Homological mirror symmetry for Brieskorn-Pham singularities

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1 Introduction

String theory is an attempt to the theory of everything, but it seems to have been more successful in unifying mathematics instead of physics, relating seemingly distant fields of mathematics through *string dualities*. *Mirror symmetry* is one of them, which lead to surprising mathematical conjectures about mysterious relationship between symplectic geometry of one Calabi-Yau manifold and complex geometry of another Calabi-Yau manifold, called the *mirror manifold*.

Another duality, which is nearly as old as mirror symmetry but has just started to be taken up seriously in mathematics, is *Calabi-Yau/Landau-Ginzburg correspondence*. A *Landau-Ginzburg model* is a pair of a smooth algebraic variety V and a holomorphic function $W : V \to \mathbb{C}$. One typical example is the function

$$\begin{array}{cccc} W: & (\mathbb{C}^\times)^2 & \to & \mathbb{C} \\ & \Psi & & \Psi \\ & (x,y) & \mapsto & x+y+\frac{1}{xy} \end{array}$$

on an algebraic torus, which is well-know to be mirror to the projective plane. Another example is the function

$$\begin{array}{cccc} W: & \mathbb{C}^5 & \to & \mathbb{C} \\ & \Psi & & \Psi \\ & (x_1,\dots,x_5) & \mapsto & x_1^5+\dots+x_5^5, \end{array}$$

on an affine space, and Calabi-Yau/Landau-Ginzburg correspondence states that this is dual to the quintic Calabi-Yau 3-fold

$$\{[x_1 : \dots : x_5] \in \mathbb{P}^4 \mid x_1^5 + \dots + x_5^5 = 0\}$$

in the projective space. More generally, Calabi-Yau/Landau-Ginzburg correspondence states that a Landau-Ginzburg model on $\mathbb{C}^{n+2}$ defined by a weighted homogeneous polynomial W satisfying the Calabi-Yau condition

$$\deg x_1 + \dots + \deg x_{n+2} = \deg W$$

is dual to the Calabi-Yau n-fold defined by W in the corresponding weighted projective space. One can also incorporate finite group actions, which will be important in applications to mirror symmetry.

Calabi-Yau/Landau-Ginzburg correspondence is a duality at the level of $N = 2$ superconformal field theories, which induces a duality on topological string theories obtained from them by the operation called the *topological twist*. There are two ways to perform this twist, and the resulting topological string theories are called A-models and B-models. One can consider either theories with or without boundaries, called open and closed string theories respectively, for both A-models and B-models.

The closed string sector of the A-model is the theory of Gromov-Witten invariants on the Calabi-Yau side. The corresponding theory on the Landau-Ginzburg side is recently developed by Fan, Jarvis and Ruan [7, 6] who call it Fan-Jarvis-Ruan-Witten theory. The closed string sectors of the B-model (at genus zero) are the theory of variations of Hodge structures on the Calabi-Yau side, and the theory of Kyoji Saito's primitive forms on the Landau-Ginzburg side.

Before going to open string sectors, let us discuss a mirror construction by Berglund and Hübsch [1], which generalizes a work of Greene and Plesser [10]. An invertible $n \times n$-matrix $A = (a_j)_{i,j=1}^n$ with integer entries defines a polynomial $W \in \mathbb{C}[x_1, \dots, x_n]$ by

$$W = \sum_{i=1}^{n} x_1^{a_{i1}} \cdots x_n^{a_{in}}. \tag{1}$$

Note that non-zero coefficients of W can be absorbed by rescaling x_i. W obtained in this way is called an *invertible polynomial*. We assume that W has an isolated critical point at the origin. Such polynomials are known by Kreuzer and Skarke [13] to be a decoupled sum of the following three types:

- Fermat: $W = x^p$.
- chain: $W = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n}$.
- loop: $W = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n} x_1$.

For example, the defining polynomial of the E_8-singularity

$$W = x^2 + y^3 + z^5$$

is the sum of three Fermat-type polynomials, and that for the D_n-singularity

$$W = x^{n-1} + xy^2 + z^2$$

is the sum of a chain-type polynomial $x^{n-1} + xy^2$ and a Fermat-type polynomial z^2.

