

Galois extensions and maps on local cohomology

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Abstract: We improve the Main theorem in "Absolute integral closure in positive characteristic" Adv. Math. 210 (2007) by C. Huneke and G. Lyubeznik, and as a application, we construct a new big Cohen-Macaulay algebra.

1 Main theorem

1.1 theorem

Let R be a domain of prime characteristic.

- (1) Let $\mathfrak{a}$ be an ideal of R and $[\eta]$ an element of $H_{\mathfrak{a}}^i(R)_{\text{nil}}$. Then there exists a finitely generated generically Galois extension S of R , with $\text{Gal}(S/R)$ a solvable group, such that $[\eta]$ maps to zero under the induced map $H_{\mathfrak{a}}^i(R) \longrightarrow H_{\mathfrak{a}}^i(S)$.
- (2) Suppose $(R, \mathfrak{m})$ is a homomorphic image of a Gorenstein ring. Then there exists a finitely generated generically Galois extension S of R such that the induced map $H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{m}}^i(S)$ is zero for each $i < \dim R$.

Here we say " S is a generically Galois extension of R " if S is an extension domain that is integral over R and the extension of fractions fields is Galois. In this case, $\text{Gal}(S/R)$ will denote the Galois group of the corresponding extension of fraction fields. And $H_{\mathfrak{a}}^i(R)_{\text{nil}}$ means the union of $\ker F^e: H_{\mathfrak{a}}^i(R) \longrightarrow H_{\mathfrak{a}}^i(R)$ for all positive integer e .

1.2 remark

(2) shows the condition is close to characteristic 0, but the statement of this type is never valid in characteristic 0.

For example, take R normal ring in characteristic 0, then for every finite extension S , the inclusion map splits as R -module by the trace map. Therefore the maps on local cohomology also split, especially is injection.

2 History and corollary

C. Huneke and G. Lyubeznik proved a weaker version of (2) above. They proved existence of S which is just finite over R to prove the existence of big Cohen-Macaulay algebra. Big Cohen-Macaulay algebra is a R -algebra S (not necessarily Noether) such that every system of parameter of R is regular sequence on S . They proved the integral closure in an algebraic closure of the quotient field of R (denoted R^+) is big Cohen-Macaulay algebra. With the same discussion, we can prove the existence of another big Cohen-Macaulay algebra.

2.1 Existence of another Big Cohen-Macaulay algebra

Let $(R, \mathfrak{m})$ is a homomorphic image of a Gorenstein ring. and R^{+sep} be the separable closure in an algebraic closure of the quotient field of R . Then R^{+sep} is big Cohen-Macaulay algebra.

3 Application of existence of big Cohen-Macaulay algebra

Big Cohen-Macaulay algebra is first introduced by M. Hochster and C. Huneke in "Infinite integral extensions and big Cohen-Macaulay algebras", Ann. of Math(1992). The existence is known in the case of positive characteristic, equicharacteristic or the dimension is lower than 4. It is known that the existence of big Cohen-Macaulay algebra implies various conjectures below.

3.1 Direct summand conjecture

Let R be a regular Noetherian ring and S be a module-finite R -algebra, then R is a direct summand of S as an R -module.

3.2 Monomial conjecture

Let S be a local ring and $x_1, \dots, x_n$ be a system of parameters for S , then for every integer k , $(x_1 \cdots x_n) \notin (x_1^{k+1}, \dots, x_n^{k+1})S$.

3.3 Cohen-Macaulayness of direct summands of regular rings

Let R be a Noetherian ring and S be a R -algebra. Assume R is a direct summand of S as an R -module, then R is Cohen-Macaulay.

4 Graded case

Let R^{+GR} be the subalgebra of R^+ which is generated by homogeneous integral element over R . So R^{+GR} is $\mathbb{Q}$ -graded. Hochster and Huneke proved this algebra is big Cohen-Macaulay algebra. Since we proved R^{+sep} is big Cohen-Macaulay, it is natural to ask whether $R^{+GR, sep} = R^{+GR} \cap R^{+sep}$ is big Cohen-Macaulay algebra.

4.1 Examples

Let R be the Rees ring

$$\frac{\overline{\mathbb{F}}_2[x, y, z]}{(x^3 + y^3 + z^3)}[xt, yt, zt]$$

with the $\mathbb{N}$ -grading where the generators x, y, z, xt, yt, zt have degree 1.

- (1) the ring $R^{+GR, sep}$ is not big Cohen-Macaulay R -algebra.

- (2) let S be a graded Cohen-Macaulay ring with $R \subseteq S \subseteq R^{+GR}$. We prove that S is not finitely generated over R .

(2) means nonexistence of small (Noether) Cohen-Macaulay algebra in graded case, which is open in general case.

Huneke and Lyubeznik calculated only lower cohomology, but we calculate the top cohomology.

4.2 Proposition on top local cohomology

Let R be an $\mathbb{N}$ -graded domain that is finitely generated over a field R_0 of positive characteristic. Set $d = \dim R$. Then the submodule $[H_{\mathfrak{m}}^d(R)]_{\geq 0}$ maps to zero under the induced map

$$H_{\mathfrak{m}}^d(R) \longrightarrow H_{\mathfrak{m}}^d(R^{+GR}).$$

Hence, there exists a finitely generated graded extension S of R , contained in R^{+GR} , such that $[H_{\mathfrak{m}}^d(R)]_{\geq 0} \longrightarrow H_{\mathfrak{m}}^d(S)$ is zero.

When R is a section ring of a projective variety, we can understand this proposition more geometrically by using the correspondence between sheaf cohomology and local cohomology.

4.3 Geometrical application

Let X denote a projective variety over a field K , then there is a finite surjective morphism of projective varieties $f: Y \rightarrow X$, such that the induced map $H^d(X, \mathcal{O}_X(t)) \rightarrow H^d(X, f^* \mathcal{O}_X(t))$ is zero map for all nonnegative integer t .

5 Thanks for reading this poster and feel free to ask questions.