

THE RING OF DIFFERENTIAL OPERATORS ON AN AFFINE TORIC VARIETY

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1. INTRODUCTION AND MOTIVATION

The ring of differential operators was introduced by Grothendieck [6]. Although it may be ugly in general [1], the ring of differential operators on an affine toric variety has some good features. The aim of this article is to exhibit some of them, in particular, a good structure of the spectrum of its graded ring (with respect to the order filtration) on a scored affine toric variety. In the final section, we consider the characteristic varieties of critical modules, which live in the spectrum of the graded ring.

Let $A := (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = (a_{ij})$ be a $d \times n$ matrix with coefficients in $\mathbb{Z}$. We sometimes identify A with the set of its column vectors. We assume that $\mathbb{Z}A = \mathbb{Z}^d$, where $\mathbb{Z}A$ is the abelian group generated by A .

For $\boldsymbol{\beta} \in \mathbb{C}^d$, the A -hypergeometric system with parameter $\boldsymbol{\beta}$ is defined by

$$M_A(\boldsymbol{\beta}) := D/DI_A(\partial_x) + D\langle A\theta - \boldsymbol{\beta} \rangle,$$

where

- $D := \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle$: the n th Weyl algebra.
- $I_A(\partial_x) := \langle \partial_x^{\mathbf{u}} - \partial_x^{\mathbf{v}} : A\mathbf{u} = A\mathbf{v}, \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \rangle$: the toric ideal.
- $\langle A\theta - \boldsymbol{\beta} \rangle := \sum_{i=1}^d \mathbb{C}[\theta] \sum_{j=1}^n (a_{ij}\theta_j - \beta_i)$: the Euler operators
- $\mathbb{C}[\theta] := \mathbb{C}[\theta_1, \dots, \theta_n]$, $\theta_j = x_j \partial_{x_j}$.

After the systematic study of the A -hypergeometric systems by Gel'fand and his collaborators ([3], [4], etc.), the systems are also known as GKZ-hypergeometric systems.

In this section, we see that the ring of differential operators on an affine toric variety naturally arises as the algebra of contiguity operators of A -hypergeometric systems [19].

Suppose that $P \in D$ satisfies

- $I_A(\partial_x)P \subseteq DI_A(\partial_x)$,
- $\langle A\theta - \boldsymbol{\beta} - \mathbf{a} \rangle P = P\langle A\theta - \boldsymbol{\beta} \rangle$.

Then there exists a D -module homomorphism

$$M_A(\boldsymbol{\beta} + \mathbf{a}) \xrightarrow{\times P} M_A(\boldsymbol{\beta})$$

or P is a **contiguity operator** shifting parameters by $\mathbf{a}$

$$\mathrm{Hom}_D(M_A(\boldsymbol{\beta}), \mathcal{O}) \ni f \mapsto Pf \in \mathrm{Hom}_D(M_A(\boldsymbol{\beta} + \mathbf{a}), \mathcal{O}),$$

where $\mathcal{O}$ is a D -module of some functions; $\mathrm{Hom}_D(M_A(\boldsymbol{\beta}), \mathcal{O})$ may be identified with the space of solutions of $M_A(\boldsymbol{\beta})$ in $\mathcal{O}$.

Consider the algebra of contiguity operators

$$\{P \in D : I_A(\partial_x)P \subseteq DI_A(\partial_x)\}.$$

Since $I_A(\partial_x)$ operates trivially on $M_A(\beta)$ for all β , we consider

$$\text{Sym}_A := \{P \in D : I_A(\partial_x)P \subseteq DI_A(\partial_x)\} / DI_A(\partial_x).$$

Then Sym_A is an algebra, and

$$\text{Sym}_A = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \text{Sym}_{A,\mathbf{a}},$$

where

$$\text{Sym}_{A,\mathbf{a}} = \{P \in \text{Sym}_A : \langle A\theta \rangle P = P \langle A\theta + \mathbf{a} \rangle\}.$$

Let ι be the anti-automorphism of D defined by

- $\iota(x_j) = \partial_{x_j}, \quad \iota(\partial_{x_j}) = x_j \quad (\forall j),$
- $\iota(PQ) = \iota(Q)\iota(P).$

Note that $\iota(\theta_j) = \iota(x_j \partial_{x_j}) = \iota(\partial_{x_j})\iota(x_j) = x_j \partial_{x_j} = \theta_j$.