Any invertible polynomial W is naturally graded by an abelian group L generated by $n+1$ elements $\vec{x}_i$ and $\vec{c}$ with relations

$$a_{i1}\vec{x}_1 + \cdots + a_{in}\vec{x}_n = \vec{c}, \qquad i = 1, \dots, n.$$

L is the group of characters of K defined by

$$K = \{(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{n1}} \cdots \alpha_n^{a_{nn}}\}.$$

The *group $G_{\max}$ of maximal diagonal symmetries* is defined as the kernel of the map

$$\begin{array}{ccc} K & \to & \mathbb{C}^\times \\ \Psi & & \Psi \\ (\alpha_1, \dots, \alpha_n) & \mapsto & \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}}, \end{array}$$

so that there is an exact sequence

$$1 \to G_{\max} \to K \to \mathbb{C}^\times \to 1.$$

This exact sequence induces an exact sequence

$$1 \to \mathbb{Z} \to L \to G_{\max}^\vee \to 1$$

of the corresponding character groups, where

$$G_{\max}^\vee = \operatorname{Hom}(G_{\max}, \mathbb{C}^\times)$$

is (non-canonically) isomorphic to $G_{\max}$. Write

$$A^{-1} = \begin{pmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(n)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n^{(1)} & \varphi_n^{(2)} & \cdots & \varphi_n^{(n)} \end{pmatrix}.$$

Then the group $G_{\max}$ is generated by

$$\rho_k = \left(\exp\left(2\pi\sqrt{-1}\varphi_1^{(k)}\right), \dots, \exp\left(2\pi\sqrt{-1}\varphi_n^{(k)}\right) \right), \qquad k = 1, \dots, n.$$

A *Landau-Ginzburg orbifold* is a pair of a Landau-Ginzburg model $W : V \to \mathbb{C}$ and a group G acting on V which preserves W. A pair of an invertible polynomial W and a subgroup $G \subset G_{\max}$ of the group of maximal diagonal symmetries provides a typical example. Put

$$\varphi_i = \varphi_i^{(1)} + \cdots + \varphi_i^{(n)}, \qquad i = 1, \dots, n$$

and define a homomorphism

$$\varphi : \mathbb{C}^\times \to K$$

by

$$\varphi(\alpha) = \left(\alpha^{\ell\varphi_1}, \dots, \alpha^{\ell\varphi_n}\right),$$

where ℓ is the smallest integer such that $\ell\varphi_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Then φ is injective and one has an exact sequence

$$1 \to \mathbb{C}^\times \xrightarrow{\varphi} K \to \overline{G}_{\max} \to 1,$$

where $\overline{G}_{\max} := \operatorname{coker} \varphi$. W is quasi-homogeneous of degree ℓ with respect to the grading

$$\deg x_i = \ell\varphi_i, \qquad i = 1, \dots, n.$$

Since W has an isolated critical point at the origin, the weighted projective hypersurface

$$Y = \{[x_1 : \cdots : x_n] \in \mathbb{P}(\ell\varphi_1, \ldots, \ell\varphi_n) \mid W(x_1, \ldots, x_n) = 0\}$$

is a smooth Deligne-Mumford stack. It is Calabi-Yau if

$$\deg x_1 + \cdots + \deg x_n = \ell. \tag{2}$$

The intersection $\operatorname{Im}\varphi \cap G$ is generated by

$$J = \left(\exp\left(2\pi\sqrt{-1}\varphi_1\right), \ldots, \exp\left(2\pi\sqrt{-1}\varphi_n\right)\right).$$

Assume $J \in G$ and let $\overline{G} = G/\langle J\rangle$ be the image of G in $\overline{G}_{\max}$. The inverse image of $\overline{G}$ by $K \to \overline{G}_{\max}$ will be denoted by H, and its group of characters by M. Then $\overline{G}$ acts naturally on Y, and one can form the quotient stack $[Y/\overline{G}]$. The origin of the following conjecture goes at least as far back as [9]:

Conjecture 1 (Calabi-Yau/Landau-Ginzburg correspondence)**.** *The Calabi-Yau orbifold* $[Y/\overline{G}]$ *is dual to the Landau-Ginzburg orbifold* (W, G)*.*

Being *dual* here means that the generating function for Gromov-Witten invariants of $[Y/\overline{G}]$ and that for Fan-Jarvis-Ruan-Witten invariants of $(W : \mathbb{C}^n \to \mathbb{C}, G)$ are related by an analytic continuation just as in the case of *crepant resolution conjecture* (see e.g. [3]).

