

Mirror symmetry and K3 surfaces

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Abstract

We review some of the interplay between mirror symmetry and K3 surfaces.

1 Introduction

Mirror symmetry is a mysterious relationship, originally suggested by string theorists, between symplectic geometry of one Calabi-Yau manifold and complex geometry of another Calabi-Yau manifold. One of the earliest prediction in mirror symmetry is that Calabi-Yau 3-fold should come in pairs (Y, \check{Y}) in such a way that their Hodge numbers satisfy $h^{1,1}(Y) = h^{1,2}(\check{Y})$ and $h^{1,2}(Y) = h^{1,1}(\check{Y})$. Note that $h^{1,1}(Y)$ is the number of parameters for the Kähler structures, and $h^{1,2}(Y)$ is the dimension of the moduli space of complex structures. More generally, a pair (Y, \check{Y}) of Calabi-Yau n -folds is said to be a *topological mirror pair* if their Hodge numbers satisfy

$$h^{i,j}(Y) = h^{i,n-j}(\check{Y}) \quad (1.1)$$

for any i, j . Although topological mirror symmetry is much weaker than other versions of mirror symmetry, construction of a topological mirror partner of a given Calabi-Yau manifold is already a highly non-trivial problem. One subtlety is the existence of rigid Calabi-Yau manifolds, as the mirror partner of such manifolds can not exist as a Calabi-Yau manifold. Another subtlety comes in if one allows Calabi-Yau n -folds to have singularities, which motivated the theory of stringy Hodge numbers and motivic integration.

For K3 surfaces, topological mirror symmetry seems to be trivial at first sight, since every K3 surface has the identical Hodge numbers. The moduli space of complex structures and the moduli space of Kähler structures are somehow ‘mixed’, as they both live in $H^2(Y; \mathbb{C})$. In a sense, it is more natural to work with the moduli space of hyperKähler structures, instead of those of complex structures and Kähler structures. However, ‘topological’ mirror symmetry for K3 surfaces can be formulated, not as an exchange of the Hodge numbers, but as an exchange of the algebraic lattice and the transcendental lattice inside the total cohomology group. As such, it is much more subtle than topological mirror symmetry for Calabi-Yau 3-folds, partly because lattices have more structures than Hodge numbers, and partly because these lattices are sensitive to the complex structure of the K3 surface.

Classical mirror symmetry is the mysterious relationship between Gromov-Witten invariants of Y and period integrals of its mirror manifold \check{Y} . It is also known as *Hodge-theoretic mirror symmetry*, since it can be formulated as an isomorphism between two variations of Hodge structures. One of them, called the A-model VHS, is defined on

the ‘moduli space of complexified Kähler structures’ of Y , and encode the information of Gromov-Witten invariants of Y . There is no satisfactory definition of the moduli space of complexified Kähler structures of Y ; the space of stability conditions on the derived category of coherent sheaves [Bri07, Bri08] is expected to be the universal cover of this moduli space. The other one, called the B-model VHS, is the variation of Hodge structures on the moduli space of complex structures on \check{Y} defined by the usual Hodge theory. Classical mirror symmetry started with the prediction that the numbers of rational curves on a general quintic hypersurface in \mathbb{P}^3 can be computed by period integrals of its mirror family [CdLOGP91], and attracted much attention from mathematicians.

Deformation invariance of Gromov-Witten invariants implies that Gromov-Witten invariants of K3 surfaces are trivial, since a generic K3 surface does not have any holomorphic curve at all. Hence classical mirror symmetry for K3 surfaces reduces essentially to the study of period maps. In particular, the Yukawa coupling can be identified with the cup product on the mirror [Dol96].

Homological mirror symmetry is introduced by Kontsevich [Kon95] to give a deeper understanding of mirror symmetry. A pair (Y, \check{Y}) of Calabi-Yau manifolds is a *homological mirror pair* if one has an equivalence

$$D^\pi \mathfrak{Fuk} Y \cong D^b \text{coh } \check{Y} \tag{1.2}$$

of derived categories. It is hard to find a homological mirror pair, and the only known homological mirror pair of K3 surfaces is the case when Y is a quartic hypersurface in \mathbb{P}^3 , due to Seidel [Sei11]. There is a fascinating interplay between homological mirror symmetry and monodromy of period maps, also pioneered by Kontsevich. It is closely related to stability conditions on triangulated categories introduced by Bridgeland [Bri07].

A geometric picture for mirror symmetry is provided by the *Strominger-Yau-Zaslow conjecture* [SYZ96], which states that any Calabi-Yau manifold has a structure of a special Lagrangian torus fibration $\pi: Y \rightarrow B$, and its mirror is obtained as the dual special Lagrangian torus fibration $\check{\pi}: \check{Y} \rightarrow B$. This motivated the construction of mirror manifolds using integral affine manifolds with singularities [KS06, GS11]. We will not review this here, and refer the interested reader to the excellent review [Gro] and references therein.

This review is organized as follows: In Section 2, we recall strange duality and its generalizations, which are now understood as incarnations of mirror symmetry. In Section 3, we discuss transposition mirror construction by Berglund and Hübsch. In Section 4, we discuss mirror symmetry for K3 surfaces following Aspinwall and Morrison. In Section 5, we discuss the notion of lattice-polarized K3 surfaces, which is introduced by Nikulin and used by Dolgachev to study mirror symmetry for K3 surfaces. In Section 6, we discuss mirror construction due to Batyrev using polar duality of reflexive polytopes. In Section 7, we discuss classical mirror symmetry for anti-canonical hypersurfaces in toric weak Fano 3-folds. In Section 8, we discuss a conjecture of Dolgachev on the relation between Batyrev mirrors and Dolgachev mirrors of K3 surfaces. In Section 9, we discuss Bridgeland stability conditions on K3 surfaces. In Section 10, we discuss mirror construction, due independently to Borcea and Voisin, for Calabi-Yau 3-folds associated with 2-elementary K3 surfaces.

2 Strange duality

2.1 The modality of a singularity

Two germs $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ of hypersurface singularities defined by convergent power series $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ and $g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ are *right equivalent* if there is a holomorphic change of coordinates $\varphi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $f = g \circ \varphi$. If the critical point of the germ $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ of a holomorphic function is isolated, then it is right equivalent to a polynomial of degree at most $\mu + 1$ [Tou68]. Here μ is the *Milnor number* of f , defined as the dimension of the Jacobi ring $\mathbb{C}\{\{z_1, \dots, z_n\}\}/(\partial_{z_1} f, \dots, \partial_{z_n} f)$ of f . This *finite-determinacy* of isolated singularities allows one to reduce the classification of isolated critical points of holomorphic functions up to right equivalence to a finite-dimensional problem.

Let G be a Lie group acting on a manifold M . The *modality* of a point $f \in M$ is the smallest integer m such that a sufficiently small neighborhood of f can be covered by a finite number of m -parameter families of orbits. The modality of the germ of a function is the modality of its jet in the space of k -jets with respect to the right action of the group of coordinate transformations for sufficiently large k . If f is the defining equation of an isolated singularity, then the modality of f is called the modality of the singularity. In other words, the modality of a singularity is the minimal number of continuous parameters needed to parametrize right equivalence classes of singularities that are close to p . Singularities of modality zero are called *simple singularities*, and have well-known ADE classification. Next in line come *unimodal singularities*, which are classified by Arnold [Arn75] into an infinite series

$$T_{p,q,r} : x^p + y^q + z^r + axyz \tag{2.1}$$

and 14 exceptional cases. Here (p, q, r) is a triple of natural numbers satisfying $1/p + 1/q + 1/r \leq 1$, and a is a general complex parameter. The $T_{p,q,r}$ -singularity is called a *simple elliptic singularity* if $1/p + 1/q + 1/r = 1$, and a *cusp singularity* if $1/p + 1/q + 1/r < 1$. *Exceptional unimodal singularities* are listed in Table 2.1, where a is again a general complex parameter. The defining polynomial is weighted homogeneous for $a = 0$, and the weights in Table 2.1 are the primitive weights of the variables and the defining polynomial.

There are two natural ways to study singularities; one is to *resolve* the singularity, and the other is to *deform* the singularity. A resolution of a surface singularity is *good* if the exceptional locus is a simple normal crossing divisor. A good resolution is *minimal* if any contraction of an exceptional curve gives a non-good resolution. The minimal good resolution of an exceptional unimodal singularity consists of four rational curves; one is an exceptional curve of the first kind, and the others are mutually disjoint rational curves, each intersecting the first curve in one point. Figure 2.1 shows the dual graph of the exceptional divisor. The self-intersection numbers $\delta = (\delta_1, \delta_2, \delta_3)$ of the three exceptional curves is called the *Dolgachev number* of the singularity.

The *Milnor fiber* of a singularity is the intersection $f^{-1}(\epsilon) \cap B_\delta$ of the deformation $f^{-1}(\epsilon)$ of the singularity with a sufficiently small ball $B_\delta = \{(x, y, z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 \leq \delta\}$. Here ϵ is a sufficiently small number which may depend on δ . The diffeomorphism type of the Milnor fiber is independent of the choice of δ and ϵ . The Milnor fiber is homotopy-equivalent to the bouquet $(S^2)^{\vee \mu}$ of μ 2-spheres, where μ is

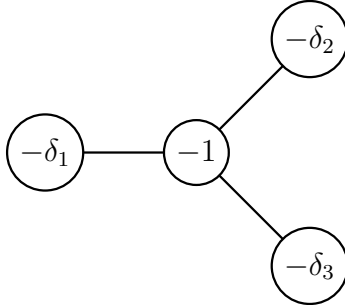


Figure 2.1: The diagram $T(\delta_1, \delta_2, \delta_3)$

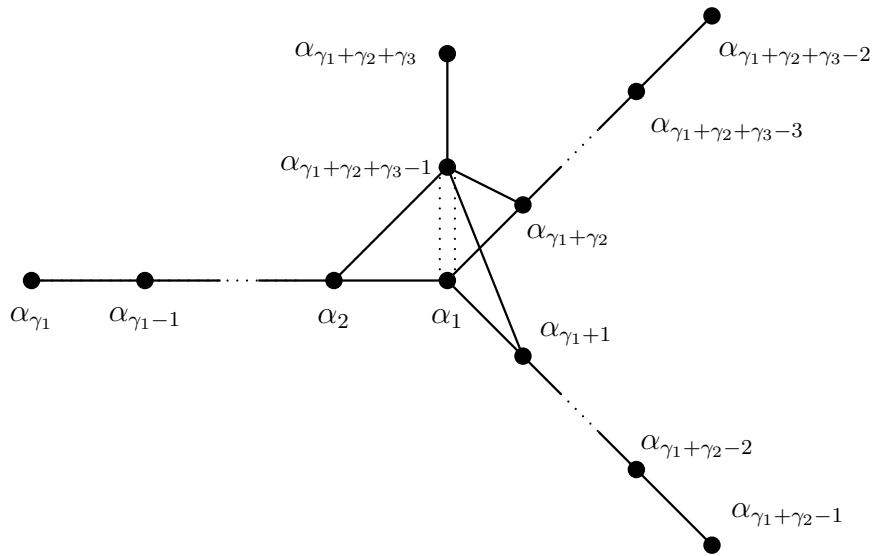


Figure 2.2: The diagram $\widehat{T}(\gamma_1, \gamma_2, \gamma_3)$

name	normal form	weight	δ	γ	dual
E_{12}	$x^3 + y^7 + z^2 + axy^5$	(6, 14, 21; 42)	(2, 3, 7)	(2, 3, 7)	E_{12}
E_{13}	$x^3 + xy^5 + z^2 + ay^8$	(4, 10, 15; 30)	(2, 4, 5)	(2, 3, 8)	Z_{11}
E_{14}	$x^3 + y^8 + z^2 + axy^6$	(3, 8, 12; 24)	(3, 3, 4)	(2, 3, 9)	Q_{10}
Z_{11}	$x^3y + y^5 + z^2 + axy^4$	(6, 8, 15; 30)	(2, 3, 8)	(2, 4, 5)	E_{13}
Z_{12}	$x^3y + xy^4 + z^2 + ax^2y^3$	(4, 6, 11; 22)	(2, 4, 6)	(2, 4, 6)	Z_{12}
Z_{13}	$x^3y + y^6 + z^2 + axy^5$	(3, 5, 9, 18)	(3, 3, 5)	(2, 4, 7)	Q_{11}
W_{12}	$x^4 + y^5 + z^2 + ax^2y^3$	(4, 5, 10; 20)	(2, 5, 5)	(2, 5, 5)	W_{12}
W_{13}	$x^4 + xy^4 + z^2 + ay^6$	(3, 4, 8, 16)	(3, 4, 4)	(2, 5, 6)	S_{11}
Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	(6, 8, 9; 24)	(2, 3, 9)	(3, 3, 4)	E_{14}
Q_{11}	$x^3 + y^2z + xz^3 + az^5$	(4, 6, 7; 18)	(2, 4, 7)	(3, 3, 5)	Z_{13}
Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	(3, 5, 6; 15)	(3, 3, 6)	(3, 3, 6)	Q_{12}
S_{11}	$x^4 + y^2z + xz^2 + ax^3z$	(4, 5, 6, 16)	(2, 5, 6)	(3, 4, 4)	W_{13}
S_{12}	$x^3y + y^2z + xz^3 + az^5$	(3, 4, 5; 13)	(3, 4, 5)	(3, 4, 5)	S_{12}
U_{12}	$x^3 + y^3 + z^4 + axyz^2$	(3, 4, 4; 12)	(4, 4, 4)	(4, 4, 4)	U_{12}

