A note on derived categories of Fermat varieties

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Abstract

We show that the quotient stack of a Fermat variety with respect to a natural abelian group action has a full strong exceptional collection consisting of invertible sheaves. We also discuss a description of the derived category of the Fermat variety in terms of the coherent action of the group of characters on the derived category of the quotient stack.

1 Introduction

The *Fermat variety* of degree m and dimension n is defined by

$$X_m^n = \{[x_1 : \cdots : x_{n+2}] \in \mathbb{P}^{n+1} \mid x_1^m + \cdots + x_{n+2}^m = 0\},$$

which can be constructed from X_m^r for $r < n$ by taking direct products, blow-ups, quotients by cyclic groups and blow-downs [SK79, Theorem I]. It follows that any Fermat variety can be constructed inductively from 'Fermat points' X_m^0 and Fermat curves X_m^1. The number of $\mathbb{F}_q$-rational points on X_m^n can be counted by Jacobi sum, which motivated Weil to propose his conjectures [Wei49]. Fermat varieties have also provided a testing ground for the Hodge conjecture and the Tate conjecture [Shi79b, Shi79a, Ran81].

The Fermat variety X_m^n has a natural action of the abelian group

$$G_m^n = \{[\mathrm{diag}(a_1, \ldots, a_{n+2})] \in PGL_{n+2}(\mathbb{C}) \mid a_1^d = \cdots = a_{n+2}^d = 1\},$$

and let $\mathbb{X}_m^n = [X_m^n / G_m^n]$ be the quotient stack. We prove the following in this paper:

Theorem 1.1. *$\mathbb{X}_m^n$ has a full strong exceptional colletion consisting of invertible sheaves.*

King conjectured that a smooth complete toric variety has a full strong exceptional collection consisting of invertible sheaves [Kin97]. It is true for toric weak Fano stacks in dimension two [BH09, IU], but fails for toric surfaces [HP06] and higher-dimensional toric Fano varieties [Efi10]. The existence of full exceptional collections on toric varieties consisting of sheaves is proved by Kawamata [Kaw06]. The case $n = 1$ in Theorem 1.1 is contained in [GL87], which gives a full strong exceptional collection of invertible sheaves on any rational orbifold curve. The existence of a full exceptional collection on $\mathbb{X}_m^n$ follows from [Orl09, Theorem 2.5] and [FU11, Theorem 1.2]. The existence of a full exceptional collection consisting of sheaves follows from the fact that $\mathbb{X}_m^n$ can be obtained from $\mathbb{P}^n$ by iterated root constructions. On the other hand, if m is greater than $n + 1$, then the canonical divisor of X_m^n is free, so that $D^b \operatorname{coh} X_m^n$ has no exceptional object at all.

Let $\mathscr{D}^b \operatorname{coh} \mathbb{X}_m^n$ be the pretriangulated dg category defined in Section 3 whose cohomology category is equivalent to the derived category $D^b \operatorname{coh} \mathbb{X}_m^n$ of coherent sheaves on $\mathbb{X}_m^n$. In other words, $\mathscr{D}^b \operatorname{coh} \mathbb{X}_m^n$ gives an enhancement of $D^b \operatorname{coh} \mathbb{X}_m^n$ in the sense of Bondal and Kapranov [BK90]. Let further $\mathscr{D}^b \operatorname{coh} X_m^n$ be the enhancement of $D^b \operatorname{coh} X_m^n$ defined similarly. There is a coherent action of the group $G_m^{n\vee}$ of characters of G_m^n on $\mathscr{D}^b \operatorname{coh} \mathbb{X}_m^n$, and one can show the following:

Theorem 1.2. *The pretriangulated dg category $\mathscr{D}^b \operatorname{coh} X_m^n$ is equivalent to the idempotent completion of the orbit category of $\mathscr{D}^b \operatorname{coh} \mathbb{X}_m^n$ with respect to the coherent action of the group $G_m^{n\vee}$.*

The idea to use dg categories to study the triangulated structure on the orbit category is due to Keller [Kel05]. Theorems 1.1 and 1.2 show that one can study the derived category of a Fermat variety in terms of a coherent action of a finite abelian group on an enhancement of a triangulated category which has a full strong exceptional collection.

