ gives noncommutative del Pezzo surfaces of degree 1.

Noncommutative cubic surfaces (continued)

$\overline{M}_{\mathrm{rel}}$ is birational to the moduli stack of decuples $(E, (L_{ij})_{i,j=0}^2)$ consisting of a genus one curve E and nine line bundles L_{ij} of degree j:

- Given relations (i.e., a two-sided ideal of the path algebra) of the quiver, the moduli space E of stable representations (with respect to a suitable stability condition) together with the tautological bundles $(L_{ij})_{i,j=0}^2$ gives a decuple.
- Given a decuple $(E, (L_{ij})_{i,j=0}^2)$, the algebra $\operatorname{End}\left(\bigoplus_{i,j=0}^2 L_{ij}\right)$ is described by the quiver with relations.

Spherical helix

- $\mathcal{C}$: a proper dg category with a Serre functor $\mathbb{S}$
- $S \in \mathcal{C}$ is *spherical of dimension d* if $\mathbb{S}(S) = S[d]$ and

$$\mathrm{Hom}^i(S,S) \cong \begin{cases} \mathbf{k} & i = 0, d, \\ 0 & \text{otherwise.} \end{cases} \tag{0.1}$$

- $T_S \coloneqq \mathrm{Cone}\,(\mathrm{ev}\colon\ \hom(S,-) \otimes S \to \mathrm{id}) \in \mathrm{Aut}(\mathcal{C})$
- A sequence $\mathbf{S} = (S_i)_{i=1}^{\ell}$ of spherical objects is a *spherical collection* if $\mathbb{S}|_{\text{the full subcat consisting of }\mathbf{S}} \simeq (-)[d]$.
- It extends to the *spherical helix* $(S_i)_{i\in\mathbb{Z}}$ by

$$S_{i-\ell} = T_{S_{i-\ell+1}} \circ T_{S_{i-\ell+2}} \circ \cdots \circ T_{S_{i-1}}(S_i)[-d-1].$$

- A spherical helix $(S_i)_{i\in\mathbb{Z}}$ is *acyclic* if $\mathrm{Hom}^k(S_i, S_j) = 0$ for any $i < j$ and $k \neq 0$.

Okawa–U (2007.07620)

- An acyclic spherical helix produces an AS-regular $\mathbb{Z}$-algebra.
- One can construct noncommutative del Pezzo surfaces in three steps:
  1. Take an acyclic helix $(E_i)_{i \in \mathbb{Z}}$ on a del Pezzo surface.
  2. The restriction $(S_i := E_i|_D)_{i \in \mathbb{Z}}$ to an anti-canonical divisor D is an acyclic spherical helix.
  3. Deform $(S_i)_{i=1}^{\ell}$ generically. It will generate an acyclic spherical helix.