Then

$$\begin{aligned} \iota(\text{Sym}_A) &= \iota(\{P \in D : I_A(\partial_x)P \subseteq DI_A(\partial_x)\}) / \iota(DI_A(\partial_x)) \\ &= \{P \in D : PI_A(x) \subseteq I_A(x)D\} / I_A(x)D. \end{aligned}$$

This is identified with the ring $D(R_A)$ of differential operators on the affine toric variety defined by A (cf. [10, Theorem 5.13]).

2. DEFINITIONS

In this section, we give some basic definitions. Let $\mathbb{N}A$ be the monoid generated by A . Let R_A denote the semigroup algebra of $\mathbb{N}A$, i.e.,

$$R_A := \mathbb{C}[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} \mathbb{C}t^{\mathbf{a}} \subseteq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}].$$

Here and hereafter we use multi-index notation; $t^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d)$. The ring of differential operators of the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ equals

$$\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle, \quad \text{where } \partial_i = \partial_{t_i}.$$

Then the ring of differential operators of R_A (or on the affine toric variety defined by A) can be given as a subalgebra of the ring of differential operators on the big torus:

$$D(R_A) = \{P \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle : P(R_A) \subset R_A\}.$$

Let $D_k(R_A)$ be the subspace of differential operators of order less or equal to k in $D(R_A)$. Then the graded ring with respect to the order filtration $\{D_k(R_A)\}$ is commutative:

$$G := \text{Gr } D(R_A) = \bigoplus_{k=0}^{\infty} D_k(R_A) / D_{k-1}(R_A) \subseteq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}, \xi_1, \dots, \xi_d],$$

where ξ_i denotes the image of ∂_i .

3. FINITENESS

In general, the ring of differential operators on an affine variety may be neither left or right Noetherian nor finitely generated as an algebra [1]. In this section, we give some results on finiteness of $D(R_A)$.

Theorem 3.1 ([20]). $D(R_A)$ is a finitely generated $\mathbb{C}$ -algebra.

Theorem 3.2 ([18]). (1) $D(R_A)$ is right Noetherian.

(2) $D(R_A)$ is left Noetherian if $\mathbb{N}A$ is S_2 .

In [18], we also gave a necessary condition for $D(R_A)$ being left Noetherian.

Definition 3.3. A semigroup $\mathbb{N}A$ is S_2 if $\mathbb{N}A = \bigcap_{\sigma: \text{facet of } \mathbb{R}_{\geq 0}A} [\mathbb{N}A + \mathbb{Z}(A \cap \sigma)]$.

The following is an example of $\mathbb{N}A$ that does not satisfy the S_2 condition.

Example 1 (non- S_2).

$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then

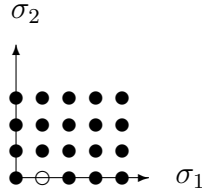


FIGURE 1. The semigroup $\mathbb{N}A$

In this case,

$$\mathbb{N}A = \mathbb{N}^2 \setminus \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{whereas} \quad \bigcap_{\sigma: \text{facet of } \mathbb{R}_{\geq 0}A} [\mathbb{N}A + \mathbb{Z}(A \cap \sigma)] = \mathbb{N}^2.$$

Theorem 3.4 ([19]).

$$\text{Gr } D(R_A) \text{ is Noetherian} \quad \Leftrightarrow \quad \mathbb{N}A \text{ is scored.}$$

Let $\mathcal{F}$ be the set of facets of $\mathbb{R}_{\geq 0}A$. For a facet $\sigma \in \mathcal{F}$, we define the **primitive integral support function** F_σ of σ as the linear form on $\mathbb{R}^d$ uniquely determined by the conditions:

- (1) $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$,
- (2) $F_\sigma(\sigma) = 0$,
- (3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Definition 3.5. The semigroup $\mathbb{N}A$ is said to be **scored** if

$$\mathbb{N}A = \bigcap_{\sigma: \text{facet}} \{ \mathbf{a} \in \mathbb{Z}^d : F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A) \}.$$

Remark 3.6.

$$\mathbb{N}A: \text{ scored} \Rightarrow \mathbb{N}A: S_2.$$

Proof. For each facet σ ,

$$\mathbb{N}A \subseteq \mathbb{N}A + \mathbb{Z}(A \cap \sigma) \subseteq \{ \mathbf{a} \in \mathbb{Z}^d : F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A) \}.$$