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2 A full strong exceptional collection on $\mathbb{X}_m^n$

We prove Theorem 1.1 in this section. The stack $\mathbb{X} = \mathbb{X}_m^n$ is the closed substack of the toric stack $\mathbb{P} = [(\mathbb{C}^{n+2} \setminus \{0\})/K]$, where

$$K = \{(\alpha_1, \dots, \alpha_{n+2}) \in (\mathbb{C}^\times)^{n+2} \mid \alpha_1^m = \dots = \alpha_{n+2}^m\}.$$

An element $\vec{a}$ of the group

$$L = \mathbb{Z}\vec{x}_1 \oplus \dots \oplus \mathbb{Z}\vec{x}_{n+2} \oplus \mathbb{Z}\vec{c}/(m\vec{x}_i - \vec{c})_{i=1}^{n+2}$$

of characters of K gives an invertible sheaf $\mathcal{O}_\mathbb{P}(\vec{a})$ on $\mathbb{P}$, which restricts to an invertible sheaf $\mathcal{O}_\mathbb{X}(\vec{a})$ on $\mathbb{X}$. The canonical bundles are given by $\omega_\mathbb{P} = \mathcal{O}_\mathbb{P}(-\vec{x}_1 - \dots - \vec{x}_{n+2})$ and $\omega_\mathbb{X} = \mathcal{O}_\mathbb{X}(-\vec{x}_1 - \dots - \vec{x}_{n+2} + \vec{c})$. Since $\mathbb{P}$ is a toric Fano stack of Picard number one, the cohomology $H^i(\mathcal{O}_\mathbb{P}(\vec{a}))$ of an invertible sheaf is non-trivial only if $i = 0, n+1$. The long exact sequence associated with

$$0 \to \mathcal{O}_\mathbb{P}(\vec{a} - \vec{c}) \to \mathcal{O}_\mathbb{P}(\vec{a}) \to \mathcal{O}_\mathbb{X}(\vec{a}) \to 0$$

implies that $H^i(\mathcal{O}_\mathbb{X}(\vec{a}))$ is zero if $i \neq 0, n$. Serre duality gives

$$\operatorname{Ext}^n(\mathcal{O}_\mathbb{X}(\vec{a}), \mathcal{O}_\mathbb{X}) = (H^0(\mathcal{O}_\mathbb{X}(\vec{\omega} + \vec{a})))^\vee$$

where $\vec{\omega} = -\vec{x}_1 - \dots - \vec{x}_{n+2} + \vec{c}$. Any $\vec{a} \in L$ can be presented uniquely as

$$\vec{a} = a_1\vec{x}_1 + \dots + a_{n+2}\vec{x}_{n+2} + e\vec{c}$$

where $0 \le a_i \le m-1$ and $e \in \mathbb{Z}$. One has $\operatorname{Ext}^n(\mathcal{O}_\mathbb{X}(\vec{a}), \mathcal{O}_\mathbb{X}) \neq 0$ if and only if

$$\vec{\omega} + \vec{a} = (a_1 - 1)\vec{x}_1 + \dots + (a_{n+2} - 1)\vec{x}_{n+2} + (e+1)\vec{c}$$

can be presented in such a way that all the coefficients are positive. This happens if

- $0 \leq e \leq n$ and at least $n-e+1$ of a_i are greater than or equal to one.

Define a finite subset of L by

$$A = \{a_{i_1}\vec{x}_{i_1} + \cdots + a_{i_k}\vec{x}_{i_k} + e\vec{c} \mid 0 \leq k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n+2,\ 0 \leq e \leq n-k\},$$

where $0 < a_{i_\ell} \leq m-1$ for $\ell = 1, \dots, k$. One can see that $\operatorname{Ext}^i(\mathcal{O}_{\mathbb{X}}(\vec{a}), \mathcal{O}_{\mathbb{X}}(\vec{b})) = 0$ for $\vec{a}, \vec{b} \in A$ and $i \neq 0$, so that $(\mathcal{O}_{\mathbb{X}}(\vec{a}))_{\vec{a} \in A}$ is a strong exceptional collection.