Table 2.1: 14 exceptional unimodal singularities

the Milnor number [Mil68]. The middle-dimensional homology group of the Milnor fiber, equipped with the intersection form, is called the *Milnor lattice*. The Milnor lattice has a distinguished basis $\{\alpha_i\}_{i=1}^{\mu}$ consisting of *vanishing cycles*, which is well-defined up to an action of a braid group. By a suitable choice of a distinguished basis of vanishing cycles, the Coxeter-Dynkin diagram of an exceptional unimodal singularity can be written in the form \widehat{T}_γ given in Figure 2.2 [Gab74]. The triple $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is called the *Gabrielov number* of the singularity. The *strange duality* is an observation by Arnold that exceptional unimodal singularities come in pairs in such a way that the Dolgachev number and the Gabrielov number are interchanged. For more on foundations of singularity theory, one can see [AGZV85, AGZV88, Mil68] and references therein.

2.2 Triangle singularities

For a sequence $(p, q, r) \in (\mathbb{Z}^+)^3$ of positive integers, the *triangle group*

$$\Delta_{p,q,r} := \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle \quad (2.2)$$

is the group generated by reflections in the triangle of angles π/p , π/q and π/r . The triangle group is called *spherical*, *Euclidean*, or *hyperbolic* depending on whether $1/p + 1/q + 1/r$ is greater than, equal to, or less than one respectively. Let $\Gamma_{p,q,r} \subset \Delta_{p,q,r}$ be the *von Dyck group*, which is the index two subgroup of the triangle group consisting of compositions of even numbers of reflections. It is written as

$$\Gamma_{p,q,r} := \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^p = \bar{y}^q = \bar{z}^r = \overline{xyz} = 1 \rangle \quad (2.3)$$

where the inclusion $\Gamma_{p,q,r} \hookrightarrow \Delta_{p,q,r}$ sends \bar{x} , \bar{y} , and \bar{z} to ab , bc and ca respectively. The sequence (p, q, r) is called the *signature* of the triangle group.

type	A_{p+q}	D_{n+2}	E_6	E_7	E_8
signature	$(1, p, q)$	$(2, 2, n)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$

Table 2.2: Spherical signatures

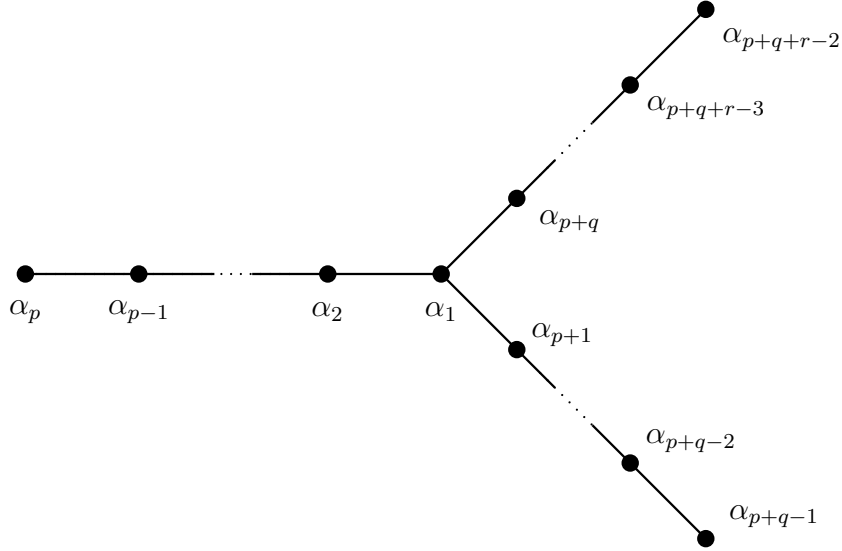


Figure 2.3: The diagram $T_{p,q,r}$

2.2.1 Spherical case

Spherical signatures are classified in Table 2.2. The von Dyck group $\Gamma_{p,q,r}$ is a polyhedral group, which is a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ (or $SO_3(\mathbb{R})$). The Kleinian group $\Pi_{p,q,r}$ obtained as the pull-back of $\Gamma_{p,q,r}$ by the universal covering map $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is a binary polyhedral group, which is described as an abstract group as

$$\Pi_{p,q,r} = \langle x, y, z \mid x^p = y^q = z^r = xyz \rangle.$$

Let $\mathbb{X} = \mathbb{X}_{p,q,r} := [\mathbb{P}^1/\Gamma_{p,q,r}]$ be the quotient stack, and $\mathbb{K} = \mathbb{K}_{p,q,r}$ be the total space of the canonical (orbi-)bundle. The coarse moduli space X of \mathbb{X} is isomorphic to \mathbb{P}^1 , and \mathbb{X} has three orbifold points with stabilizer groups $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}/r\mathbb{Z}$. Let

$$R = H^0(\mathcal{O}_{\mathbb{K}}) = \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_{\mathbb{X}}(-k\omega_{\mathbb{X}})),$$

be the anticanonical ring, which is the ring of regular functions on \mathbb{K} . The spectrum $\bar{Y} := \mathrm{Spec} R$ of R gives the corresponding Kleinian singularity $\bar{Y} \cong \mathbb{C}^2/\Pi_{p,q,r}$, and the coarse moduli space K of \mathbb{K} is a partial resolution of Y with three simple singularities of types A_{p-1} , A_{q-1} and A_{r-1} . The minimal resolution Y of K is the minimal resolution of \bar{Y} , and the exceptional divisor is a tree of (-2) -curves where three chains of (-2) -curves consisting of $p-1$, $q-1$ and $r-1$ rational curves is connected to the (-2) -curve obtained as the strict transform of the zero-section in K . The dual graph of the exceptional curves

type	\widetilde{E}_6	\widetilde{E}_7	\widetilde{E}_8
signature	(3, 3, 3)	(2, 4, 4)	(2, 3, 6)
weight	(1, 1, 1)	(1, 1, 2)	(1, 2, 3)
$E \cdot E$	-3	-2	-1

Table 2.3: Euclidean signatures

in the resolution $Y \rightarrow \overline{Y}$ is the $T_{p,q,r}$ -graph shown in Figure 2.3. One has a derived equivalence

$$D^b \text{coh } \mathbb{K} \cong D^b \text{coh } Y$$

by Kapranov and Vasserot [KV00].

2.2.2 Euclidean case

Euclidean signatures are classified in Table 2.3, which correspond to the tessellation of \mathbb{C} by triangles of angles $2\pi/p$, $2\pi/q$, and $2\pi/r$. The von Dyck group is a subgroup of $SO_2(\mathbb{R}) \times \mathbb{R}^2 \cong U(1) \times \mathbb{C}$ acting naturally on \mathbb{C} . The quotient $E = \mathbb{C}/\Gamma_{p,q,r}$ is a smooth elliptic curve, which can be written as $\{[x : y : z] \in \mathbb{P}(a, b, c) \mid x^p + y^q + z^r = 0\}$ where the weights (a, b, c) are given by $(1, 1, 1)$, $(1, 1, 2)$, and $(1, 2, 3)$ respectively.

Recall that a normal surface singularity is a *simple elliptic singularity* if the exceptional divisor E of the minimal resolution is a smooth elliptic curve. Simple elliptic singularities, which are also hypersurface singularities, are classified by Saito [Sai74] into three types \widetilde{E}_6 , \widetilde{E}_7 and \widetilde{E}_8 . They are $T_{p,q,r}$ -singularities (2.1) for $(p, q, r) = (3, 3, 3)$, $(2, 4, 4)$, and $(2, 3, 6)$, which are triangle singularities if $a = 0$. The triple (p, q, r) is the signature of the corresponding triangle group.

2.2.3 Hyperbolic case

A hyperbolic von Dyck group is a Fuchsian group, i.e., a discrete subgroup of $\text{PSL}_2(\mathbb{R}) \cong \text{Aut } \mathbb{H}$. A holomorphic function $f: \mathbb{H} \rightarrow \mathbb{C}$ is an *automorphic form of weight k* with respect to a Fuchsian group Γ if

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad (2.4)$$

and f is holomorphic at the cusp. The orbifold quotient $\mathbb{X}^\circ = [\mathbb{H}/\Gamma]$ can be compactified to a smooth orbifold \mathbb{X} in such a way that $\mathbb{X} \setminus \mathbb{X}^\circ$ has the trivial stabilizer. The space $A_k(\Gamma)$ of automorphic forms of weight k can be identified with the space

$$H^0(\mathbb{X}, (T^*\mathbb{X})^{\otimes k}) \cong H^0(\mathbb{H}, (T^*\mathbb{H})^{\otimes k})^\Gamma \quad (2.5)$$

of sections of the k -th tensor power of the cotangent bundle. Let $A(\Gamma) = \bigoplus_{n=0}^{\infty} A_n(\Gamma)$ be the ring of automorphic forms. The spectrum $\overline{V}_\Gamma = \text{Spec } A(\Gamma)$ is a cone over the orbifold \mathbb{X} , and has an isolated singularity at the origin called a *hyperbolic triangle singularity*. It is known by Dolgachev that a hyperbolic triangle singularity is a hypersurface singularity if and only if it is an exceptional unimodal singularity [Šće78]. The signature (p, q, r) of the von Dyck group coincides with the Dolgachev number of the corresponding exceptional unimodal singularity.

2.3 Strange duality and K3 surfaces

Pinkham [Pin77] and Dolgachev and Nikulin [Dol83, Nik79b] gave the following beautiful interpretation of the strange duality in terms of the Picard lattices and the transcendental lattices of K3 surfaces obtained as compactifications of the Milnor fibers.

Let Γ be a hyperbolic von Dyck group of signature $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)$ corresponding to an exceptional unimodal singularity. Let further V_Γ be the minimal smooth normal-crossing compactification of the minimal good resolution of the surface \overline{V}_Γ . The complement $V_\Gamma \setminus (\overline{V}_\Gamma \setminus \mathbf{0})$ has two connected components. One is E_0 , which is the exceptional locus of the minimal good resolution of the exceptional unimodal singularity. The other is E_∞ , which is a tree of (-2) -curves whose dual graph is the $T_{p,q,r}$ -graph shown in Figure 2.3.

