Mirror symmetry for Calabi-Yau manifolds and mirror symmetry for singularities

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The lattice of vanishing cycles equipped with the intersection form is called the *Milnor lattice*, which is one of the central objects in singularity theory. The Milnor lattice admits a categorification called the *Fukaya-Seidel category*, which is an A_∞-category whose objects are vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes.

Fukaya-Seidel categories appear in homological mirror symmetry for Fano manifolds. If we take the projective space $\mathbb{P}^n$ as an example, then the mirror is given by the Laurent polynomial

$$W = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}$$

defining a regular map $W : (\mathbb{C}^\times)^n \to \mathbb{C}$, and one has an equivalence

$$D^b \operatorname{coh} \mathbb{P}^n \cong D^b \mathfrak{Fuk}\, W \tag{1}$$

of triangulated categories [7, 2].

Fukaya-Seidel categories also appear in homological mirror symmetry for singulairites. If we take a Brieskorn-Pham polynomial

$$f = x_1^{p_1} + \cdots + x_n^{p_n}$$

as an example, then the mirror is the Brieskorn-Pham singularity $R = \mathbb{C}[x_1, \dots, x_n]/(f)$ equipped with a grading by the abelian group

$$L = \mathbb{Z}\vec{x}_1 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n \oplus \mathbb{Z}\vec{c}/(p_1\vec{x}_1 - \vec{c}, \dots, p_n\vec{x}_n - \vec{c})$$

of rank one, and one has an equivalence

$$D^b \mathfrak{Fuk}\, f \cong D^b_{\mathrm{sing}}(\operatorname{gr} R) \tag{2}$$

of triangulated categories [3]. Here, the category on the right hand side is the *stable derived category*, defined as the quotient category $D^b(\operatorname{gr} R)/D^{\mathrm{perf}}(\operatorname{gr} R)$ of the bounded derived category of finitely-generated L-graded R-modules by the full subcategory consisting of bounded complexes of projective modules. Similar result has been proved also for arbitrary Sebastiani-Thom sum of singularities of types A and D [1].

Now assume that the Milnor fiber of f can be compactified to a Calabi-Yau manifold Y. A typical example is the case when $n = 3$ and $f = x^2 + y^3 + z^7$, which defines one of Arnold's 14 exceptional unimodal singularities called the E_{12}-singularity. The mirror Calabi-Yau manifold $\check{Y}$ of Y is obtained as (a crepant resolution of) the quotient of Y by a suitable abelian group, and one expects an equivalence

$$D^b \mathfrak{Fuk}\, Y \cong D^b \operatorname{coh} \check{Y} \tag{3}$$

of triangulated categories [5]. The Fukaya-Seidel category $\mathfrak{Fuk}\, f$ is a *directed subcategory* of $\mathfrak{Fuk}\, Y$, and the stable derived category $D^b_{\mathrm{sing}}(\mathrm{gr}\, R)$ is a directed subcategory of $D^b \operatorname{coh} \check{Y}$ [11], so that it is natural to expect the existence of a commutative diagram

$$\begin{tikzcd} D^b \mathfrak{Fuk}\, f \arrow[r, hook] \arrow[d, "\wr"] & D^b \mathfrak{Fuk}\, Y \arrow[d, "\wr"] \\ D^b_{\mathrm{sing}}(\mathrm{gr}\, R) \arrow[r, hook] & D^b \operatorname{coh} \check{Y} \end{tikzcd}$$

where horizontal arrows are embeddings of directed subcategories and vertical arrows are homological mirror symmetry. This helps, for instance, to understand strange duality for Arnold's 14 exceptional unimodal singularities in the context of mirror symmetry for K3 surfaces [4].

On the other hand, the compatibility

$$\begin{tikzcd} D^b \mathfrak{Fuk}\, W \arrow[r, hook] \arrow[d, "\wr"] & D^b \mathfrak{Fuk}\, Y \arrow[d, "\wr"] \\ D^b \operatorname{coh} \mathbb{P}^n \arrow[r, hook] & D^b \operatorname{coh} \check{Y} \end{tikzcd}$$

of homological mirror symmetry for the projective space and that for its Calabi-Yau hypersurface is known by [8, 9, 10, 6], and it is an interesting problem to generalize this to, say, complete intersections in toric stacks.

References

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