Let $p_i = [x_1 : \cdots : x_{n+2}]$ be the point such that $x_i = 1$ and $x_j = 0$ for $i \neq j$. An exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \bigoplus_{k \neq i} \mathcal{O}_{\mathbb{P}}(\vec{x}_k) \to \bigoplus_{i \neq k,\ell} \mathcal{O}_{\mathbb{P}}(\vec{x}_k + \vec{x}_l) \to \cdots \to \mathcal{O}_{\mathbb{P}}\left(\sum_{k \neq i} \vec{x}_k\right) \to \mathcal{O}_{p_i} \to 0$$

of $\mathcal{O}_{\mathbb{P}}$-modules restricts to an exact sequence

$$0 \to \mathcal{O}_{\mathbb{X}} \to \bigoplus_{k \neq i} \mathcal{O}_{\mathbb{X}}(\vec{x}_k) \to \bigoplus_{i \neq k,\ell} \mathcal{O}_{\mathbb{X}}(\vec{x}_k + \vec{x}_l) \to \cdots \to \mathcal{O}_{\mathbb{X}}\left(\sum_{k \neq i} \vec{x}_k\right) \to 0 \tag{2.1}$$

of $\mathcal{O}_{\mathbb{X}}$-modules. By using these sequences and their translates, the fact that one can construct any invertible sheaf from $(\mathcal{O}_{\mathbb{X}}(\vec{a}))_{\vec{a} \in A}$ by taking shifts and cones is reduced to the following lemma:

Lemma 2.1. *Let B be the subset of L containing A satisfying the following for any $0 \leq i \leq n$ and any $\vec{a} \in L$:*

- *Let $\mathcal{P}$ be the power set of $\{1, \dots, n+2\} \setminus \{i\}$. If B contains $\vec{a} + \sum_{k \in K} \vec{x}_k$ for any $K \in \mathcal{P}$ except K_0, then B also contains $\vec{a} + \sum_{k \in K_0} \vec{x}_k$.*

Then B coincides with the whole of L.

Proof. We use the following two extreme cases:

(1) If B contains $\vec{a} + \sum_{k \in K} \vec{x}_k$ for all subset K strictly contained in $\{1, \dots, n+2\} \setminus \{i\}$, then B also contains $\vec{a} + \sum_{k \neq i} \vec{x}_k$.

(2) If B contains $\vec{a} + \sum_{k \in K} \vec{x}_k$ for any non-empty subset $K \subset \{1, \dots, n+2\} \setminus \{i\}$, then B also contains $\vec{a}$.

If we replace $\vec{c}$ by $m\vec{x}_i$ for suitable i, we can write

$$A = \{a_1\vec{x}_1 + \cdots + a_{n+2}\vec{x}_{n+2} \mid 0 \leq a_i \leq m, \text{ at least two of } a_i\text{'s are zero}\}.$$

We show that $\vec{v} := a_1\vec{x}_1 + \cdots + a_{n+2}\vec{x}_{n+2} \in B$ for all $(a_1, \dots, a_{n+2}) \in \mathbb{Z}^{n+2}$.

We first consider the case $a_i \geq 0$ for all i. In this case, we may assume either $0 \leq a_i \leq m$ for all i or $0 < a_i$ for all i. In the former case, we may assume at most one of a_i's is zero and we can show $\vec{v} \in B$ by the induction on $\sum_i a_i$, using (1). Then the same induction also shows the latter case.

If $\sum_i a_i \geq m(n+2)$, then we may assume $a_i \geq 0$ for all i and obtain $\vec{v} \in B$. Finally, the reverse induction on $\sum_i a_i$ and (2) prove $\vec{v} \in B$ for all $(a_1, \dots, a_{n+2})$. □

Let $\mathcal{T}$ be the full triangulated subcategory of $D^b \operatorname{coh} \mathbb{X}$ generated by $(\mathcal{O}_{\mathbb{X}}(\vec{a}))_{\vec{a} \in A}$. Since the condition that $\operatorname{Hom}(\mathcal{O}_{\mathbb{X}}(\vec{a}), \mathcal{E}) = 0$ for a coherent sheaf $\mathcal{E}$ and any $\vec{a} \in L$ implies $\mathcal{E} \cong 0$, the right semiorthogonal complement $\mathcal{T}^{\perp}$ is trivial so that the collection $(\mathcal{O}_{\mathbb{X}}(\vec{a}))_{\vec{a} \in A}$ is full.