Choose homogeneous generators x, y and z of $A(\Gamma)$ of degrees a, b , and c respectively, and write $A(\Gamma) = \mathbb{C}[x, y, z]/(f)$, where $f \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree h . Let w be an indeterminant of degree $h - a - b - c$ and F be a homogeneous element of $\mathbb{C}[x, y, z, w]$ of degree h such that $F(x, y, z, 0) = f(x, y, z)$. Assume that F does not have a critical point outside of the origin, and let $S = \mathbb{C}[x, y, z, w]/(F)$ be the quotient ring of $\mathbb{C}[x, y, z, w]$ by the ideal generated by F . Then the stack $\mathbb{Y} = \mathbb{P}\text{roj } S := [(\text{Spec } S \setminus \mathbf{0})/\mathbb{C}^\times]$ is a smooth Deligne-Mumford stack, which has the trivial canonical bundle since $\deg F = \deg x + \deg y + \deg z + \deg w$. Let \overline{Y} be the coarse moduli scheme of \mathbb{Y} and $Y \rightarrow \overline{Y}$ be the minimal resolution of Y . The scheme Y is a smooth compactification of the Milnor fiber of f . The condition $F(x, y, z, 0) = f(x, y, z)$ implies that the ring $R = S/(w)$ is isomorphic to $A(\Gamma)$, so that the divisor $\{w = 0\} = \mathbb{P}\text{roj } R \subset \mathbb{Y}$ at infinity is isomorphic to the orbifold curve \mathbb{X} . The Calabi-Yau property of \mathbb{Y} and the adjunction formula implies that the normal bundle of \mathbb{X} in \mathbb{Y} is isomorphic to the cotangent bundle of \mathbb{X} . Since \mathbb{X} has three orbifold points of orders δ_1, δ_2 , and δ_3 , the coarse moduli scheme Y has simple singularities of types $A_{\delta_1-1}, A_{\delta_2-1}$, and A_{δ_3-1} at these orbifold points. It follows that the minimal resolution Y has a configuration of (-2) -curves whose dual graph is the T_δ -graph shown in Figure 2.3. Here, the central node corresponds to the strict transform of the coarse moduli scheme X of \mathbb{X} , and three legs comes from resolutions of simple singularities of type A . The scheme Y is a K3 surface, whose Néron-Severi lattice is generated by these (-2) -curves for very general F . On the other hand, the Milnor lattice of f is given by $\widehat{T}_\gamma \cong T_\gamma \perp U$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the Gabrielov number of f . Since E_∞ is disjoint from the Milnor fiber of f , the transcendental lattice

$$T(Y) = \text{NS}(Y)^\perp \subset H^2(Y; \mathbb{Z}) \quad (2.6)$$

of Y clearly contains the Milnor lattice \widehat{T}_γ . One can show that the embedding of the \widehat{T}_γ -lattice into the orthogonal lattice of T_δ in $H^2(Y; \mathbb{Z}) \cong E_8 \perp E_8 \perp U \perp U \perp U$ is unique and surjective. It follows that $T(Y) \cong \widehat{T}_\gamma$ for very general F .

Let \check{f} be the defining polynomial of the weighted homogeneous exceptional unimodal singularity, which is related to f by the strange duality so that its Dolgachev and Gabrielov numbers $\check{\boldsymbol{\delta}}$ and $\check{\boldsymbol{\gamma}}$ satisfy $\check{\boldsymbol{\delta}} = \boldsymbol{\gamma}$ and $\check{\boldsymbol{\gamma}} = \boldsymbol{\delta}$. Then the transcendental lattice and the algebraic lattice of the corresponding K3 surface \check{Y} satisfy

$$\text{NS}(Y) \perp U \cong T(\check{Y}), \quad T(Y) \cong \text{NS}(\check{Y}) \perp U. \quad (2.7)$$

2.4 Categorifications of strange duality

The Grothendieck group $K(\mathcal{T})$ of a triangulated category \mathcal{T} has a structure of a lattice with respect to the Euler pairing

$$\langle X, Y \rangle := \sum_{n \in \mathbb{Z}} (-1)^n \dim \operatorname{Ext}^n(X, Y). \quad (2.8)$$

A *categorification* of a lattice is a triangulated category whose Grothendieck group is isometric to the lattice.

Let $f \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial in three variables defining an exceptional unimodal singularity, and Y be a compactification of the Milnor fiber of f as in Section 2.3. The *numerical Grothendieck group* $\mathcal{N}(Y)$ is the quotient of the Grothendieck group $K(Y)$ by the radical of the Euler form. Riemann-Roch theorem implies that $\mathcal{N}(Y)$ is isomorphic to the lattice $H^0(Y; \mathbb{Z}) \oplus \operatorname{NS}(Y) \oplus H^4(Y; \mathbb{Z})$ of algebraic cycles equipped with the Mukai pairing

$$((a_0, a_2, a_4), (b_0, b_2, b_4)) = a_2 \cdot b_2 - a_0 b_4 - a_4 b_0. \quad (2.9)$$

This allows one to think of $D^b \operatorname{coh} Y$ as a categorification of the algebraic lattice, in a slightly weak sense in that one considers the numerical Grothendieck group instead of the Grothendieck group.

On the dual side, the *Fukaya category* $\mathfrak{Fuk} \check{Y}$ is an A_∞ -category whose objects are Lagrangian submanifolds of \check{Y} and whose spaces of morphisms are Lagrangian intersection Floer complexes [Fuk93, FOOO09]. Since the Euler number of the Lagrangian intersection Floer complex is the algebraic intersection number of the Lagrangian submanifolds, the numerical Grothendieck group of the Fukaya category is a sublattice of the transcendental lattice. Since the transcendental lattice of \check{Y} is generated by vanishing cycles, the Fukaya category $\mathfrak{Fuk} \check{Y}$ is a categorification of the Milnor lattice, and strange duality is a categorification of (conjectural) homological mirror symmetry (1.2).

There is an alternative way to categorify strange duality, using stable derived categories [Eis80, Buc87, Orl04] and Fukaya-Seidel categories [Sei01b, Sei01a, Sei08]. Let $R = \mathbb{C}[x, y, z]/(f)$ be the coordinate ring of a weighted homogeneous exceptional unimodal singularity, and $\operatorname{gr} R$ be the category of finitely-generated \mathbb{Z} -graded R -modules. A complex of \mathbb{Z} -graded R -module is *perfect* if it is quasi-isomorphic to a bounded complex of finitely-generated projective modules. The *stable derived category* $D_{\operatorname{sing}}^b(\operatorname{gr} R)$ is defined as the quotient category of the bounded derived category $D^b(\operatorname{gr} R)$ by the full subcategory consisting of perfect complexes. On the dual side, the *Fukaya-Seidel category* is the A_∞ -category whose objects are vanishing cycles of \check{f} and whose spaces of morphisms are Lagrangian intersection Floer complexes.

Conjecture 2.1. *There is an equivalence*

$$D_{\operatorname{sing}}^b(\operatorname{gr} R) \cong D^b \mathcal{F}(\check{f}) \quad (2.10)$$

of triangulated categories.

The left hand side of (2.10) is a categorification of the algebraic lattice $\mathcal{N}(Y) \cong \operatorname{NS}(Y) \perp U \cong \widehat{T}_\delta$, in the sense that the Grothendieck group of $D_{\operatorname{sing}}^b(\operatorname{gr} R)$ equipped with

the symmetrized Euler pairing $(X, Y) = \langle X, Y \rangle + \langle Y, X \rangle$ is isometric to $\mathcal{N}(Y)$ [KMU13, Ued]. Similarly, the right hand side of (2.10) is a categorification of the Milnor lattice of f . Conjecture 2.1 is known for a disconnected sum of polynomials of type A or D [FU11, FU13]. Fukaya category of unimodal singularities has been studied by Keating, who in particular has proved homological mirror symmetry [Kea, Theorem 7.1]

$$D^b \mathcal{F}(T_{p,q,r}) \cong D^b \text{coh } \mathbb{X}_{p,r,q} \quad (2.11)$$

between $T_{p,q,r}$ -singularity and orbifold rational curves conjectured by Takahashi [Tak10, Conjecture 7.4].

2.5 Generalizations of strange duality

2.5.1 Cusp singularities

An isolated singularity in dimension 2 is a *cusp singularity* if the exceptional locus of the minimal resolution is either a cycle of non-singular rational curves or a nodal rational curve. A cusp singularity is a hypersurface singularity if and only if it is a $T_{p,q,r}$ -singularity with $1/p + 1/q + 1/r < 1$ [Kar77]. Let C_1, \dots, C_n be the irreducible components of the exceptional locus. The off-diagonal part of the intersection matrix $((C_i, C_j))_{i,j=1}^n$ is given by $C_i \cdot C_j = 1$ for $|i - j| = 1$ and $C_i \cdot C_j = 0$ for $|i - j| > 1$, and only the diagonal part

$$b_i := -C_i \cdot C_i \quad (2.12)$$

contains a non-trivial information. The *cycle number* of a cusp singularity is the sequence of integers defined by

$$(d_1, \dots, d_n) = \begin{cases} (b_1, \dots, b_n) & n \geq 2, \\ (b_1 + 2) & n = 1. \end{cases} \quad (2.13)$$

Let

$$\omega = \overline{[b_1, b_2, \dots, b_n]} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \frac{1}{\dots}}} \quad (2.14)$$

be the totally-real quadratic irrational number defined by an infinite continued fraction. The *dual cycle number* of the cusp singularity is the sequence (e_1, \dots, e_k) of integers defined by the continued fraction expansion

$$1/\omega = [f_1, \dots, f_s, \overline{e_1, \dots, e_k}]. \quad (2.15)$$

Then a certain class of cusp singularities come in pairs in such a way that the cycle numbers and the dual cycle numbers are interchanged.

The duality of cusp singularities can be described in terms of hyperbolic Inoue surfaces, just as the strange duality of Arnold can be interpreted in terms of K3 surfaces. In the Enriques-Kodaira classification of compact complex surfaces, a surface of class VII is a

non-Kähler elliptic surface with first Betti number 1. Inoue surfaces form an important class of surfaces of class VII, which can further be divided into three subclasses; hyperbolic, half, and parabolic.

A *hyperbolic Inoue surface* is constructed as follows. Let K be a real quadratic field, and M be a free \mathbb{Z} -module of rank 2 in K . Define

$$U_+(M) = \{x \in K \mid xM = M, x > 0, x' > 0\}, \quad (2.16)$$

where $(-)' : K \rightarrow K$ is the conjugation. Let V be a subgroup of $U_+(M)$ of finite index. The group

$$G(M, V) = \left\{ \begin{pmatrix} \varepsilon & m \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \mid \varepsilon \in V, m \in M \right\} \quad (2.17)$$

acts on $\mathbb{H} \times \mathbb{C}$ by

$$\begin{pmatrix} \varepsilon & m \\ 0 & 1 \end{pmatrix} : (z_1, z_2) \mapsto (\varepsilon z_1 + m, \varepsilon' z_2 + m'). \quad (2.18)$$

The quotient space $Y^\circ(M, V) = (\mathbb{H} \times \mathbb{C})/G(M, V)$ can be compactified to a compact complex surface $\overline{Y}(M, V)$ by adding two points ∞ and ∞' , which are cusp singularities. The minimal resolution $Y(M, V)$ of $\overline{Y}(M, V)$ is a surface of class VII, whose second Betti number is the sum of the length of the chains of rational curves obtained from the cusp singularities ∞ and ∞' [Ino77]. Any minimal surface of class VII with two cycles of rational curves is isomorphic to a hyperbolic Inoue surface [Nak84]. A pair of cusp singularities are dual if they come from two cusps of a hyperbolic Inoue surface [Nak80]. The relation between cusp singularities and mirror symmetry is investigated in [GHK] using the techniques developed in [KS06, GS11].

2.5.2 Ebeling-Wall duality

Strange duality is extended in [EW85] to encompass certain classes of bimodal singularities, isolated complete intersection singularities, and quadrilateral singularities. Since a well-written review already exists [Ebe99], we do not discuss it here.

2.5.3 Saito duality

A sequence $W = (a, b, c; h)$ of positive integers with $h > \max\{a, b, c\}$ is a *regular weight system* if

$$\frac{(1 - T^{h-a})(1 - T^{h-b})(1 - T^{h-c})}{(1 - T^a)(1 - T^b - 1)(1 - T^c)} \quad (2.19)$$

is a polynomial in T [Sai87]. The integers (a, b, c) and h are called *weights* and the *Coxeter number* respectively. For a regular weight system W , define a sequence (m_1, \dots, m_μ) of not necessarily distinct integers by

$$\chi_W(T) := T^\epsilon \frac{(1 - T^{h-a})(1 - T^{h-b})(1 - T^{h-c})}{(1 - T^a)(1 - T^b - 1)(1 - T^c)} = T^{m_1} + \dots + T^{m_\mu}, \quad (2.20)$$

where

$$\epsilon = \epsilon_W := a + b + c - h \quad (2.21)$$

is the *minimal exponent* of W , and

$$\mu = \mu_W := \chi_W(1) = (h - a)(h - b)(h - c)/abc \quad (2.22)$$

is the *rank* of W . The *characteristic polynomial* of W is defined by

$$\varphi_W(\lambda) = \prod_{i=1}^{\mu} (\lambda - \omega_i), \quad (2.23)$$

where

$$\omega_i = \exp(2\pi\sqrt{-1}m_i/h), \quad i = 1, \dots, \mu. \quad (2.24)$$

One can write

$$\varphi(\lambda) = \prod_{i|h} (\lambda^i - 1)^{e(i)} \quad (2.25)$$

for some sequence $(e(i))_{i|h}$ of integers. A pair (W, W^*) of regular weight systems with the identical Coxeter number are said to be *dual* in the sense of Saito [Sai98a] if their characteristic polynomials

$$\varphi_W(\lambda) = \prod_{i|h} (\lambda^i - 1)^{e(i)}, \quad \varphi_{W^*}(\lambda) = \prod_{i|h} (\lambda^i - 1)^{e^*(i)}, \quad (2.26)$$

satisfy

$$e(i) + e^*(h/i) = 0 \quad (2.27)$$

for all $i|h$.