Hence

$$\mathbb{N}A \subseteq \bigcap_{\sigma \in \mathcal{F}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma)) \subseteq \bigcap_{\sigma \in \mathcal{F}} \{ \mathbf{a} \in \mathbb{Z}^d : F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A) \}.$$

□

Example 2 (Scored).

$$A_3 = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}. \text{ Then}$$

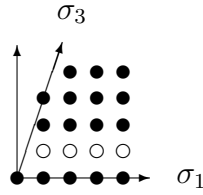


FIGURE 2. The semigroup $\mathbb{N}A_3$

$$\begin{aligned} \mathcal{F} &= \{ \sigma_1 = \mathbb{R}_{\geq 0} \mathbf{a}_1, \sigma_3 = \mathbb{R}_{\geq 0} \mathbf{a}_3 \}, \\ F_{\sigma_1}(s_1, s_2) &= s_2, F_{\sigma_3}(s_1, s_2) = 3s_1 - s_2. \\ F_{\sigma_1}(\mathbb{N}A) &= \mathbb{N} \setminus \{1\}, F_{\sigma_3}(\mathbb{N}A) = \mathbb{N}. \end{aligned}$$

We have

$$\mathbb{N}A = \{ \mathbf{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\mathbf{a}) \in \mathbb{N} \setminus \{1\}, F_{\sigma_3}(\mathbf{a}) \in \mathbb{N} \}.$$

Hence $\mathbb{N}A$ is scored.

Example 3 (S_2 but non-scored). $A_2 = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then

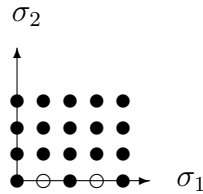


FIGURE 3. The semigroup $\mathbb{N}A_2$

$$\begin{aligned} \mathcal{F} &= \{ \sigma_1 = \mathbb{R}_{\geq 0} \mathbf{a}_1, \sigma_2 = \mathbb{R}_{\geq 0} \mathbf{a}_2 \}, \\ F_{\sigma_1}(s_1, s_2) &= s_2, F_{\sigma_2}(s_1, s_2) = s_1. \\ F_{\sigma_1}(\mathbb{N}A) &= \mathbb{N}, F_{\sigma_2}(\mathbb{N}A) = \mathbb{N}. \end{aligned}$$

We have

$$\mathbb{N}A \subsetneq \{ \mathbf{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\mathbf{a}) \in \mathbb{N}, F_{\sigma_2}(\mathbf{a}) \in \mathbb{N} \} = \mathbb{N}^2.$$

Hence $\mathbb{N}A$ is not scored.

The Running Example-1.

$d = 1, n = 2, A = (2, 3).$

This is the smallest non-trivial example; we use this as a running example.

We have the following:

- $\mathbb{N}A = \{0, 2, 3, 4, \dots\} = \mathbb{N} \setminus \{1\}. \quad \mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}.$
- $\mathcal{F} = \{\{0\}\}, \quad F_{\{0\}}(s) = s; \quad \mathbb{N}A$ is scored.
- $R_A = \mathbb{C}[t^2, t^3].$
- $D(R_A) = \{P \in \mathbb{C}[t^{\pm 1}] \langle \partial \rangle : P(\mathbb{C}[t^2, t^3]) \subseteq \mathbb{C}[t^2, t^3]\}.$
- $D(R_A) = \bigoplus_{a \in \mathbb{Z}} D(R_A)_a, \quad \text{where}$

$$D(R_A)_a = \{P = \sum_{k \in \mathbb{Z}, l \in \mathbb{N}, k-l=a} c_{k,l} t^k \partial^l \in D(R_A)\}.$$

4. THE SPECTRUM

By Theorem 3.4, the spectrum of $\text{Gr } D(R_A)$ is in question, when $\mathbb{N}A$ is scored.