Remark 2.2. Let $\mathbf{m} = (m_1, \ldots, m_{n+2})$ be a sequence of positive integers and $\mathbb{X}_{\mathbf{m}} = [Z_{\mathbf{m}}/K_{\mathbf{m}}]$ be the quotient stack of $Z_{\mathbf{m}} = \{(x_1, \ldots, x_{n+2}) \in \mathbb{C}^{n+2} \setminus \{0\} \mid x_1^{m_1} + \cdots + x_{n+2}^{m_{n+2}} = 0\}$ by $K_{\mathbf{m}} = \{(\alpha_1, \ldots, \alpha_{n+2}) \in (\mathbb{C}^{\times})^{n+2} \mid \alpha_1^{m_1} = \cdots = \alpha_{n+2}^{m_{n+2}}\}$. Then it is straightforward to generalize the proof of Theorem 1.1 above to show that $\mathbb{X}_{\mathbf{m}}$ has a full strong exceptional collection consisting of invertible sheaves.

3 Coherent actions and orbit categories

Let G be a group and $\mathcal{C}$ be a category. A *coherent action* $(F_g, \alpha_{g,h})$ of G on $\mathcal{C}$ consists of endofunctors $F_g : \mathcal{C} \to \mathcal{C}$ for each $g \in G$ together with natural isomorphisms $\alpha_{g,h} : F_g \circ F_h \to F_{gh}$ of endofunctors for $g, h \in G$ such that

- F_e is the identity functor of $\mathcal{C}$,
- $\alpha_{g,h}$ is the identity if $g = e$ or $h = e$, and
- the diagram

$$\begin{array}{ccc} F_g \circ F_h \circ F_k & \xrightarrow{\alpha_{g,h} \circ F_k} & F_{gh} \circ F_k \\ {\scriptstyle F_g \circ \alpha_{h,k}} \downarrow & & \downarrow {\scriptstyle \alpha_{gh,k}} \\ F_g \circ F_{hk} & \xrightarrow{\alpha_{g,hk}} & F_{ghk} \end{array}$$

  is commutative.

For a coherent action $(F_g, \alpha_{g,h})$ of a finite group G on an additive category $\mathcal{C}$, the corresponding *orbit category* $\mathcal{C}/G$ is defined as follows:

- An object of $\mathcal{C}/G$ is an object of $\mathcal{C}$.
- The space of morphisms from x to y is given by

$$\operatorname{Hom}_{\mathcal{C}/G}(x, y) = \bigoplus_{g \in G} \operatorname{Hom}_{\mathcal{C}}(x, F_g(y)).$$

- The composition of $\varphi \in \operatorname{Hom}_{\mathcal{C}}(x, F_g(y))$ and $\psi \in \operatorname{Hom}_{\mathcal{C}}(y, F_h(z))$ is given by

$$x \xrightarrow{\varphi} F_g(y) \xrightarrow{F_g(\psi)} F_g \circ F_h(z) \xrightarrow{\alpha_{g,h}(z)} F_{gh}(z).$$

Now we recall the notion of enhanced triangulated categories by Bondal and Kapranov [BK90]. See [Kel06] and references therein for basic definitions and results on dg categories. A *dg category* $\mathscr{D}$ is an additive category such that the spaces of morphisms are cochain complexes, the identity morphisms are cocycles, and the compositions satisfy the Leibniz rule. The *cohomology category* $H^0(\mathscr{D})$ has the same objects as $\mathscr{D}$ and the spaces of morphisms are zero-th cohomology groups. A *twisted complex over $\mathscr{D}$* is a collection $\{(E_i)_{i\in\mathbb{Z}}, (q_{ij} : E_i \to E_j)_{i,j\in\mathbb{Z}}\}$ of objects E_i of $\mathscr{D}$ and morphisms q_{ij} of degree $i-j+1$ such that $E_i = 0$ for almost all $i \in \mathbb{Z}$ and the Maurer-Cartan equation $dq_{ij} + \sum_k q_{kj}q_{ik} = 0$ is satisfied for any $i, j \in \mathbb{Z}$. We always assume that a twisted complex is *one-sided* in the sense that $q_{ij} = 0$ for $i \geq j$. Twisted complexes form a dg category $\mathcal{T}w(\mathscr{D})$, equipped with a full and faithful functor $\iota : \mathscr{D} \to \mathcal{T}w(\mathscr{D})$ sending an object E to the twisted complex $\{(E_i)_{i\in\mathbb{Z}}, (q_{ij})_{i,j\in\mathbb{Z}}\}$ such that $E_i = 0$ for $i \neq 0$, $E_0 = E$ and $q_{ij} = 0$ for any $i, j \in \mathbb{Z}$. A *dg module* over a dg category $\mathscr{D}$ is a dg functor from $\mathscr{D}$ to the dg category of chain complexes of $\mathbb{C}$-vector spaces. Dg modules over a dg category $\mathscr{D}$ form a dg category $\operatorname{mod}(\mathscr{D})$, and $\mathscr{D}$ has the Yoneda embedding into $\operatorname{mod}(\mathscr{D})$. The *idempotent completion* of $\mathscr{D}$ is defined as the smallest full subcategory of $\operatorname{mod}(\mathscr{D})$ containing $\mathscr{D}$ and closed under direct summands. A twisted complex X defines a dg functor $\hom_{\mathcal{T}w(\mathscr{D})}(\iota(\bullet), X)$, which gives an object of $\operatorname{mod}(\mathscr{D})$, and $\mathscr{D}$ is said to be *pretriangulated* if this functor is always representable by an object of $\mathscr{D}$. The cohomology category of a pretriangulated dg category is triangulated, and an *enhanced triangulated category* is a triple $(\mathcal{T}, \mathscr{D}, \phi : H^0(\mathscr{D}) \to \mathcal{T})$ consisting of a triangulated category $\mathcal{T}$ together with a pretriangulated dg category $\mathscr{D}$ and an equivalence $\phi : H^0(\mathscr{D}) \to \mathcal{T}$ of triangulated categories.