As an example, consider the case of the quintic 3-fold, where the Landau-Ginzburg potential $W : \mathbb{C}^5 \to \mathbb{C}$ is given by

$$W(x_1, \ldots, x_5) = x_1^5 + \cdots + x_5^5.$$

The group $G_{\max}$ of maximal diagonal symmetries is generated by

$$\rho_1 = (\zeta, 1, 1, 1, 1), \ \rho_2 = (1, \zeta, 1, 1, 1), \ \ldots, \ \rho_5 = (1, 1, 1, 1, \zeta),$$

where $\zeta = \exp(2\pi\sqrt{-1}/5)$, and one has

$$J = (\zeta, \zeta, \zeta, \zeta, \zeta).$$

If we choose $G = \langle J\rangle$, then Calabi-Yau/Landau-Ginzburg correspondence predicts a relation between Gromov-Witten invariants of the quintic 3-fold and by Fan-Jarvis-Ruan-Witten invariants of $(W, \langle J\rangle)$, which is proved by Chiodo and Ruan [2].

Now let us explain the construction of Berglund and Hübsch. The *transpose* W^* of the invertible polynomial W in (1) is defined by

$$W^T = \sum_{i=1}^n x_1^{a_{1i}} \cdots x_n^{a_{ni}}.$$

Note that the exponent matrix A^T of W^T is the transpose of A. The group $G_{\max}^T$ of maximal diagonal symmetries of W^T is generated by the column vectors $\overline{\rho}_i$ of $(A^T)^{-1} = (A^{-1})^T$. The *transpose* G^T of a subgroup $G \subset G_{\max}$ is defined by Krawitz [12] as

$$G^T = \left\{ \prod_{i=1}^n \overline{\rho}_i^{r_i} \;\middle|\; \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix} A^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z} \text{ for all } \prod_{i=1}^n \rho_i^{a_i} \in G \right\}.$$

note that the correspondence $G \leftrightarrow G^T$ interchanges the maximal group $G_{\max}$ of diagonal symmetry with the trivial group $\{e\}$, and the group generated by J with the intersection $G_{\max} \cap \mathrm{SL}_n(\mathbb{C})$ of $G_{\max}$ with the special linear group.

Conjecture 2 (transposition mirror symmetry)**.** *The Landau-Ginzburg orbifolds* (W, G) *and* (W^T, G^T) *are mirror dual to each other.*

This translates into a construction of mirror pairs of (orbifolds of) Calabi-Yau hypersurfaces in weighted projective spaces by Calabi-Yau/Landau-Ginzburg correspondence. In the case of the quintic 3-fold, the mirror Landau-Ginzburg potential W^T is the same as W, but the group G^T is $G^T_{\max} \cap \mathrm{SL}_5(\mathbb{C}) \cong (\mathbb{Z}/5\mathbb{Z})^4$, so that the corresponding Calabi-Yau 3-fold is the orbifold of the quintic 3-fold by $\overline{G}^T \cong (\mathbb{Z}/5\mathbb{Z})^3$. This reproduces the mirror construction by Greene and Plesser [10].

Now let us discuss open string sector of the topological strings, which is the theory of cyclic A_∞-categories from mathematical point of view (see [4] and references therein). On the Calabi-Yau side of the A-model, the relevant A_∞-category is the Fukaya category, whose objects are Lagrangian submanifolds and whose spaces of morphisms are Lagrangian intersection Floer complexes [8]. On the Landau-Ginzburg side of the A-model, the corresponding A_∞-category is an adaptation of the Fukaya category to Lefschetz fibrations by Seidel [17].

One major obstacle to formulate Calabi-Yau/Landau-Ginzburg correspondence for A-model open string is the lack of definition for "orbifold Fukaya categories";

Problem 3. Define orbifold Fukaya categories $D^b \mathfrak{Fuk}[Y/\overline{G}]$ and $D^b \mathfrak{Fuk}^G W$ for Calabi-Yau orbifolds $[Y/\overline{G}]$ and Landau-Ginzburg orbifolds (W, G).

One way to circumvent this problem is to assume an analogue of crepant resolution conjecture for Fukaya categories and work on a crepant resolution. In the case of the Fermat quintic 3-fold with $G = \langle J \rangle$, the total space $K_{\mathbb{P}^4}$ of the canonical bundle of $\mathbb{P}^4$ gives a crepant resolution of the quotient $\mathbb{C}^5/G$, and the critical point of W pulled back to $K_{\mathbb{P}^4}$ is precisely the quintic 3-fold inside the zero-section. Although this is not a Lefschetz fibration as critical points are not isolated, it is tempting to hope that the definition of Fukaya categories for Lefschetz fibrations can be extended to this case and gives an A_∞-category (quasi-)equivalent to the Fukaya category of the quintic 3-fold.