4.1. Weight Decomposition. It is easy to see $s_i := t_i \partial_i \in D(R_A) \quad (i = 1, \dots, d).$

For $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$, set

$$D(R_A)_{\mathbf{a}} := \{P \in D(R_A) : [s_i, P] = a_i P \quad \text{for } i = 1, 2, \dots, d\}.$$

Then $t_i \in D(R_A)_{e_i}, \quad \mathbf{e}_i = {}^t(0, \dots, \overset{i}{1}, \dots, 0).$

Lemma 4.1. (1) $D(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D(R_A)_{\mathbf{a}}.$

(2) $D_k(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D_k(R_A) \cap D(R_A)_{\mathbf{a}}.$

(3) $\text{Gr } D(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \text{Gr } D(R_A)_{\mathbf{a}}.$

Theorem 4.2 ([11]).

$$D(R_A)_{\mathbf{a}} = t^{\mathbf{a}} \mathbb{I}(\Omega(\mathbf{a})) \quad \text{for all } \mathbf{a} \in \mathbb{Z}^d,$$

where

$$\Omega(\mathbf{a}) := \{\mathbf{b} \in \mathbb{N}A : \mathbf{b} + \mathbf{a} \notin \mathbb{N}A\} = \mathbb{N}A \setminus (-\mathbf{a} + \mathbb{N}A),$$

$$\mathbb{I}(\Omega(\mathbf{a})) := \{f(s) \in \mathbb{C}[s] := \mathbb{C}[s_1, \dots, s_d] : f \text{ vanishes on } \Omega(\mathbf{a})\}.$$

In particular, $D(R_A)_{\mathbf{0}} = \mathbb{C}[s].$

The Running Example-2.

$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}.$

$a \in \mathbb{Z}. \quad \Omega(a) = \mathbb{N}A \setminus (-a + \mathbb{N}A). \quad D(R_A)_a = t^a \mathbb{I}(\Omega(a)).$

- $\Omega(a) = \emptyset \quad (a \in \mathbb{N}A), \quad D(R_A)_a = t^a \mathbb{C}[s].$
- $\Omega(1) = \{0\}, \quad D(R_A)_1 = ts \mathbb{C}[s] = t^2 \partial \mathbb{C}[s].$
- $\Omega(-1) = \{0, 2\}, \quad D(R_A)_{-1} = t^{-1} s(s-2) \mathbb{C}[s].$
- $\Omega(-2) = \{0, 3\}, \quad D(R_A)_{-2} = t^{-2} s(s-3) \mathbb{C}[s].$
- $\Omega(-k) = \{0, 2, \dots, k-1\} \cup \{k+1\} \quad (k \geq 3),$
 $D(R_A)_{-k} = t^{-k} s(s-2) \cdots (s-(k-1))(s-(k+1)) \mathbb{C}[s].$

Note that $|\Omega(-k)| = k$ if $k \in \mathbb{N}A.$

4.2. $\mathbb{Z}^d$ -graded Prime Ideals. From now on, we assume that $\mathbb{N}A$ is **scored**, and set $G := \text{Gr } D(R_A)$. By Lemma 4.1, we work on $\mathbb{Z}^d$ -graded prime ideals of G .

Corollary 4.3 (to Theorem 4.2).

$$G = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \overline{t^{\mathbf{a}} \mathbb{I}(\Omega(\mathbf{a}))} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \overline{P_{\mathbf{a}}} \mathbb{C}[s],$$

where

$$\begin{aligned} p_{\mathbf{a}} &:= \prod_{\sigma} \prod_{m \in F_{\sigma}(\mathbb{N}A) \setminus (-F_{\sigma}(\mathbf{a}) + F_{\sigma}(\mathbb{N}A))} (F_{\sigma}(s) - m), \\ P_{\mathbf{a}} &:= t^{\mathbf{a}} \cdot p_{\mathbf{a}}(s), \\ \overline{P_{\mathbf{a}}} &= t^{\mathbf{a}} \cdot \prod_{\sigma} F_{\sigma}(s)^{\sharp(F_{\sigma}(\mathbb{N}A) \setminus (-F_{\sigma}(\mathbf{a}) + F_{\sigma}(\mathbb{N}A)))}. \end{aligned}$$

Since $G_0 = \mathbb{C}[s]$ is a subalgebra of G , the following lemma is immediate.

Lemma 4.4. *Let $\mathfrak{P} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \mathfrak{P}_{\mathbf{a}}$ be a $\mathbb{Z}^d$ -graded prime ideal of G . Then $\mathfrak{P}_0$ is a prime ideal of $G_0 = \mathbb{C}[s]$.*

Given a prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we shall classify all $\mathbb{Z}^d$ -graded prime ideals $\mathfrak{P}$ of G with $\mathfrak{P}_0 = \mathfrak{p}$.