Let $\sigma : G \times X \to X$ be an action of a finite group G on an algebraic variety X. We write the group law of G as $\mu_G : G \times G \to G$. A *G-linearization* of a line bundle $\pi : L \to X$ on X is an action $\Sigma : G \times L \to L$ of G on L such that $\pi(\Sigma(g, l)) = \sigma(g, \pi(l))$ for any $g \in G$ and $l \in L$. The space $H^0(X, L)$ of sections of L has a natural G-action defined by $(g \cdot s)(x) = \Sigma(g, s(\sigma(g^{-1}, x))$ for $s \in H^0(X, L)$, $g \in G$ and $x \in X$. In terms of the invertible sheaf $\mathcal{L}$ associated with L, a G-linearization is an isomorphism $\phi : \sigma^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$ on $G \times X$ satisfying the cocycle condition

$$(p_{23}^*\phi) \circ [(\mathrm{id}_G \times \sigma)^*\phi] = (\mu_G \times \mathrm{id}_X)^*\phi : [\sigma \circ (\mathrm{id}_G \times \sigma)]^*\mathcal{L} \to [p_2 \circ p_{23}]^*\mathcal{L}$$

on $G \times G \times X$ (see e.g. [MFK94, §3]). A *G-equivariant coherent sheaf on X* is a pair $(\mathcal{E}, \phi)$ of a coherenet sheaf on X and a G-linearization $\phi : \sigma^*\mathcal{E} \xrightarrow{\sim} p_2^*\mathcal{E}$ satisfying the cocycle condition above. It follows from the definition that a G-equivariant coherent sheaf on X is equivalent to a coherent sheaf on the quotient stack $[X/G]$.

Let X be a smooth projective variety and G be a finite group acting on X. Choose an affine open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ such that Λ has a G-action satisfying $g(U_\lambda) = U_{g\lambda}$ for $g \in G$ and $\lambda \in \Lambda$. Such a covering can be constructed from any affine covering $X = \bigcup_{\xi\in\Xi} V_\xi$ by setting $\Lambda = G \times \Xi$ and $U_\lambda = gV_\xi$ for $\lambda = (g, \xi)$. We introduce a dg category $\operatorname{vect}_G X$ as follows:

- An object of $\operatorname{vect}_G X$ is a G-equivariant locally-free sheaf on X.
- The space of morphisms between two objects $\mathcal{E}$ and $\mathcal{F}$ is the Čech complex;

$$\hom^i_{\operatorname{vect}_G X}(\mathcal{E}, \mathcal{F}) = \bigoplus_{\lambda_0,\ldots,\lambda_i\in\Lambda} \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_i}, \mathcal{E}^\vee \otimes \mathcal{F}).$$