The A_∞-category for the B-model on the Calabi-Yau side is the derived category of coherent sheaves. The corresponding A_∞-category on the Landau-Ginzburg side is the differential graded category $\mathrm{MF}^M W$ defined as follows; objects are infinite sequence

$$K^\bullet = \{\cdots \to K^i \xrightarrow{k^i} K^{i+1} \xrightarrow{k^{i+1}} K^{i+2} \to \cdots\}$$

of morphisms of M-graded free $\mathbb{C}[x_1, \dots, x_n]$-modules such that $k^{i+1} \circ k^i = W$ and $K^\bullet[2] = K^\bullet(\vec{c})$. The first condition means that the composition of two consecutive morphisms is equal to multiplication by W, and the second condition means that $K^\bullet$ is quasi-2-periodic, in the sense that shifting the sequence by two to the left is equal to changing the M-grading by $\vec{c}$. A morphism between two objects $K^\bullet$ and $L^\bullet$ is a family of morphisms $f^i : K^i \to L^i$ of M-graded modules such that $f^{i+2} = f^i(\vec{c})$. The composition and the differential of morphisms is defined in just the same way as the case of the differential graded category of (unbounded) complexes of M-graded modules. MF stands for matrix factorizations, which refers to the property $k^i \circ k^{i+1} = k^{i+1} \circ k^i = W$. Matrix factorizations are introduced by Eisenbud [5] to study Cohen-Macaulay modules on hypersurfaces. The idea to use matrix factorizations to study B-branes in Landau-Ginzburg models seems to be due to Kontsevich (see Orlov [14]). The homotopy category $\mathrm{HMF}^M W$ of $\mathrm{MF}^M W$ is triangulated, and Calabi-Yau/Landau-Ginzburg correspondence is a theorem in this case:

Theorem 4 (Orlov [15])**.** *If Y is a Calabi-Yau orbifold, then one has an equivalence*

$$D^b \operatorname{coh}[Y/\overline{G}] \cong \mathrm{HMF}^M W$$

of triangulated categories.

The open string sector of mirror symmetry at the topological level is called *homological mirror symmetry.* It is proposed by Kontsevich [11] as an attempt for a conceptual understanding of mirror symmetry. It should take the form

$$D^b \operatorname{coh}[Y/\overline{G}] \cong D^b \mathfrak{Fuk}[Y^T/\overline{G}^T]$$

for Calabi-Yau orbifolds and

$$\mathrm{HMF}^M W \cong D^b \mathfrak{Fuk}^{G^T} W^T$$

for Landau-Ginzburg orbifolds, although the former makes sense only when $\overline{G}^T = \{e\}$ and the latter makes sense only when $G^T = \{e\}$ due to the lack of definition for orbifold Fukaya categories. Homological mirror symmetry for Landau-Ginzburg models with $G^T = \{e\}$ is studied by Atsushi Takahashi [18], who emphasizes its relation with Kyoji Saito's duality for regular systems of weights [16]. If W satisfies the Calabi-Yau condition, then the left hand side is equivalent to the derived category of coherent sheaves on the quotient stack $[Y/\overline{G}_{\max}]$ by Calabi-Yau/Landau-Ginzburg correspondence. The coarse moduli space of $[Y/\overline{G}_{\max}]$ is the projective space, and one can show that $D^b \operatorname{coh}[Y/\overline{G}_{\max}]$ has a full exceptional collection by an inductive process of reducing generic stabilizers of subspaces.

2 Statement of the main result

Let n be a natural number. An element of $\mathbb{C}[x_1, \dots, x_n]$ is said to be a *Brieskorn-Pham polynomial* if it is of the form

$$f_{\boldsymbol{p}} = x_1^{p_1} + \cdots + x_n^{p_n}$$

for positive integers $p_1, \dots, p_n$.

For a positive integer p, let $\mathcal{A}_p$ be the differential graded category with

$$\mathfrak{Ob}(\mathcal{A}_p) = (C_1, \dots, C_p),$$

$$\hom(C_i, C_j) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_{C_i} & \text{if } i = j, \\ \mathbb{C}[-1] & \text{if } i = j - 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the trivial differential. The trivial category without any objects is denoted by $\mathcal{A}_0$.

