Homological Mirror Symmetry and McKay Correspondence

Kazushi Ueda

Let A be a finite abelian subgroup of $SL_3(\mathbb{C})$. A has a natural action on $\mathbb{C}^3$ and the quotient $\mathbb{C}^3/A$ has the crepant resolution $A\text{-Hilb}\,\mathbb{C}^3$ by Nakamura [4]. By Bridgeland, King, and Reid [3], there exists an equivalence of triangulated categories

$$D^b\mathrm{coh}_0\, A\text{-Hilb}\,\mathbb{C}^3 \cong D^b\mathrm{coh}_0^A\,\mathbb{C}^3$$

between the derived category of coherent sheaves on $A\text{-Hilb}\,\mathbb{C}^3$ supported on the exceptional set and the derived category of A-equivariant coherent sheaves on $\mathbb{C}^3$ supported at the origin.

The quotient $\mathbb{C}^3/A$ has a structure of a toric variety, determined by a fan whose one-dimensional cones are generated by elements of $\mathbb{Z}^3$ of the following forms:

$$\overline{v_1} = (v_{1,1}, v_{1,2}, 1),\ \overline{v_2} = (v_{2,1}, v_{2,2}, 1),\ \overline{v_3} = (v_{3,1}, v_{3,2}, 1).$$

Here, $v_i = (v_{i,1}, v_{i,2})$, $i = 1, 2, 3$, are elements of $\mathbb{Z}^2$. We assume that the convex hull of $\{v_i\}_{i=1}^3$ contains at least one lattice point in its interior. Such v_i's can be normalized by the actions of $SL_2(\mathbb{Z})$ and translations to

$$v_1 = (n-1, -1),\ v_2 = (-1, n-1),\ v_3 = (-1, -1), \tag{1}$$

for $n = 3, 4, \ldots$, or

$$v_1 = (n, 0),\ v_2 = (0, n),\ v_3 = (-na, -nb) \tag{2}$$

for $a, b, n = 1, 2, \ldots$. Now take a generic Laurent polynomial W in two variables x and y whose Newton polygon is the convex hull of $\{v_i\}_{i=1}^3$, and endow $(\mathbb{C}^\times)^2$ with a symplectic structure by $\omega = d\arg x \wedge d|x|/|x| + d\arg y \wedge d|y|/|y|$. Then W considered as a map from $(\mathbb{C}^\times)^2$ to $\mathbb{C}$ is an exact Lefschetz fibration in the sense of Seidel [8], and one can define its *directed Fukaya category* $\mathfrak{Fuk}^{\to} W$ whose objects are vanishing cycles and whose morphisms are Lagrangian intersection Floer cohomology [6].

Although $\mathfrak{Fuk}^{\to} W$ is not an honest category but merely an A_∞-category in general, it turns out to be a differential graded category with a trivial differential for the above W and a suitable choice of a distinguished basis of vanishing cycles. Then, one can consider its trivial extension category $\mathfrak{Fuk}\, W$ as in [9], (10a). Define the derived category $D^b\mathfrak{Fuk}\, W$ of $\mathfrak{Fuk}\, W$ by using twisted complexes [2].

Theorem 1. *In the above situation, we have an equivalence of triangulated categories*

$$D^b\mathfrak{Fuk}\, W \cong D^b\mathrm{coh}_0^A\,\mathbb{C}^3.$$

The proof is given by choosing an explicit correspondence between generators of both sides and comparing morphisms between them.

The generators of $D^b\mathrm{coh}_0^A\mathbb{C}^3$ are given by the structure sheaf of the origin tensored with irreducible representations of A, and morphisms between them can be computed by the Koszul resolution of the origin. In $D^b\mathfrak{Fuk}\, W$, we can draw the pictures of the vanishing cycles of W and compute their Floer cohomologies by "painting triangles."

When v_i's are as in (2), drawing vanishing cycles of W and computing Floer cohomologies among them can be reduced by the n^2-fold cover $(\mathbb{C})^2 \ni (x, y) \mapsto (x^n, y^n) \in (\mathbb{C})$ to the case of $n = 1$, which are treated by Seidel [7] when $a = b = 1$ and by Auroux, Katzarkov and Orlov [1] in the general case. The case when v_i's are as in (1) requires a separate treatment. See [5] for the case when $n = 3$.

Theorem 1 can be used to compute the Stokes matrix for certain hypergeometric series of Gelfand-Kapranov-Zelevinsky type [5].

References

[1] D. Auroux, L. Katzarkov, and D. Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, to appear in Annals of Math., Arxiv:math.AG/0404281

[2] A. Bondal and M. Kapranov, *Enhanced triangulated categories*, Math. Sb., 181(5) (1990), 669-683.

[3] T, Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc., 14(3) (2001), 535-554.

[4] I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom., 10(4) (2001), 757-779.

[5] K. Ueda, *Stokes Matrix for the Quantum Cohomology of Cubic Surfaces*, ArXiv:math.AG/0505350.

[6] P. Seidel, *Vanishing cycles and mutation*, In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, Progr. Math. 202 (2001), 65-85.

[7] P. Seidel, *More about vanishing cycles and mutation*, In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, World Sci. Publishing, 2001, 429-465.

[8] P. Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003), 1003-1063.

[9] P. Seidel, *Homological Mirror Symmetry for the quartic surface*, ArXiv:math.AG/0310414.