The Čech complex has the G-action coming from the G-linearizations of $\mathcal{E}$ and $\mathcal{F}$, which sends $s \in \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_i}, \mathcal{E}^\vee \otimes \mathcal{F})$ to $g \cdot s \in \Gamma(U_{g\cdot\lambda_0} \cap \cdots \cap U_{g\cdot\lambda_i}, \mathcal{E}^\vee \otimes \mathcal{F})$. We do not take the space of G-invariants in the definition of the spaces of morphisms in $\operatorname{vect}_G X$, so that the isomorphism class of an object does not depend on the choice of a G-linearization. Let $\mathscr{D}^b \operatorname{coh} X$ be the idempotent completion of the dg category $\mathcal{T}w(\operatorname{vect}_G X)$ consisting of twisted complexes over $\operatorname{vect}_G X$. The following lemma shows that $\mathscr{D}^b \operatorname{coh} X$ gives an enhancement of $D^b \operatorname{coh} X$:

Lemma 3.1. *Let G be a finite group acting on a smooth projective variety X. Then any object in $D^b \operatorname{coh} X$ is a direct summand of a complex of G-linearizable locally-free sheaves.*

Proof. Let D be an ample divisor on X. Then $\sum_{g \in G} g \cdot D$ is a G-invariant ample divisor on X, and the corresponding invertible sheaf $\mathcal{L}$ is G-linearizable. For any coherent sheaf $\mathcal{E}$, there is a surjection

$$\varphi_0 : (\mathcal{L}^{\otimes(-n_0)})^{\oplus k_0} \to \mathcal{E}$$

for sufficiently large n_0 and k_0. Let $\mathcal{E}_1 = \ker \varphi_0$ be the kernel of this morphism. Then there is a surjection

$$\varphi_1 : (\mathcal{L}^{\otimes(-n_1)})^{\oplus k_1} \to \mathcal{E}_1$$

for sufficiently large n_1 and k_1, and one can set $\mathcal{E}_2 = \ker \varphi_1$. By repeating this process, one obtains a distinguished triangle

$$\mathcal{E}_{k+1}[k] \to \mathcal{F} \to \mathcal{E} \xrightarrow{[+1]} \mathcal{E}_{k+1}[k+1]$$

where

$$\mathcal{F} = \left\{ \left(\mathcal{L}^{\otimes(-n_k)}\right)^{\oplus m_k} \xrightarrow{\varphi_k} \left(\mathcal{L}^{\otimes(-n_{k-1})}\right)^{\oplus m_{k-1}} \xrightarrow{\varphi_{k-1}} \cdots \xrightarrow{\varphi_0} \left(\mathcal{L}^{\otimes(-n_0)}\right)^{\oplus k_0} \right\}$$

for any $k \geq 0$. Since X is smooth, the homological dimension of $\operatorname{coh} X$ is equal to the dimension of X, and this triangle splits for $k > \dim X$. It follows that any coherent sheaf is a direct summand of a complex of G-linearizable locally-free sheaves, and Lemma 3.1 is proved. □

Let $\operatorname{vect}^G X$ be the subcategory of $\operatorname{vect}_G X$ with the same set of objects and G-invariant morphisms;

$$\hom_{\operatorname{vect}^G X}(\mathcal{E}, \mathcal{F}) = (\hom_{\operatorname{vect}_G X}(\mathcal{E}, \mathcal{F}))^G.$$

The existence of a G-equivariant ample line bundle shows that any G-equivariant coherent sheaf has a resolution by a bounded complex of G-equivariant locally-free sheaves. It follows that $\mathscr{D}^b \operatorname{coh}^G X := \mathcal{T}w(\operatorname{vect}^G X)$ gives an enhancement of the derived category $D^b \operatorname{coh}^G X$ of G-equivariant coherent sheaves on X.

Now assume that G is abelian and let $G^\vee = \operatorname{Hom}(G, \mathbb{C}^\times)$ be the group of characters of G. There is a coherent action $(F_\rho, \alpha_{\rho,\sigma})$ of $G^\vee$ on $\operatorname{vect}_G X$ defined as follows:

- For a G-equivariant locally-free sheaf $\mathcal{E}$ and a character $\rho \in G^\vee$, the G-equivariant locally-free sheaf $F_\rho(\mathcal{E}) = \mathcal{E} \otimes \rho$ is given by changing the G-linearization of $\mathcal{E}$ by ρ. To be more precise, let V be a vector space and $\rho : G \to GL(V)$ be a representation. Then the coherent sheaf $\mathcal{E} \otimes V$ is the functor sending an open set $U \subset X$ to the vector space $\mathcal{E}(U) \otimes V$, and the G-linearization is given by $\phi \otimes \rho : \sigma^*(\mathcal{E} \otimes V) \to p_2^*(\mathcal{E} \otimes V)$ where ρ is considered as a $GL(V)$-valued function on $G \times X$ constant along X.