A weight system $W = (a, b, c; h)$ is regular if and only if there exists a weighted homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ of weight $\deg(x, y, z; f) = (a, b, c; h)$ with isolated critical point at the origin. The rank of a regular weight system is the Milnor number $\dim \mathbb{C}[x, y, z]/(\partial_x f, \partial_y f, \partial_z f)$ of the singularity, and the minimal exponent is the Gorenstein parameter of the coordinate ring $\mathbb{C}[x, y, z]/(f)$. The characteristic polynomial is the characteristic polynomial for the Milnor monodromy acting on the middle-dimensional homology group of the Milnor fiber.

The duality of regular weight systems generalizes the strange duality of exceptional unimodal singularities. See [Sai98b] for a summary of the theory of regular weight systems, and [Sai98a] for a thorough treatment. The relation between the duality of regular weight systems and mirror symmetry is discussed in [Kob08, Tak99, Ebe06].

2.5.4 Kobayashi duality

A *weight system* in the sense of Kobayashi [Kob08] is a sequence $W = (a_1, \dots, a_n; h)$ of positive integers satisfying $h \in \mathbb{N}a_1 + \dots + \mathbb{N}a_n$. A weight system is *reduced* if $\gcd(a_1, \dots, a_n; h) = 1$. A *weighted magic square* for a pair (W, W^*) of weight systems is an $n \times n$ matrix C with non-negative integer entries satisfying

$$C \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} h \\ \vdots \\ h \end{pmatrix} \quad \text{and} \quad (a_1^* \ \dots \ a_n^*) C = (h^* \ \dots \ h^*). \quad (2.28)$$

A weighted magic square C is *primitive* if $|\det C| = h = h^*$. The weight systems W and W^* are *dual* in the sense of Kobayashi [Kob08] if there exists a primitive weighted magic square for (W, W^*) . This also gives a generalization of the strange duality of exceptional unimodal singularities. This notion of duality is further investigated in [Ebe06].

3 Berglund-Hübsch transpose

A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is *invertible* if the following two conditions are satisfied:

1. There is an integer matrix $A = (a_{ij})_{i,j=1}^n$ such that

$$f = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}. \quad (3.1)$$

2. f has an isolated critical point at the origin.

The second condition implies that the matrix A has a non-zero determinant. It follows that any invertible polynomial is weighted homogeneous, and the corresponding reduced weight system

$$(a_1, \dots, a_n; h) := \deg(x_1, \dots, x_n; f) \quad (3.2)$$

is determined uniquely.

An invertible polynomial is a Sebastiani-Thom (or decoupled) sum of polynomials of the following three types [KS92]:

- Fermat: x^p .
- Chain: $x_1^{p_1} x_2 + x_2^{p_2} x_3 + \dots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n}$.
- Loop: $x_1^{p_1} x_2 + x_2^{p_2} x_3 + \dots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n} x_1$.

A *Landau-Ginzburg model* is a pair (V, f) of an algebraic variety V and a holomorphic function $f: V \rightarrow \mathbb{C}$ on V . An invertible polynomial f gives an example (\mathbb{C}^n, f) of a Landau-Ginzburg model.

An invertible polynomial is naturally graded by the abelian group L of rank one generated by $n + 1$ elements $\vec{x}_1, \dots, \vec{x}_n$, and \vec{c} with n relations

$$a_{i1}\vec{x}_1 + \dots + a_{in}\vec{x}_n = \vec{c}, \quad i = 1, \dots, n. \quad (3.3)$$

The group L is the group of characters of the group K defined by

$$K = \{(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{a_{11}} \dots \alpha_n^{a_{1n}} = \dots = \alpha_1^{a_{n1}} \dots \alpha_n^{a_{nn}}\}. \quad (3.4)$$

The group of maximal diagonal symmetries is the kernel G_{\max} of the map

$$\begin{array}{ccc} K & \rightarrow & \mathbb{C}^\times \\ \Psi & & \Psi \\ (\alpha_1, \dots, \alpha_n) & \mapsto & \alpha_1 \dots \alpha_n, \end{array} \quad (3.5)$$

so that there is an exact sequence

$$1 \rightarrow G_{\max} \rightarrow K \rightarrow \mathbb{C}^\times \rightarrow 1. \quad (3.6)$$

This exact sequence induces an exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow L \rightarrow G_{\max}^\vee \rightarrow 1 \quad (3.7)$$

of the corresponding groups of characters, where

$$G_{\max}^\vee = \text{Hom}(G_{\max}, \mathbb{C}^\times) \quad (3.8)$$

is (non-canonically) isomorphic to G_{\max} . Write

$$A^{-1} = \begin{pmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \dots & \varphi_1^{(n)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \dots & \varphi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n^{(1)} & \varphi_n^{(2)} & \dots & \varphi_n^{(n)} \end{pmatrix}. \quad (3.9)$$

Then the group G_{\max} is generated by

$$\rho_k = \left(\exp\left(2\pi\sqrt{-1}\varphi_1^{(k)}\right), \dots, \exp\left(2\pi\sqrt{-1}\varphi_n^{(k)}\right) \right), \quad k = 1, \dots, n. \quad (3.10)$$

A *Landau-Ginzburg orbifold* is a pair $((V, f), G)$ of a Landau-Ginzburg model (V, f) and a finite group G acting on V in such a way that the function $f: V \rightarrow \mathbb{C}$ is G -invariant. An invertible polynomial f and a subgroup G of G_{\max} gives an example $((f, \mathbb{C}^n), G)$ of a Landau-Ginzburg orbifold.

The *transpose* of f is the invertible polynomial

$$\check{f} = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ji}}. \quad (3.11)$$

Note that the exponent matrix \check{A} is the transpose matrix of A . The group \check{G}_{\max} of maximal diagonal symmetries of \check{f} is generated by the column vectors $\check{\rho}_i$ of \check{A}^{-1} . The *transpose* of a subgroup $G \subset G_{\max}$ is defined in [BH95, Kra] as

$$\check{G} = \left\{ \prod_{i=1}^n \check{\rho}_i^{r_i} \mid (r_1 \ \cdots \ r_n) A^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z} \text{ for all } \prod_{i=1}^n \rho_i^{a_i} \in G \right\}. \quad (3.12)$$

The *transpose* of a pair (f, G) of an invertible polynomial f and a subgroup G of the group G_{\max} of maximal diagonal symmetries is defined as (\check{f}, \check{G}) . In particular, the transpose of the pair $(f, 1)$ of an invertible polynomial and the trivial group is the pair $(\check{f}, \check{G}_{\max})$ of the transpose polynomial and the group of maximal diagonal symmetries.

Transposition is introduced in [BH93] as a generalization of the orbifold mirror construction [GP90]. Interpretation of strange duality as mirror symmetry for Landau-Ginzburg orbifolds goes back at least to [LS90, Mar90]. There has been a renewed interest in transposition mirror construction recently, coming in part from the development of cohomological field theories associated with invertible polynomials [FJR13, PV]. In particular, classical mirror symmetry for exceptional unimodal singularities is proved in [LLSS]. Transposition mirror construction as a generalization of strange duality is explored in [ET11].

To relate mirror symmetry of Landau-Ginzburg orbifolds to mirror symmetry of Calabi-Yau manifolds, one needs the correspondence between Calabi-Yau manifolds and Landau-Ginzburg orbifolds. Let f be an invertible polynomial and G be a subgroup of G_{\max} . Put

$$\varphi_i = \varphi_i^{(1)} + \cdots + \varphi_i^{(n)}, \quad i = 1, \dots, n, \quad (3.13)$$

and define a homomorphism $\varphi: \mathbb{C}^\times \rightarrow K$ by

$$\varphi(\alpha) = (\alpha^{h\varphi_1}, \dots, \alpha^{h\varphi_n}), \quad (3.14)$$

where h is the smallest positive integer such that $a_i := h\varphi_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Then φ is injective and one has an exact sequence

$$1 \rightarrow \mathbb{C}^\times \xrightarrow{\varphi} K \rightarrow \overline{G}_{\max} \rightarrow 1, \quad (3.15)$$

where $\overline{G}_{\max} := \text{coker } \varphi$. The polynomial f is weighted homogeneous with respect to the weight system $(a_1, \dots, a_n; h)$. Since f has an isolated singularity at the origin, the weighted projective hypersurface

$$\mathbb{Y} = \{[x_1 : \cdots : x_n] \in \mathbb{P}(a_1, \dots, a_n) \mid f(x_1, \dots, x_n) = 0\} \quad (3.16)$$

is a smooth Deligne-Mumford stack. It is Calabi-Yau if $a_1 + \cdots + a_n = h$. The intersection $\text{Im } \varphi \cap G_{\max}$ is generated by

$$J = (\exp(2\pi\sqrt{-1}\varphi_1), \dots, \exp(2\pi\sqrt{-1}\varphi_n)). \quad (3.17)$$

Assume $J \in G$ and let $\overline{G} = G / \langle J \rangle$ be the image of G in \overline{G}_{\max} . Then \overline{G} acts naturally on \mathbb{Y} , and one can form the quotient stack $[\mathbb{Y}/\overline{G}]$. The *Calabi-Yau/Landau-Ginzburg*

correspondence, or the CY/LG correspondence for short, is an idea which goes back at least to [Gep87], that the Landau-Ginzburg orbifold (f, G) is ‘dual’ to the orbifold $[\mathbb{Y}/\overline{G}]$. On the symplectic side, this is proved at the level of topological mirror symmetry in [CR10]. On the complex side, the equivalence of derived categories is established in [Orl09]. The relation between categorical equivalence and analytic continuation of periods is discussed in [CIR].

Transposition mirror construction gives a candidate for the mirror of a Landau-Ginzburg orbifold, which produces a candidate for the mirror of a Calabi-Yau manifold by the CY/LG correspondence. For K3 surfaces, this construction is known to give mirror pairs in the sense of Definition 5.2 below for an invertible polynomial of the form $x^p + f(y, z, w)$ for $p = 2$ in [ABS] and for prime p in [CLPS].

4 Aspinwall-Morrison mirrors

Let Y be a K3 surface. The *extended K3 lattice* is the free abelian group $H^*(Y; \mathbb{Z})$ equipped with the Mukai pairing (2.9). The complex structure of Y is determined by the class $[\Omega] \in \mathbb{P}(H^2(Y, \mathbb{C}))$ of the holomorphic 2-form $\Omega \in H^{2,0}(Y)$ satisfying the Hodge-Riemann bilinear relations

$$(\Omega, \Omega) = 0, \quad (\Omega, \overline{\Omega}) > 0. \quad (4.1)$$

A *complexified Kähler structure* on Y is an element $\mathcal{U} \in H^*(Y, \mathbb{C})$ of the form

$$\begin{aligned} \mathcal{U} &= \exp(B + \sqrt{-1}\omega) \\ &= \left(1, B + \sqrt{-1}\omega, \frac{1}{2}(B + \sqrt{-1}\omega)^2 \right) \\ &\in H^0(Y; \mathbb{C}) \oplus H^2(Y; \mathbb{C}) \oplus H^4(Y; \mathbb{C}) \end{aligned}$$

satisfying

$$(\mathcal{U}, \mathcal{U}) = 0, \quad (\mathcal{U}, \overline{\mathcal{U}}) > 0 \quad (4.2)$$

and

$$(\mathcal{U}, \Omega) = 0, \quad (\mathcal{U}, \overline{\Omega}) = 0. \quad (4.3)$$

The class $\omega \in H^2(Y, \mathbb{R})$ is (a slight generalization of) the Kähler class, which satisfies $\omega \in H^2(Y, \mathbb{R}) \cap H^{1,1}(Y)$ and $\omega \cdot \omega > 0$. The class $B \in H^2(Y, \mathbb{R})$ is called the *B-field*, which satisfies $B \in H^2(Y, \mathbb{R}) \cap H^{1,1}(Y)$. Note that ω and B are determined by the class $[\mathcal{U}] \in \mathbb{P}(H^*(Y, \mathbb{C}))$.