On the symplectic side, consider the polynomial

$$W_{\boldsymbol{p}}(x_1, \dots, x_n) = f_{\boldsymbol{p}}(x_1, \dots, x_n) + (\text{lower order terms})$$

obtained by Morsifying $f_{\boldsymbol{p}}$. Then $W_{\boldsymbol{p}}$ gives an exact Lefschetz fibration

$$W_{\boldsymbol{p}} : \mathbb{C}^n \to \mathbb{C}$$

with respect to the standard Euclidean Kähler structure on $\mathbb{C}^n$. Let $\mathfrak{Fuk}\, W_{\boldsymbol{p}}$ be the Fukaya category of Lefschetz fibration in the sense of Seidel [17].

Theorem 5. *For any sequence $\boldsymbol{p} = (p_1, \dots, p_n)$ of positive integers, one has an equivalence*

$$D^b \mathfrak{Fuk}\, W_{\boldsymbol{p}} \cong D^b(\mathcal{A}_{p_1-1} \otimes \cdots \otimes \mathcal{A}_{p_n-1})$$

of triangulated categories.

On the complex side, let $L(\boldsymbol{p})$ be the abelian group generate by elements $\vec{x_1}, \dots, \vec{x_n}, \vec{c}$ with relations

$$p_1 \vec{x_1} = \cdots = p_n \vec{x_n} = \vec{c}$$

and consider the triangulated category $\mathrm{HMF}^{L(\boldsymbol{p})} f_{\boldsymbol{p}}$ of $L(\boldsymbol{p})$-graded matrix factorizations.

Theorem 6. *For any sequence $\boldsymbol{p} = (p_1, \dots, p_n)$ of positive integers, one has an equivalence*

$$\mathrm{HMF}^{L(\boldsymbol{p})} f_{\boldsymbol{p}} \cong D^b(\mathcal{A}_{p_1-1} \otimes \cdots \otimes \mathcal{A}_{p_n-1})$$

of triangulated categories.

Let ℓ be the least common multiple of $(p_1, \dots, p_n)$ and equip $A(\boldsymbol{p})$ with a $\mathbb{Z}$-grading given by

$$\deg x_i = a_i = \frac{\ell}{p_i}, \qquad i = 1, \dots, n.$$

Then $Y = \operatorname{Proj} A(\boldsymbol{p})$ is a hypersurface of degree ℓ in the weighted projective space $\mathbb{P}(a_1, \dots, a_n)$. Put

$$K = \{(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{p_1} = \cdots = \alpha^{p_n}\}$$

and define a homomorphism

$$\varphi : \mathbb{C}^\times \to K$$

by

$$\varphi(\alpha) = (\alpha^{a_1}, \dots, \alpha^{a_n}).$$

Then the cokernel

$$G = \operatorname{coker} \varphi$$

of φ is a finite abelian group acting on Y. The Calabi-Yau/Landau-Ginzburg correspondence proved by Orlov [15, Theorem 2.5] gives the following:

Theorem 7. *If Y is Calabi-Yau in the sense that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

then one has an equivalence

$$\operatorname{HMF}^{L(\boldsymbol{p})} W_{\boldsymbol{p}} \cong D^b \operatorname{coh}[Y/G]$$

of triangulated categories.

By combining Theorems 5, 6 and 7, one has a derived equivalence

$$D^b \mathfrak{Fuk}\, W_{\boldsymbol{p}} \cong D^b \operatorname{coh}[Y/G]$$

between the Fukaya category of Lefschetz fibration $W_{\boldsymbol{p}}$ and the category of coherent sheaves on the quotient stack of the Calabi-Yau hypersurface Y in a weighted projective space by a finite group G.

3 A few words on the proof

The proof of Theorem 5 is an induction on n using Picard-Lefschetz theory of Seidel [17]. Assume that one has

$$\mathfrak{Fuk}\, W_{\boldsymbol{p}} \cong \mathcal{A}_{p_1} \otimes \cdots \otimes \mathcal{A}_{p_n}.$$

for a suitable choice of $W_{\boldsymbol{p}}$ and a distinguished basis of vanishing cycles, and consider the Fukaya category for

$$W = W_{\boldsymbol{p}} + W_{p_{n+1}}.$$

The last projection

$$\begin{array}{cccc} \pi : & W_{\boldsymbol{p}}^{-1}(0) & \to & \mathbb{C} \\ & \Cup & & \Cup \\ & (x_1, \ldots, x_{n+1}) & \mapsto & x_{n+1} \end{array}$$

is an exact Lefschetz fibration, whose critical values are inverse images by $W_{p_{n+1}}$ of the critical values of $W_{\boldsymbol{p}}$;

$$\operatorname{Critv}(W) = W_{p_{n+1}}^{-1}(\operatorname{Critv} W_{\boldsymbol{p}}).$$