- For a morphism $\psi \in \hom(\mathcal{E}, \mathcal{F})$ of G-equivariant locally-free sheaves (i.e. an element of the Čech complex), the morphism $F_\rho(\psi) \in \hom(\mathcal{E} \otimes \rho, \mathcal{F} \otimes \rho)$ is defined in the obvious way as $\psi \otimes \mathrm{id}$.
- For two characters $\rho, \sigma \in G^\vee$, the natural isomorphism $\alpha_{\rho,\sigma} : F_\rho \circ F_\sigma \xrightarrow{\sim} F_{\rho \otimes \sigma}$ comes from the structure of a tensor category on the category of representations of G (i.e. the natural isomorphism $(A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ for any G-linear spaces).

This coherent $G^\vee$-action on $\operatorname{vect}_G X$ induces a coherent $G^\vee$-action on the G-invariant part $\operatorname{vect}^G X$, which in turn induces a coherent $G^\vee$-action on the category $\mathscr{D}^b \operatorname{coh}^G X$ of twisted complexes over $\operatorname{vect}^G X$. It is clear that the orbit category $\operatorname{vect}^G X/G^\vee$ is equivalent to $\operatorname{vect}_G X$, so that the orbit category $\mathscr{D}^b \operatorname{coh}^G X/G^\vee$ is equivalent to $\mathcal{T}w(\operatorname{vect}_G X)$. Since $\mathscr{D}^b \operatorname{coh} X$ is the idempotent completion of $\mathcal{T}w(\operatorname{vect}_G X)$, Theorem 1.2 is proved.

References

[BH09] Lev Borisov and Zheng Hua, *On the conjecture of King for smooth toric Deligne-Mumford stacks*, Adv. Math. **221** (2009), no. 1, 277–301. MR MR2509327

[BK90] A. I. Bondal and M. M. Kapranov, *Enhanced triangulated categories*, Mat. Sb. **181** (1990), no. 5, 669–683. MR MR1055981 (91g:18010)

[Efi10] Alexander I. Efimov, *Maximal lengths of exceptional collections of line bundles*, arXiv:1010.3755, 2010.

[FU11] Masahiro Futaki and Kazushi Ueda, *Homological mirror symmetry for Brieskorn-Pham singularities*, Selecta Math. (N.S.) **17** (2011), no. 2, 435–452.

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR MR915180 (89b:14049)

[HP06] Lutz Hille and Markus Perling, *A counterexample to King's conjecture*, Compos. Math. **142** (2006), no. 6, 1507–1521. MR MR2278758 (2007h:14074)

[IU] Akira Ishii and Kazushi Ueda, *Dimer models and exceptional collections*, arXiv:0911.4529.

[Kaw06] Yujiro Kawamata, *Derived categories of toric varieties*, Michigan Math. J. **54** (2006), no. 3, 517–535. MR MR2280493 (2008d:14079)

[Kel05] Bernhard Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581. MR 2184464 (2007c:18006)

[Kel06] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190. MR MR2275593 (2008g:18015)

[Kin97] Alastair King, *Tilting bundles on some rational surfaces*, preprint available at `http://www.maths.bath.ac.uk/~masadk/papers/`, 1997.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR MR1304906 (95m:14012)

[Orl09] Dmitri Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., vol. 270, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 503–531. MR 2641200 (2011c:14050)

[Ran81] Ziv Ran, *Cycles on Fermat hypersurfaces*, Compositio Math. **42** (1980/81), no. 1, 121–142. MR 594486 (82d:14005)

[Shi79a] Tetsuji Shioda, *The Hodge conjecture and the Tate conjecture for Fermat varieties*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 3, 111–114. MR 531455 (80e:14006)

[Shi79b] ———, *The Hodge conjecture for Fermat varieties*, Math. Ann. **245** (1979), no. 2, 175–184. MR 552586 (80k:14035)

[SK79] Tetsuji Shioda and Toshiyuki Katsura, *On Fermat varieties*, Tôhoku Math. J. (2) **31** (1979), no. 1, 97–115. MR 526513 (80g:14033)

[Wei49] André Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497–508. MR 0029393 (10,592e)

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