Let \check{Y} be another K3 surface and $(\check{\Omega}, \check{\mathcal{U}})$ be a pair of 2-forms satisfying the above conditions. The following definition is due to Aspinwall and Morrison [AM97]:

Definition 4.1. A pair $((Y, (\Omega, \mathcal{U})), (\check{Y}, (\check{\Omega}, \check{\mathcal{U}})))$ of K3 surfaces equipped with complexified Kähler structures is a *mirror pair* if there is an isometry of extended K3 lattices

$$\varphi : H^*(Y; \mathbb{Z}) \xrightarrow{\sim} H^*(\check{Y}; \mathbb{Z}) \quad (4.4)$$

such that

$$(\varphi(\Omega), \varphi(\mathcal{U})) = (\check{\mathcal{U}}, \check{\Omega}). \quad (4.5)$$

5 Dolgachev mirrors

Let Y be a K3 surface. The *K3 lattice* is the free abelian group $H^2(Y, \mathbb{Z})$ equipped with the intersection form. It has rank 22 and signature $(3, 19)$, and isomorphic to

$$L = E_8 \perp E_8 \perp U \perp U \perp U \quad (5.1)$$

as an abstract lattice. Here E_8 is the negative-definite even unimodular lattice of type E_8 and U is the even unimodular indefinite lattice of rank two. For a K3 surface Y , set

$$\Delta(Y) = \{\delta \in \text{Pic}(Y) \mid (\delta, \delta) = -2\}. \quad (5.2)$$

Let \mathcal{L} be a line bundle such that $[\mathcal{L}] = \delta \in \text{Pic}(Y)$. Riemann-Roch theorem gives

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^\vee) \geq 2 + \frac{1}{2}(\delta, \delta), \quad (5.3)$$

so that either \mathcal{L} or \mathcal{L}^\vee has a non-trivial section, and hence either δ or $-\delta$ is effective;

$$\Delta(Y) = \Delta(Y)^+ \amalg \Delta(Y)^-, \quad (5.4)$$

$$\Delta(Y)^+ = \{\delta \in \Delta(Y) \mid \delta \text{ is effective}\}, \quad (5.5)$$

$$\Delta(Y)^- = -\Delta(Y)^+. \quad (5.6)$$

The subgroup $W(Y) \subset O(L)$ generated by reflections with respect to elements in $\Delta(Y)$ acts properly discontinuously on the connected component V^+ of

$$V(Y) = \{x \in H^{1,1}(Y) \cap H^2(Y, \mathbb{R}) \mid (x, x) > 0\} \quad (5.7)$$

containing the Kähler class. The fundamental domain is given by

$$C(Y) = \{x \in V(Y)^+ \mid (x, \delta) \geq 0 \text{ for any } \delta \in \Delta(Y)^+\}, \quad (5.8)$$

and the Kähler cone is given (cf. e.g. [BHPVdV04, Corollary VIII.3.9]) by

$$C(Y)^+ = \{x \in V(Y)^+ \mid (x, \delta) > 0 \text{ for any } \delta \in \Delta(Y)^+\}. \quad (5.9)$$

Recall that

$$\text{Pic}(Y) = H^{1,1}(Y) \cap H^2(Y; \mathbb{Z}) \quad (5.10)$$

by the Lefschetz theorem. Set

$$\text{Pic}(Y)^+ = C(Y) \cap H^2(Y; \mathbb{Z}), \quad (5.11)$$

$$\text{Pic}(Y)^{++} = C(Y)^+ \cap H^2(Y; \mathbb{Z}). \quad (5.12)$$

Let M be an even non-degenerate lattice of signature $(1, t)$ where $0 \leq t \leq 19$. Choose one of two connected components of

$$V(M) = \{x \in M_{\mathbb{R}} \mid (x, x) > 0\} \quad (5.13)$$

and call it $V(M)^+$. Choose a subset $\Delta(M)^+$ of

$$\Delta(M) = \{\delta \in M \mid (\delta, \delta) = -2\} \quad (5.14)$$

such that

1. $\Delta(M) = \Delta(M)^+ \amalg \Delta(M)^-$ where $\Delta(M)^- = \{-\delta \mid \delta \in \Delta(M)^+\}$, and
2. $\Delta(M)^+$ is closed under addition (but not subtraction).

Define

$$C(M)^+ = \{h \in V(M)^+ \cap M \mid (h, \delta) > 0 \text{ for all } \delta \in \Delta(M)^+\}. \quad (5.15)$$

The notion of a lattice-polarized K3 surface is introduced by Nikulin [Nik79a]. The following definition is taken from [Dol96, Section 1]:

Definition 5.1. An M -polarized K3 surface is a pair (Y, j) where Y is a K3 surface and $j : M \hookrightarrow \text{Pic}(Y)$ is a primitive lattice embedding. An *isomorphism* of M -polarized K3 surfaces (Y, j) and (Y', j') is an isomorphism $f : Y \rightarrow Y'$ of K3 surfaces such that $j = f^* \circ j'$. An M -polarized K3 surface is *pseudo-ample* if

$$j(C(M)^+) \cap \text{Pic}(Y)^+ \neq \emptyset, \quad (5.16)$$

and *ample* if

$$j(C(M)^+) \cap \text{Pic}(Y)^{++} \neq \emptyset. \quad (5.17)$$

Assume that for any two primitive embeddings $\iota_1, \iota_2 : M \hookrightarrow L$, there exists an isometry $\sigma : L \rightarrow L$ such that $\sigma \circ \iota_1 = \iota_2$. Fix a primitive lattice embedding $i_M : M \hookrightarrow L$ and let $N = M^\perp$ be the orthogonal complement. The period domain

$$\mathcal{D}(M) = \{[\Omega] \in \mathbb{P}(N \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0, (\Omega, \bar{\Omega}) > 0\} \quad (5.18)$$

can be identified with the symmetric homogeneous space $O(2, 19-t)/SO(2) \times O(19-t)$ of oriented positive-definite 2-planes in $N_{\mathbb{R}}$. It consists of two connected components $\mathcal{D}^+(M)$ and $\mathcal{D}^-(M)$, each of which is isomorphic to a bounded Hermitian domain of type IV. Set

$$\Gamma(M) = \{\sigma \in O(L) \mid \sigma(m) = m \text{ for any } m \in M\} \quad (5.19)$$

and Γ_M be its image under the natural injective homomorphism $\Gamma(M) \hookrightarrow O(N)$. Global Torelli theorem and surjectivity of the period map show that $\mathcal{D}_0^+(M)/\Gamma_M$ is the coarse moduli space of ample M -polarized K3 surfaces, where

$$\mathcal{D}_0^+(M) = \mathcal{D}^+(M) \setminus \left(\bigcup_{\delta \in \Delta(N)} \delta^\perp \right) \quad (5.20)$$

is the complement of reflection hyperplanes

$$\delta^\perp = \{z \in \mathcal{D}^+(M) \mid (z, \delta) = 0\}. \quad (5.21)$$

The closure of $\mathcal{D}^+(M)$ in the *compact dual*

$$\tilde{\mathcal{D}}(M) = \{[\Omega] \in \mathbb{P}(N \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0\} \quad (5.22)$$

of the period domain is denoted by $\overline{\mathcal{D}}(M)$. Its topological boundary is given by

$$\overline{\mathcal{D}}(M) \setminus \mathcal{D}^+(M) = \bigcup_{I : \text{isotropic subspace of } M_{\mathbb{R}}} \mathbb{P}(I_{\mathbb{C}}) \cap \overline{\mathcal{D}}(M).$$

Since the signature of M is $(2, 19-t)$, one either has rank $I = 1$ or 2 , so that $\mathbb{P}(I_{\mathbb{C}}) \cap \overline{\mathcal{D}}(M)$ is one point or isomorphic to the upper half plane. The boundary component is *rational* if I is defined over \mathbb{Q} . The *Satake-Baily-Borel compactification* is defined by

$$\left(\mathcal{D}^+(M) \cup \bigcup_{I : \text{rational}} \mathbb{P}(I_{\mathbb{C}}) \cap \overline{\mathcal{D}}(M) \right) / \Gamma_M. \quad (5.23)$$

For a vector f in a lattice S , the positive integer $\text{div } f$ is defined as the greatest common divisor of $(f, g) \in \mathbb{Z}$ for all $g \in S$. An isotropic vector f is called *m-admissible* if $\text{div } f = m$ and there exists another isotropic vector g such that $(f, g) = m$ and $\text{div } g = m$. If M^{\perp} has an *m-admissible* vector f , then one has $M^{\perp} = U(m) \perp M$, where $U(m)$ is the lattice generated by f and g .

Definition 5.2 (Dolgachev [Dol96, Section 6]). The *mirror moduli space* is the moduli space D_M° of ample M -polarized K3 surfaces.

The case $m = 1$ is of particular interest. Two sublattices M and \check{M} of the K3 lattice L are said to be *K3-dual* if one has

$$M^{\perp} = \check{M} \perp U$$

for a unimodular hyperbolic plane U in L . Nikulin [Nik79b] has shown that two hyperbolic lattices M and \check{M} are K3-dual if $\text{rank } M + \text{rank } \check{M} = 20$ and the discriminant group $A(M) = M^*/M$ equipped with the discriminant form q_M is isomorphic to $A(\check{M})$ equipped with $-q_{\check{M}}$. This duality has been investigated by Belcastro [Bel02] in the case of K3 surfaces associated with weighted projective spaces. There are 95 weights where the minimal model of a general anti-canonical hypersurface of the corresponding weighted projective space gives a K3 surface [Yon90]. Belcastro computed the Picard lattice of very general K3 surfaces obtained in this way, and discovered that some of them are K3-dual with each other. The relation of this duality with the Kobayashi duality is studied in [Ebe06].

6 Batyrev mirrors

Let Y be a smooth anti-canonical hypersurface in a smooth toric weak Fano 3-fold X . Here, a *weak Fano* manifold is a projective manifold whose anti-canonical bundle is nef and big. It follows from the adjunction formula that Y has the trivial canonical bundle. Lefschetz hyperplane theorem shows that Y is simply-connected, so that Y is a K3 surface. Batyrev [Bat94] introduced the following construction of a candidate for the mirror of Y .

Let $\mathbb{T} \subset X$ be the dense torus and $\mathbf{M} = \text{Hom}(\mathbb{T}, \mathbb{C}^{\times})$ be the group of characters so that $\mathbb{T} = \text{Spec } \mathbb{C}[\mathbf{M}]$. Let further

$$\Delta = \text{Conv}\{m \in \mathbf{M} \mid x^m \in H^0(\mathcal{O}_X(-K_X))\} \quad (6.1)$$

be the Newton polytope of the anti-canonical bundle of X , where the canonical divisor K_X is the sum of all prime toric divisors, and $x^m \in \mathbb{C}[\mathbf{M}]$ is the rational function on X corresponding to $m \in \mathbf{M}$. Since X is weak Fano, the polytope Δ is reflexive, i.e., the polar dual polytope

$$\check{\Delta} = \{n \in \mathbf{N} \mid \langle n, m \rangle \geq -1 \text{ for any } m \in \Delta\} \quad (6.2)$$

is also a lattice polytope (i.e., the convex hull of a finite subset of \mathbf{N}). Here $\mathbf{N} = \text{Hom}(\mathbf{M}, \mathbb{Z})$ is the group of one-parameter subgroups of \mathbb{T} .

Recall that the *fan polytope* of a toric variety is the convex hull of primitive generators of one-dimensional cones of the fan. Let $\check{\Sigma}$ be any unimodular fan in $\mathbf{M}_{\mathbb{R}} = \mathbf{M} \otimes \mathbb{R}$ whose fan polytope coincides with Δ . Note that the fan $\check{\Sigma}$ lives in $\mathbf{M}_{\mathbb{R}}$, whereas one usually considers a fan inside $\mathbf{N}_{\mathbb{R}}$. This comes from the fact that we are working on the mirror side, where the roles of the torus and the dual torus are interchanged. Let \check{X} be the toric variety associated with the fan $\check{\Sigma}$. The toric variety \check{X} is weak Fano since Δ is reflexive, and a general member $\check{Y} \in |-K_{\check{X}}|$ of the anti-canonical linear system is a smooth K3 surface.