Put $I_p = \{1, 2, \ldots, p\}$ for a natural number p. One can choose a base point on the x_{n+1}-plane and a distinguished set $(\gamma_{\boldsymbol{i},j})_{\boldsymbol{i},j}$ of vanishing paths, where $\boldsymbol{i} \in I_{\boldsymbol{p}} = \prod_{k=1}^n I_{p_k}$ and $j \in I_{p_{k+1}+1}$. We put the lexicographic order on $I_{\boldsymbol{p}}$, and the order on the set $I_{\boldsymbol{p}} \times I_{p_{n+1}+1}$ is such that $(\boldsymbol{i}, j) \le (\boldsymbol{k}, l)$ if $j \ge l$ or $j = l$ and $\boldsymbol{i} \le \boldsymbol{k}$. Let $\Delta_{\boldsymbol{i},j} \subset \pi^{-1}(0)$ be the vanishing cycle of π along $\gamma_{\boldsymbol{i},j}$. Then one has

$$\mathfrak{Ob}(\mathfrak{Fuk}\, \pi) = (\Delta_{\boldsymbol{i},k})_{(\boldsymbol{i},k) \in I_{\boldsymbol{p}} \times I_{p_{n+1}+1}},$$

$$\hom(\Delta_{\boldsymbol{i},k}, \Delta_{\boldsymbol{j},l}) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_{\Delta_{\boldsymbol{i},k}} & \text{if } (\boldsymbol{i}, k) = (\boldsymbol{j}, l), \\ \hom_{\mathcal{A}_{\boldsymbol{p}}}(C_{\boldsymbol{i}}, C_{\boldsymbol{j}}) \oplus \hom_{\mathcal{A}_{\boldsymbol{p}}}(C_{\boldsymbol{j}}, C_{\boldsymbol{i}})^\vee[-n] & \text{if } (\boldsymbol{i}, k) < (\boldsymbol{j}, l), \\ 0 & \text{otherwise.} \end{cases}$$

A theorem of Seidel [17] shows that there is a full and faithful functor

$$\mathfrak{Fuk}\, W^{-1}(0) \to D^b\, \mathfrak{Fuk}\, \pi$$

such that the vanishing cycle $C_{\boldsymbol{i},j}$ for $(\boldsymbol{i}, j) \in I_{\boldsymbol{p}} \times I_{p_{n+1}}$ is mapped to

$$\mathrm{Cone}(\Delta_{\boldsymbol{i},j+1} \to \Delta_{\boldsymbol{i},j}),$$

and this allows to compute $\hom(C_{\boldsymbol{i},j}, C_{\boldsymbol{i}',j'})$ and A_∞-operations on them in $\mathfrak{Fuk}\, W^{-1}(0)$. The A_∞-category $\mathfrak{Fuk}\, W$ is just the directed subcategory of $\mathfrak{Fuk}\, W^{-1}(0)$ with respect to the collection $(C_{\boldsymbol{i},j})_{\boldsymbol{i},j}$.

As for the proof of Theorem 6, let $A(\boldsymbol{p}) = \mathbb{C}[x_1, \dots, x_n]/(f_{\boldsymbol{p}})$ be the coordinate ring of the affine hypersurface, $k = A(\boldsymbol{p})/(x_1, \dots, x_n)$ be the structure sheaf of the origin, and

$$I = \{a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n \in L \mid -p_1 + 2 \le a_1 \le 0, \dots, -p_n + 2 \le a_n \le 0\}$$

be a finite subset of $L(\boldsymbol{p})$. One can use an argument of Orlov [15] to give a full and faithful functor

$$\Psi : \mathrm{HMF}^{L(\boldsymbol{p})}\, f_{\boldsymbol{p}} \to D^b(\mathrm{gr}\text{–}A(\boldsymbol{p}))$$

from the triangulated category of $L(\boldsymbol{p})$-graded matrix factorizations to the derived category of finitely-generated $L(\boldsymbol{p})$-graded $A(\boldsymbol{p})$-modules, whose essential image is generated by $(k(\vec{m}))_{\vec{m} \in I}$. The full differential graded subcategory of (the enhancement of) $D^b(\mathrm{gr}\text{–}A(\boldsymbol{p}))$ consisting of $(k(\vec{m}))_{\vec{m} \in I}$ is quasi-equivalent to $\mathcal{A}_{p_1-1} \otimes \cdots \otimes \mathcal{A}_{p_n-1}$, and Theorem 6 follows.

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