Dimer models and matrix factorizations

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(joint work with Akira Ishii)

A dimer model is a bicolored graph on a real 2-torus which encodes the information of a quiver with relations. One can show that for any smooth quasi-projective toric Calabi-Yau 3-fold Y, there is a dimer model G such that

- the moduli space $\mathcal{M}_\theta$ of θ-stable representations of the quiver Γ with relations associated with G of dimension vector $(1, \ldots, 1)$ is isomorphic to Y for a suitable choice of a stability parameter θ, and
- the direct sum $\mathcal{E} = \bigoplus_v \mathcal{E}_v$ of the tautological bundles is a tilting object whose endomorphism algebra is isomorphic to the path algebra $\mathbb{C}\Gamma$ of the quiver Γ with relations.

This gives a combinatorial description of the derived category of toric Calabi-Yau 3-fold;

$$D^b \operatorname{coh} Y \cong D^b \operatorname{mod} \mathbb{C}\Gamma.$$

See e.g. [1] and references therein for the proof of these facts.

Let Y_0 be the union of all the toric divisors of Y, and $W \in H^0(\mathcal{O}_\mathcal{M})$ be the defining function of Y_0, which can also be considered as a central element of $\operatorname{End} \mathcal{E} \cong \mathbb{C}\Gamma$. The restriction $\mathcal{E}|_{Y_0}$ is a tilting object in $D^b \operatorname{coh} Y_0$, and one has an isomorphism $\operatorname{End}(\mathcal{E}|_{Y_0}) \cong \mathbb{C}\Gamma/(W)$ of algebras [2]. This gives an equivalence

$$D^b \operatorname{coh} Y_0 \cong D^b \operatorname{mod} \mathbb{C}\Gamma/(W),$$

of derived categories, which in turn induces an equivalence

(1) $$D^b_{\operatorname{sing}}(\operatorname{coh} Y_0) \cong D^b_{\operatorname{sing}}(\operatorname{mod} \mathbb{C}\Gamma/(W)).$$

of singularity categories. The left hand side of (1) is equivalent to the the triangulated category of non-affine matrix factorizations of W on Y, whereas the right hand side is equivalent to the triangulated category of non-commutative matrix factorizations of W over $\mathbb{C}\Gamma$.

REFERENCES

[1] Akira Ishii and Kazushi Ueda, *Dimer models and the special McKay correspondence*, arXiv:0905.0059.
[2] Akira Ishii and Kazushi Ueda, *Dimer models and exceptional collections*, arXiv:0911.4529.