7 Classical mirror symmetry

Let $\mathbf{N} \cong \mathbb{Z}^3$ be a free abelian group of rank three and $\mathbf{M} = \text{Hom}(\mathbf{N}, \mathbb{Z})$ be the dual group. Let further $(\Sigma, \check{\Sigma})$ be a pair of unimodular fans in $\mathbf{N}_{\mathbb{R}} = \mathbf{N} \otimes \mathbb{R}$ and $\mathbf{M}_{\mathbb{R}} = \mathbf{M} \otimes \mathbb{R}$ whose fan polytopes $(\Delta, \check{\Delta})$ are polar dual to each other. The set of generators of one-dimensional cones of the fan Σ will be denoted by $\{b_1, \dots, b_m\} \subset \mathbf{N}$. One has the *fan sequence*

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N} \rightarrow 0 \quad (7.1)$$

and the *divisor sequence*

$$0 \rightarrow \mathbf{M} \xrightarrow{\beta^*} (\mathbb{Z}^m)^* \rightarrow \mathbb{L}^* \rightarrow 0, \quad (7.2)$$

where β sends the i -th coordinate vector to b_i and

$$\text{Pic}(X) \cong H^2(X; \mathbb{Z}) \cong \mathbb{L}^*. \quad (7.3)$$

Set $\mathcal{M} = \mathbb{L}^* \otimes \mathbb{C}^\times$ and $\check{\mathbb{T}} = \mathbf{M} \otimes \mathbb{C}^\times$ so that one has the exact sequence

$$1 \rightarrow \check{\mathbb{T}} \rightarrow (\mathbb{C}^\times)^m \rightarrow \mathcal{M} \rightarrow 1. \quad (7.4)$$

The uncompactified mirror \check{Y}_α° of a smooth anticanonical hypersurface $Y \subset X$ is defined by

$$\check{Y}_\alpha^\circ = \{y \in \check{\mathbb{T}} \mid W_\alpha(y) = \sum_{i=1}^m \alpha_i y^{b_i} = 1\} \quad (7.5)$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{C}^\times)^m$. We will study symplectic geometry of Y , so that the defining equation of Y is irrelevant. The closure \check{Y}_α of \check{Y}_α° in \check{X} for general α is a smooth

anti-canonical K3 hypersurface, which is the compact mirror of Y . We say that \check{Y}_α is $\check{\Sigma}$ -regular if the intersection of \check{Y}_α with any toys orbit of \check{X} is a smooth subvariety of codimension one. Let $(\mathbb{C}^\times)_{\text{reg}}^m$ be the set of $\alpha \in (\mathbb{C}^\times)^m$ such that \check{Y}_α is $\check{\Sigma}$ -regular, and $\tilde{\varphi} : \tilde{\mathfrak{Y}} \rightarrow (\mathbb{C}^\times)_{\text{reg}}^m$ be the second projection from

$$\tilde{\mathfrak{Y}} = \{(y, \alpha) \in \check{X} \times (\mathbb{C}^\times)_{\text{reg}}^m \mid W_\alpha(y) = 1\}. \quad (7.6)$$

The quotient of the family $\tilde{\varphi} : \tilde{\mathfrak{Y}} \rightarrow (\mathbb{C}^\times)_{\text{reg}}^m$ by the free $\check{\mathbb{T}}$ -action

$$t \cdot (y, (\alpha_1, \dots, \alpha_m)) = (t^{-1}y, (t^{b_1}\alpha_1, \dots, t^{b_m}\alpha_m)) \quad (7.7)$$

will be denoted by $\check{\varphi} : \check{\mathfrak{Y}} \rightarrow \mathcal{M}_{\text{reg}}$, where $\mathcal{M}_{\text{reg}} = (\mathbb{C}^\times)_{\text{reg}}^m / \check{\mathbb{T}}$. The *residue part* of $H^2(\check{Y}_\alpha; \mathbb{C})$ is defined as the image of the residue map:

$$H_{\text{res}}^2(\check{Y}_\alpha; \mathbb{C}) := \text{Im}(\text{Res} : H^0(\check{X}, \Omega_{\check{X}}^3(*\check{Y}_\alpha)) \rightarrow H^2(\check{Y}_\alpha; \mathbb{C})). \quad (7.8)$$

One can show [Iri11, Section 6.3] that $H_{\text{res}}^2(\check{Y}_\alpha; \mathbb{C})$ can be identified with the lowest weight component $W_2(H^2(\check{Y}_\alpha^\circ; \mathbb{C}))$ of the mixed Hodge structure on $H^2(\check{Y}_\alpha^\circ; \mathbb{C})$, and hence comes naturally with a \mathbb{Q} -Hodge structure of weight 2. The *residual B-model VHS* $(\mathcal{H}_B, \nabla^B, H_{B, \mathbb{Q}}, \mathcal{F}_B^\bullet, Q_B)$ on \check{U} consists of

- the locally-free subsheaf \mathcal{H}_B of $(R^2\check{\varphi}_*\mathbb{C}_{\check{\mathfrak{Y}}}) \otimes \mathcal{O}_{\mathcal{M}_{\text{reg}}}$ whose fiber at $[\alpha]$ is $H_{\text{res}}^2(\check{Y}_\alpha; \mathbb{C})$,
- the Gauss-Manin connection ∇^B on \mathcal{H}_B ,
- the rational structure $H_{B, \mathbb{Q}} \subset \text{Ker } \nabla^B$ explained above,
- the standard Hodge filtration $\mathcal{F}_{B, [\alpha]}^p = \bigoplus_{j \geq p} H_{\text{res}}^{j, 2-j}(\check{Y}_\alpha; \mathbb{C})$, and
- the intersection form

$$Q_B(\omega_1, \omega_2) = \int_{\check{Y}_\alpha} \omega_1 \cup \omega_2. \quad (7.9)$$

The composition

$$H_3(\check{\mathbb{T}}, \check{Y}_\alpha^\circ; \mathbb{C}) \xrightarrow{\partial} H_2(\check{Y}_\alpha^\circ; \mathbb{C}) \rightarrow H_2(\check{Y}_\alpha; \mathbb{C}) \xrightarrow{\text{PD}} H^2(\check{Y}_\alpha; \mathbb{C}) \quad (7.10)$$

gives a surjection $\text{VC} : H_3(\check{\mathbb{T}}, \check{Y}_\alpha^\circ; \mathbb{C}) \rightarrow H_{\text{res}}^2(\check{Y}_\alpha; \mathbb{C})$ called the *vanishing cycle map*. The image of $H_n(\check{\mathbb{T}}, \check{Y}_\alpha^\circ; \mathbb{Z})$ by the vanishing cycle map defines the *vanishing cycle integral structure* $H_{B, \mathbb{Z}}^{\text{vc}} \subset H_{B, \mathbb{Q}}$ on the residual B-model VHS [Iri11, Definition 6.7].

On the A-model side, let

$$H_{\text{amb}}^\bullet(Y; \mathbb{C}) = \text{Im}(\iota^* : H^\bullet(X; \mathbb{C}) \rightarrow H^\bullet(Y; \mathbb{C})) \quad (7.11)$$

be the subspace of $H^\bullet(Y; \mathbb{C})$ coming from the cohomology classes of the ambient toric variety, and set

$$U = \{\tau \in H_{\text{amb}}^2(Y; \mathbb{C}) \mid \Re \langle \tau, d \rangle \leq -M \text{ for any non-zero } d \in \text{Eff}(Y)\} \quad (7.12)$$

for some sufficiently large M . Here $\text{Eff}(Y)$ is the semigroup of effective curves. This open subset U is considered as a neighborhood of the large radius limit point. Choose an integral basis p_1, \dots, p_r of $\text{Pic } X$ such that each p_i is nef, and let $(\tau^i)_{i=1}^r$ be the dual coordinate on $H_{\text{amb}}^2(Y; \mathbb{C})$; $\tau = \sum_{i=1}^r \tau^i p_i$.

The *ambient A -model VHS* $(\mathcal{H}_A, \nabla^A, \mathcal{F}_A^\bullet, Q_A)$ [Iri11, Definition 6.2] consists of

- the locally free sheaf $\mathcal{H}_A = H_{\text{amb}}^\bullet(Y) \otimes \mathcal{O}_U$,
- the Dubrovin connection $\nabla^A = d + \sum_{i=1}^r (p_i \circ_\tau) d\tau^i: \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \Omega_U^1$,
- the Hodge filtration $\mathcal{F}_A^p = H_{\text{amb}}^{\leq 4-2p}(Y) \otimes \mathcal{O}_U$, and
- the symmetric pairing $Q_A: \mathcal{H}_A \otimes \mathcal{H}_A \rightarrow \mathcal{O}_U$, $(\alpha, \beta) \mapsto (2\pi\sqrt{-1})^2 \left((-1)^{\frac{\deg}{2}} \alpha, \beta \right)$.

Let $L_Y(\tau)$ be the fundamental solution of the quantum differential equation, that is, the $\text{End}(H_{\text{amb}}^\bullet(Y; \mathbb{C}))$ -valued functions satisfying $\nabla_i^A L_Y(\tau) = 0$ for $i = 1, \dots, r$ and $L_Y(\tau) = \text{id} + O(\tau)$. Since Y is a K3 surface, the quantum cup product \circ_τ coincides with the ordinary cup product, and the fundamental solution is given by $L(\tau) = \exp(-\tau) \cup (-)$. Let $H_{A, \mathbb{C}} = \text{Ker } \nabla^A$ be the \mathbb{C} -local system associated with ∇^A and define the integral local subsystem $H_{A, \mathbb{Z}}^{\text{amb}} \subset H_{A, \mathbb{C}}$ as

$$H_{A, \mathbb{Z}}^{\text{amb}} = \left\{ (2\pi\sqrt{-1})^{-2} L_Y \left(\widehat{\Gamma}_Y \cup (2\pi\sqrt{-1})^{\frac{\deg}{2}} \text{ch}(\iota^* \mathcal{E}) \right) \mid \mathcal{E} \in K(X) \right\}, \quad (7.13)$$

where the $\widehat{\Gamma}_Y$ is defined in terms of the Chern roots δ_1, δ_2 of the tangent bundle TY as $\widehat{\Gamma}_Y := \Gamma(1 + \delta_1)\Gamma(1 + \delta_2)$ [Iri11, Definition 6.3].

The basis $\{p_i\}_{i=1}^r$ of $\text{Pic } X$ determines a coordinate $q = (q_1, \dots, q_r)$ on $\mathcal{M} \cong \text{Pic } X \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Let $u_i \in H^2(X; \mathbb{Z})$ be the Poincaré dual of the toric divisor corresponding to the one-dimensional cone $\mathbb{R} \cdot b_i \in \Sigma$ and $v = u_1 + \dots + u_m$ be the anticanonical class. Givental's *I-function* is defined as the series

$$I_{X, Y}(q, z) = e^{p \log q/z} \sum_{d \in \text{Eff}(X)} q^d \frac{\prod_{k=-\infty}^{(d, v)} (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^0 (u_j + kz)}{\prod_{k=-\infty}^0 (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^{(d, u_j)} (u_j + kz)},$$

which gives a multi-valued map from an open subset of $\mathcal{M} \times \mathbb{C}^\times$ to the classical cohomology ring $H^\bullet(X; \mathbb{C}[z^{-1}])$. Givental's *J-function* is defined by

$$J_Y(\tau, z) = L_Y(\tau, z)^{-1}(1) = \exp(\tau/z).$$

If we write

$$I_{X, Y}(q, z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}),$$

then Givental's mirror theorem [Giv96, Giv98, CG07] states that

$$\text{Euler}(\omega_X^{-1}) \cup I_{X, Y}(q, z) = F(q) \cdot \iota_* J_Y(\varsigma(q), z), \quad (7.14)$$

where $\text{Euler}(\omega_X^{-1}) \in H^2(X; \mathbb{Z})$ is the Euler class of the anticanonical bundle of X , and the *mirror map* $\varsigma(q): \mathcal{M} \rightarrow H_{\text{amb}}^2(Y; \mathbb{C})$ is a multi-valued map defined by

$$\varsigma(q) = \iota^* \left(\frac{G(q)}{F(q)} \right). \quad (7.15)$$

The functions $F(q)$, $G(q)$ and $H(q)$ satisfy the Gelfand–Kapranov–Zelevinsky hypergeometric differential equations, and give periods for the B-model VHS $(\mathcal{H}_B, \nabla^B, \mathcal{F}_B^\bullet, Q_B)$. Iritani [Iri11, Theorem 6.9] lifted the mirror theorem (7.14) to an isomorphism of integral variations of pure and polarized Hodge structures.

An important step in the proof of [Iri11, Theorem 6.9] is the identification, given in the proof of [Iri11, Theorem 5.7], of the monodromy of the B-model VHS along an element ℓ of $\pi_1(\mathbb{L}_{\mathbb{C}^\times}^\vee) \cong \mathbb{L}^\vee$ in the neighborhood of the large complex structure limit point, with the isometry

$$(-) \otimes \iota^*(\mathcal{L}^\vee): \mathcal{N}(\check{Y}_\alpha) \rightarrow \mathcal{N}(\check{Y}_\alpha) \quad (7.16)$$

induced by the tensor product of the restriction to \check{Y}_α of the dual of the line bundle \mathcal{L} on \check{X} with $c_1(\mathcal{L}) = \ell \in \mathbb{L}^\vee \cong \text{Pic } \check{X}$. Note that the isometry (7.16) lifts to an autoequivalence of $D^b \text{coh } \check{Y}$. The relation between monodromy of period map and autoequivalence of the derived category goes back to [Kon98, Hor].

Givental’s mirror theorem (7.14) (and its integral lift by Iritani) gives an isomorphism of the A-model VHS and the B-model VHS only in a neighborhood of the large radius limit point. A global study of the period map from the point of view of mirror symmetry has been done for the quartic mirror family of K3 surfaces in [Har13] based on earlier works [NS95, NS01]. Similar analysis for a couple of 2-parameter cases has been performed in [HNU] based on earlier works [Nag12, Nag13].

8 Dolgachev conjecture

Let $(\Delta, \check{\Delta})$ be a polar dual pair of three-dimensional reflexive polytopes. Let further X be a smooth toric weak Fano 3-fold whose fan polytope is $\check{\Delta}$, and \check{X} be another smooth toric weak Fano 3-fold whose fan polytope is Δ . In other words, Δ is the Newton polytope of $H^0(\mathcal{O}_X(-K_X))$, and $\check{\Delta}$ is the Newton polytope of $H^0(\mathcal{O}_{\check{X}}(-K_{\check{X}}))$. Let $Y \subset X$ and $\check{Y} \subset \check{X}$ be smooth anticanonical hypersurfaces. Define $M_\Delta \subset H^2(Y; \mathbb{Z})$ as the primitive sublattice generated by $H_{A, \mathbb{Z}}^{\text{amb}}(Y) \cap H^2(Y, \mathbb{Z})$, and similarly for $M_{\check{\Delta}} \subset H^2(\check{Y}, \mathbb{Z})$.

Conjecture 8.1 (Dolgachev [Dol96, Conjecture (8.6)]).

1. *There exist a lattice \check{M}_Δ and an orthogonal decomposition $M_\Delta^\perp = U \perp \check{M}_\Delta$.*
2. *There exists a primitive embedding $M_{\check{\Delta}} \subset \check{M}_\Delta$.*
3. *The equality $M_{\check{\Delta}} = \check{M}_\Delta$ holds if and only if $M_\Delta \cong \text{Pic } Y$.*

Let $H_{A, \mathbb{Z}}(Y)$ and $H_{B, \mathbb{Z}}(Y)$ be the primitive sublattices of $H^*(Y; \mathbb{Z})$ generated by $H_{A, \mathbb{Z}}^{\text{amb}}(Y)$ and $H_{B, \mathbb{Z}}^{\text{vc}}(Y)$ respectively. Let further M_Δ^0 be the orthogonal complement of M_Δ inside $\text{NS}(Y)$. One has

$$H_{A, \mathbb{Z}}(Y) = U \oplus M_\Delta \quad (8.1)$$

where U is the unimodular hyperbolic plane generated by the classes $[\mathcal{O}_Y]$ and $[\mathcal{O}_p]$ of the structure sheaf and a skyscraper sheaf. One has sublattices

$$\begin{aligned} H_{A, \mathbb{Z}}^{\text{amb}}(Y) \perp M_\Delta^0 \perp H_{B, \mathbb{Z}}^{\text{vc}}(Y) &\subset H^*(Y; \mathbb{Z}), \\ H_{B, \mathbb{Z}}^{\text{vc}}(\check{Y}) \perp M_{\check{\Delta}}^0 \perp H_{A, \mathbb{Z}}^{\text{amb}}(\check{Y}) &\subset H^*(\check{Y}; \mathbb{Z}) \end{aligned} \quad (8.2)$$

and isomorphisms

$$\begin{aligned} \text{Mir}_Y^{\mathbb{Z}}: H_{A,\mathbb{Z}}^{\text{amb}}(Y) &\xrightarrow{\sim} H_{B,\mathbb{Z}}^{\text{vc}}(\check{Y}), \\ \text{Mir}_{\check{Y}}^{\mathbb{Z}}: H_{A,\mathbb{Z}}^{\text{amb}}(\check{Y}) &\xrightarrow{\sim} H_{B,\mathbb{Z}}^{\text{vc}}(Y). \end{aligned} \quad (8.3)$$

The ranks of the sublattices are given by

$$\text{rank } H_{A,\mathbb{Z}}^{\text{amb}}(Y) = \#((\Delta^\circ)^{(1)} \cap \mathbf{N}) - 3, \quad (8.4)$$

$$\text{rank } H_{B,\mathbb{Z}}^{\text{vc}}(Y) = \#(\Delta^{(1)} \cap \mathbf{M}) - 3, \quad (8.5)$$

$$\text{rank } M_\Delta^0 = \sum_{\gamma \in \Delta^{(1)}} \ell^*(\gamma) \ell^*(\gamma^*), \quad (8.6)$$

where $\Delta^{(1)}$ denotes the 1-skeleton of the polytope Δ , and $\ell^*(\gamma)$ denotes the number of interior lattice points of an interval γ ([DK86, §5.11], [Kob08, Proposition 4.3.3]). Conjecture 8.1.1 holds since the isomorphisms (8.3) lifts to the hyperbolic plane $U = \mathbb{Z}[\mathcal{O}_{\check{Y}}] \oplus \mathbb{Z}[\mathcal{O}_p] \subset H_{B,\mathbb{Z}}(\check{Y})$ by [Iri11, Theorem 6.10]. Conjectures 8.1.2 and 8.1.3 hold if the isomorphisms (8.3) lifts to an isomorphism

$$\begin{aligned} H_{A,\mathbb{Z}}(Y) &\xrightarrow{\sim} H_{B,\mathbb{Z}}(\check{Y}), \\ H_{A,\mathbb{Z}}(\check{Y}) &\xrightarrow{\sim} H_{B,\mathbb{Z}}(Y) \end{aligned} \quad (8.7)$$

of overlattices.

9 Stability conditions

Stability conditions are introduced by Bridgeland [Bri09] motivated by stability of BPS D-branes studied by string theorists [Dou02].

Definition 9.1 ([Bri07, Definition 1.1]). A *stability condition* on a triangulated category \mathcal{T} is a pair $\sigma = (Z, \mathcal{P})$ consisting of

- a group homomorphism $Z: K(\mathcal{T}) \rightarrow \mathbb{C}$, and
- full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$

satisfying the following conditions:

1. If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
2. for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
3. for $A_j \in \mathcal{P}(\phi_j)$ ($j = 1, 2$) with $\phi_1 > \phi_2$, one has $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$,
4. for every non-zero object $E \in \mathcal{T}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{array}{ccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & & \swarrow \text{dashed} & & \swarrow \text{dashed} & & & \swarrow \text{dashed} & & \swarrow \text{dashed} \\ & & A_1 & & A_2 & & & A_n & & \end{array} \quad (9.1)$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j .

The homomorphism Z is called the *central charge*, and the diagram (9.1) is called the *Harder-Narasimhan filtration*. It follows from the definition that $\mathcal{P}(\phi)$ is an abelian category, and its non-zero object $\mathcal{E} \in \mathcal{P}(\phi)$ is said to be *semistable of phase ϕ* . An object \mathcal{E} is said to be *stable* if it is a simple object of $\mathcal{P}(\phi)$, i.e., there are no proper subobjects of \mathcal{E} in $\mathcal{P}(\phi)$. By [Bri07, Proposition 5.3], giving a stability condition on a triangulated category \mathcal{T} is equivalent to giving a bounded t -structure on \mathcal{T} and a *stability function* on its heart with the *Harder-Narasimhan property*. For the definitions of a stability function and the Harder-Narasimhan property, see [Bri07, §2].

Assume that the numerical Grothendieck group $\mathcal{N}(\mathcal{T})$ of finite rank, and take a norm $\|\cdot\|$ on the finite-dimensional vector space $\mathcal{N}(\mathcal{T}) \otimes \mathbb{R}$. A stability condition is *numerical* if the central charge $Z: K(\mathcal{T}) \rightarrow \mathbb{C}$ factors through the numerical Grothendieck group. A numerical stability condition is said to satisfy the *support condition* [KS] if there is a positive constant K such that for any $E \in \mathcal{P}(\phi)$, one has

$$|Z(E)| \geq K\|E\|. \quad (9.2)$$

The set of numerical stability conditions satisfying the support condition is denoted by $\text{Stab } \mathcal{T}$. There is a natural topology on $\text{Stab } \mathcal{T}$ such that the forgetful map

$$\begin{array}{ccc} \mathcal{Z}: \text{Stab } \mathcal{T} & \rightarrow & \text{Hom}(\mathcal{N}(\mathcal{T}), \mathbb{C}) \\ \downarrow & & \downarrow \\ (Z, \mathcal{P}) & \mapsto & Z \end{array} \quad (9.3)$$

is a local homeomorphism. This local homeomorphism induces a structure of a complex manifold on $\text{Stab } \mathcal{T}$.

Since the definition of $\text{Stab } \mathcal{T}$ depends only on the triangulated structure of \mathcal{T} , the group $\text{Aut } \mathcal{T}$ of autoequivalences of \mathcal{T} as a triangulated category acts naturally on $\text{Stab } \mathcal{T}$ from the left; for $\sigma = (Z, \mathcal{P}) \in \text{Stab } \mathcal{T}$ and $\Phi \in \text{Aut } \mathcal{T}$,

$$\Phi(\sigma) = (Z \circ \Phi_*^{-1}, \Phi(\mathcal{P})) \quad (9.4)$$

where $\Phi_*: \mathcal{N}(\mathcal{T}) \rightarrow \mathcal{N}(\mathcal{T})$ is the automorphism induced by Φ . This action commutes with the right action of the universal cover $\widetilde{\text{GL}}(2, \mathbb{R})$ of the general linear group $\text{GL}^+(2, \mathbb{R})$ with positive determinant, which “rotates” the central charge [Bri07, Lemma 8.2].

If $\mathcal{T} = D(Y)$ is the bounded derived category of coherent sheaves on a smooth projective variety Y , then $\text{Stab } \mathcal{T}$ will be denoted by $\text{Stab}(Y)$. A stability condition is *geometric* if all skyscraper sheaves \mathcal{O}_x are stable of the same phase. When Y is a K3 surface, then there is a distinguished connected component $\text{Stab}^\dagger(Y)$ containing geometric stability conditions. Let $\text{Stab}^*(Y)$ be the union of connected components of $\text{Stab}(Y)$ which are images of $\text{Stab}^\dagger(Y)$ by the action of $\text{Aut } D(Y)$.

The derived categories $D(Y)$ and $D(Y')$ of K3 surfaces Y and Y' are equivalent as a triangulated category if and only if there exists a Hodge isometry between the transcendental lattices of Y and Y' [Orl97, Theorem 3.3]. An autoequivalence $\Phi \in \text{Aut } D(Y)$ induces a Hodge isometry $\Phi_*: H^*(Y; \mathbb{Z}) \rightarrow H^*(Y; \mathbb{Z})$ making the following diagram commutative;

$$\begin{array}{ccc} D^b \text{ coh } Y & \xrightarrow{\Phi} & D^b \text{ coh } Y \\ \text{ch}(-)\sqrt{\text{td}_Y} \downarrow & & \text{ch}(-)\sqrt{\text{td}_Y} \downarrow \\ H^*(Y; \mathbb{Z}) & \xrightarrow{\Phi_*} & H^*(Y; \mathbb{Z}). \end{array} \quad (9.5)$$

This induces a group homomorphism

$$(-)_*: \text{Aut } D(Y) \rightarrow \text{Aut } H^*(Y; \mathbb{Z}) \quad (9.6)$$

to the group of Hodge isometries, whose image is the index 2 subgroup

$$\text{Aut}^+ H^*(Y; \mathbb{Z}) \subset \text{Aut } H^*(Y; \mathbb{Z}) \quad (9.7)$$

of Hodge isometries preserving the orientation of positive-definite 4-planes [HLOY04, HMS09]. The kernel of this homomorphism will be denoted by $\text{Aut}^0 D(Y)$.

Consider the space

$$\mathcal{P} = \{\Omega \in \mathcal{N}(Y) \otimes \mathbb{C} \mid (\Omega, \bar{\Omega}) > 0\}, \quad (9.8)$$

which can be identified with the space of pairs $(\Re \Omega, \Im \Omega)$ of elements in $\mathcal{N}(Y) \otimes \mathbb{R}$ which span a positive-definite 2-planes in $\mathcal{N}(Y) \otimes \mathbb{R}$. The space \mathcal{P} consists of two connected components. We write the connected component containing the vector $(1, \sqrt{-1}\omega, -\frac{1}{2}\omega^2)$ for an ample class $\omega \in \text{NS}(Y) \otimes \mathbb{R}$ as $\mathcal{P}^+(Y)$. The complement of the union of reflection hyperplanes for all (-2) -elements is denoted by

$$\mathcal{P}_0^+(Y) = \mathcal{P}^+(Y) \setminus \bigcup_{\delta \in \Delta(Y)} \delta^\perp. \quad (9.9)$$

Theorem 9.2 ([Bri08, Theorem 1.1]). *The map (9.3) induces a Galois covering*

$$\mathcal{Z}: \text{Stab}^\dagger(Y) \rightarrow \mathcal{P}_0^+(Y) \quad (9.10)$$

whose group of deck transformations is the subgroup of $\text{Aut}^0 D(Y)$ which preserves the connected component $\text{Stab}^\dagger(Y)$.

Conjecture 9.3 below implies that $\text{Aut}^0 D(Y) \cong \pi_1(\mathcal{P}_0^+(Y))$.

Conjecture 9.3. *$\text{Stab}^*(Y)$ is connected and simply connected.*

Let

$$\mathcal{Q}(Y) = \{\Omega \in \mathcal{P}(Y) \mid (\Omega, \Omega) = 0\} \quad (9.11)$$

be a quadric hypersurface of $\mathcal{P}(Y)$ and set $\mathcal{Q}_0^+(Y) = \mathcal{Q}(Y) \cap \mathcal{P}_0^+(Y)$. The group $\text{GL}_2^+(\mathbb{R})$ acts freely on $\mathcal{P}_0^+(Y)$ by rotating $(\Re \Omega, \Im \Omega)$. The group \mathbb{C}^\times acts as a subgroup of $\text{GL}_2^+(\mathbb{R})$ by rescaling Ω . Since $\text{GL}_2^+(\mathbb{R})$ -orbits in $\mathcal{P}_0^+(Y)$ intersects $\mathcal{Q}_0^+(Y)$ in a unique \mathbb{C}^\times -orbit, one has a homeomorphism

$$\mathcal{P}_0^+(Y) / \text{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}_0^+(Y) / \mathbb{C}^\times, \quad (9.12)$$

which induces a group isomorphism

$$\pi_1(\mathcal{P}_0^+(Y) / \text{GL}_2^+(\mathbb{R})) \cong \pi_1(\mathcal{Q}_0^+(Y) / \mathbb{C}^\times). \quad (9.13)$$

Since $\pi_1(\text{GL}_2^+(\mathbb{R})) \cong \pi(\mathbb{C}^\times) \cong \pi_1(S^1) \cong \mathbb{Z}$, this yields a group isomorphism

$$\pi_1(\mathcal{P}_0^+(Y)) \cong \pi_1(\mathcal{Q}_0^+(Y)). \quad (9.14)$$

Let

$$\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z}) \subset \mathrm{Aut}^+ H^*(Y; \mathbb{Z}) \quad (9.15)$$

be the subgroup consisting of Hodge isometries which acts trivially on $H^{2,0}(Y) \subset H^*(Y; \mathbb{Z}) \otimes \mathbb{C}$. Such action is usually called *symplectic*, where CY is the notation in [BB] standing for *Calabi-Yau*. An element $\phi \in \mathrm{Aut}^+ H^*(Y; \mathbb{Z})$ lies in $\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})$ if and only if ϕ acts trivially on the transcendental lattice $T(Y)$: The ‘if’ part is clear since $H^{2,0}(Y) \subset T(Y) \otimes \mathbb{C}$, and the ‘only if’ part follows from the fact that for any $\alpha \in T(Y)$, the element $\phi(\alpha) - \alpha$ is integral and orthogonal to both $H^{2,0}(Y)$ and $\mathcal{N}(Y)$, and hence zero. It follows that $\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})$ is isomorphic to the index 2 subgroup of the group $\mathrm{Aut} \mathcal{N}(Y)$ of isometries of $\mathcal{N}(Y)$ preserving orientations of positive-definite 2-planes.

Define $\mathrm{Aut}_{\mathrm{CY}}(D(Y))$ by

$$\mathrm{Aut}_{\mathrm{CY}}(D(Y)) = \{\Phi \in \mathrm{Aut}(D(Y)) \mid \Phi_* \in \mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})\}. \quad (9.16)$$

Conjecture 9.3 gives an exact sequence

$$1 \rightarrow \pi_1(\mathcal{P}_0^+(Y)) \rightarrow \mathrm{Aut} D(Y) \rightarrow \mathrm{Aut}^+ H^*(Y; \mathbb{Z}) \rightarrow 1, \quad (9.17)$$

which together with (9.14), (9.15) and (9.16) induces a short exact sequence

$$1 \rightarrow \pi_1(\mathcal{Q}_0(Y)) \rightarrow \mathrm{Aut}_{\mathrm{CY}} D(Y) \rightarrow \mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z}) \rightarrow 1 \quad (9.18)$$

and an isomorphism

$$\pi_1^{\mathrm{orb}}([\mathcal{Q}_0(Y)/\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})]) \cong \mathrm{Aut}_{\mathrm{CY}} D(Y). \quad (9.19)$$

Since the natural action of \mathbb{C}^\times on $\mathcal{Q}_0^+(Y)$ is free and commutes with the action of $\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y)$, the orbifold quotient $[\mathcal{Q}_0(Y)/\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})]$ is a principal \mathbb{C}^\times -bundle over the moduli space $\mathcal{M}_0(Y) = [\mathcal{D}_0^+(Y)/\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})]$ of ample $\mathrm{NS}(Y)$ -polarized K3 surfaces. The associated long exact sequence of homotopy groups

$$1 \rightarrow \pi_1(\mathbb{C}^\times) \rightarrow \pi_1^{\mathrm{orb}}([\mathcal{Q}_0(Y)/\mathrm{Aut}_{\mathrm{CY}}^+ H^*(Y; \mathbb{Z})]) \rightarrow \pi_1^{\mathrm{orb}}(\mathcal{M}_0(Y)) \rightarrow 1 \quad (9.20)$$

gives

$$1 \rightarrow \mathbb{Z} \rightarrow \mathrm{Aut}_{\mathrm{CY}} D(Y) \rightarrow \pi_1^{\mathrm{orb}}(\mathcal{M}_0(Y)) \rightarrow 1. \quad (9.21)$$

The map $\mathbb{Z} \rightarrow \mathrm{Aut}_{\mathrm{CY}} D(Y)$ sends $1 \in \mathbb{Z}$ to the shift $[2] \in \mathrm{Aut}_{\mathrm{CY}} D(Y)$, so that Conjecture 9.3 implies (and is in fact equivalent to) the isomorphism

$$\pi_1^{\mathrm{orb}}(\mathcal{M}_0(Y)) \cong \mathrm{Aut}_{\mathrm{CY}} D(Y)/[2]. \quad (9.22)$$

10 Borcea-Voisin mirrors

An involution $\iota: S \rightarrow S$ on a K3 surface S is said to be *symplectic* if the induced map $\iota^*: H^{2,0}(S) \rightarrow H^{2,0}(S)$ on $H^{2,0}(S)$ is the identity map, and *anti-symplectic* if it is minus

the identity map. A *2-elementary K3 surface* is a pair (K, ι) of a K3 surface and an anti-symplectic involution ι . Any 2-elementary K3 surface is algebraic.

Decompose $H^2(S; \mathbb{C})$ into the $(+1)$ - and (-1) -eigenspaces $H^2(S; \mathbb{C})^+$ and $H^2(S; \mathbb{C})^-$ of the action $\iota^*: H^2(S; \mathbb{C}) \rightarrow H^2(S; \mathbb{C})$, and set $H^2(S; \mathbb{Z})^\pm = H^2(S; \mathbb{C})^\pm \cap H^2(S; \mathbb{Z})$. The signatures of $H^2(S; \mathbb{Z})^+$ and $H^2(S; \mathbb{Z})^-$ are given by $(1, r-1)$ and $(2, 20-r)$ respectively, where r is the rank of $H^2(S; \mathbb{Z})^+$.

A lattice M of rank r is *2-elementary* if $A_M = M^\vee/M \cong (\mathbb{Z}/2\mathbb{Z})^a$, where a is the minimal number of generators of A_M . The induced quadratic form $q_M: A_M \rightarrow \mathbb{Q}/2\mathbb{Z}$ is called the *discriminant form*. The *parity* of q_M is defined by

$$\delta = \begin{cases} 0 & q_M(A_M) \subset \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases} \quad (10.1)$$

The triple (r, a, δ) is called the *main invariant* of M .

Theorem 10.1 ([Nik79c, Theorem 4.3.2]). *The isometry class of a hyperbolic even 2-elementary lattice is determined by the main invariant.*

If (S, ι) is a 2-elementary K3 surface, then $H^2(S; \mathbb{Z})^+$ is a 2-elementary lattice. The *main invariant* of a 2-elementary K3 surface (S, ι) is defined as the main invariant of the 2-elementary lattice $H^2(S; \mathbb{Z})^+$.

Theorem 10.2 ([Nik79c, Theorem 4.2.2]). *Let (S, ι) be a 2-elementary K3 surface with main invariant (r, a, δ) . Then the fixed locus $S^\iota \subset S$ is given as follows:*

- If $(r, a, \delta) = (10, 10, 0)$, then $S^\iota = \emptyset$.
- If $(r, a, \delta) = (10, 8, 0)$, then S^ι is the union of two elliptic curves.
- In all other cases, S^ι is the disjoint union $S^\iota = C^g \amalg E_1 \amalg \cdots \amalg E_k$ of a curve C^g of genus g and rational curves E_1, \dots, E_k with

$$g = 11 - \frac{r+a}{2}, \quad k = \frac{r-a}{2}. \quad (10.2)$$

Let $\iota: E \rightarrow E$ be an involution on an elliptic curve E . The induced map $\iota^*: H^{1,0}(E) \rightarrow H^{1,0}(E)$ is either id or $-\text{id}$. The quotient $E/\langle \iota \rangle$ is isomorphic to an elliptic curve if $\iota^* = \text{id}$, and to \mathbb{P}^1 if $\iota^* = -\text{id}$.

Let (S, ι_1) be a 2-elementary K3 surface, and (E, ι_2) be an elliptic curve with an involution with $E/\langle \iota_2 \rangle \cong \mathbb{P}^1$. Then the product $\iota = \iota_1 \times \iota_2$ is an involution on $S \times E$ such that the induced map $\iota^*: H^{3,0}(S \times E) \rightarrow H^{3,0}(S \times E)$ is the identity map. If we write the irreducible components of the fixed loci of ι_1 and ι_2 as C_1, \dots, C_n and p_1, \dots, p_4 respectively, then the fixed point of ι consists of $C_i \times p_j$ for $i = 1, \dots, n = k+1$ and $j = 1, \dots, 4$. The involution ι lifts naturally to an involution $\tilde{\iota}$ on the blow-up $(S \times E)^\sim$ of $S \times E$ along these $4n$ curves, and the quotient will be denoted by $Y = (S \times E)^\sim / \langle \tilde{\iota} \rangle$.

Theorem 10.3 ([Bor97, Voi93]). *Y is a smooth Calabi-Yau 3-fold with*

$$h^{1,1}(Y) = 11 + 5n - g, \quad h^{2,1}(Y) = 11 + 5g - n. \quad (10.3)$$

The classification of hyperbolic even 2-elementary lattices admitting primitive embedding in the K3 lattice is given in [Nik79c, Table 1]. One can immediately see from the table that if one restricts to $g > 0$ and $(r, a, \delta) \neq (14, 6, 0)$, then there is a 2-elementary K3 surface (S, ι) with main invariants (r, a, δ) if and only if there is a 2-elementary K3 surface $(\check{S}, \check{\iota})$ with main invariants $(\check{r}, \check{a}, \check{\delta}) = (20 - r, a, \delta)$. Take an elliptic curve $(\check{E}, \check{\iota}_2)$ with an involution such that $\check{E}/\check{\iota}_2 \cong \mathbb{P}^1$. The Calabi-Yau 3-fold associated with $((\check{S}, \check{\iota}_1), (\check{E}, \check{\iota}_2))$ will be denoted by $\check{Y} = (\check{S} \times \check{E})^\sim / \langle (\check{\iota}_1, \check{\iota}_2) \rangle$. Then one has

$$h^{1,1}(Y) = h^{2,1}(\check{Y}), \quad h^{2,1}(Y) = h^{1,1}(\check{Y}). \quad (10.4)$$

The pair (Y, \check{Y}) is called a *Borcea-Voisin mirror pair*.

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