4.3. Degree and Expected Degree. For $\sigma \in \mathcal{F}$ and $\mathbf{a} \in \mathbb{Z}^d$, set

- $\deg_{\sigma}(\mathbf{a}) := \sharp(F_{\sigma}(\mathbb{N}A) \setminus (-F_{\sigma}(\mathbf{a}) + F_{\sigma}(\mathbb{N}A)))$,
- $\expdeg_{\sigma}(\mathbf{a}) := \begin{cases} 0 & \text{if } F_{\sigma}(\mathbf{a}) \geq 0 \\ |F_{\sigma}(\mathbf{a})| & \text{if } F_{\sigma}(\mathbf{a}) \leq 0. \end{cases}$

Then

$$\overline{P_{\mathbf{a}}} = t^{\mathbf{a}} \prod_{\sigma \in \mathcal{F}} F_{\sigma}^{\deg_{\sigma}(\mathbf{a})}.$$

The Running Example-3.

$$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}.$$

$$F_{\{0\}}(s) = s.$$

a	$\cdots$	$-k$	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$\expdeg_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	1	0	0	0	0	$\cdots$
$\deg_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	$\mathbf{2}$	0	$\mathbf{1}$	0	0	$\cdots$

$$G = \bigoplus_{a \in \mathbb{Z}} t^a s^{\deg_{\{0\}}(a)} \mathbb{C}[s] \subseteq \mathbb{C}[t^{\pm 1}, \xi], \quad s = t\xi.$$

For a fixed prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we define

$$\mathcal{F}(\mathfrak{p}) := \{ \sigma \in \mathcal{F} : F_{\sigma} \in \mathfrak{p} \},$$

$$\Sigma(\mathfrak{p}) : \text{ the fan determined by the hyperplane arrangement } \{ \mathbb{R}\sigma : \sigma \in \mathcal{F}(\mathfrak{p}) \},$$

$$S(\mathfrak{p}) := \{ \mathbf{a} \in \mathbb{Z}^d : |F_{\sigma}(\mathbf{a})| \in F_{\sigma}(\mathbb{N}A) \text{ (for } \forall \sigma \in \mathcal{F}(\mathfrak{p})) \}.$$

The Running Example-4.

$$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}.$$

$$F_{\{0\}}(s) = s.$$

- $\mathfrak{p} = (s - \beta) : \text{a fixed prime ideal of } \mathbb{C}[s]$
- $\mathcal{F}((s - \beta)) = \{ \sigma \in \mathcal{F} : F_\sigma \in (s - \beta) \} = \begin{cases} \{0\} & (\beta = 0) \\ \emptyset & (\text{otherwise}). \end{cases}$
- $\Sigma((s - \beta)) = \begin{cases} \{\mathbb{R}_{\geq 0}, \{0\}, \mathbb{R}_{\leq 0}\} & (\beta = 0) \\ \{\mathbb{R}\} & (\text{otherwise}). \end{cases}$
- $S((s - \beta)) = \begin{cases} \mathbb{Z} \setminus \{\pm 1\} & (\beta = 0) \\ \mathbb{Z} & (\text{otherwise}). \end{cases}$

For $\mathbf{a} \in \mathbb{Z}^d$, put

- $\deg_{\mathfrak{p}}(\mathbf{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \deg_{\sigma}(\mathbf{a}).$
- $\expdeg_{\mathfrak{p}}(\mathbf{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \expdeg_{\sigma}(\mathbf{a}).$

Then $\deg_{\mathbf{m}}(\mathbf{a}) = \deg(p_{\mathbf{a}})$, where $\mathbf{m} = (s_1, \dots, s_d)$.

Proposition 4.5. (1) $\deg_{\mathfrak{p}}(\mathbf{a}) \geq \expdeg_{\mathfrak{p}}(\mathbf{a}).$

(2) $\deg_{\mathfrak{p}}(\mathbf{a}) = \expdeg_{\mathfrak{p}}(\mathbf{a})$ if and only if $\mathbf{a} \in S(\mathfrak{p})$.

4.4. Classification. For a cone $\tau \in \Sigma(\mathfrak{p})$, we define an ideal $\mathfrak{P}(\mathfrak{p}, \tau) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \mathfrak{P}(\mathfrak{p}, \tau)_{\mathbf{a}}$ of G by

$$\mathfrak{P}(\mathfrak{p}, \tau)_{\mathbf{a}} := \begin{cases} G_{\mathbf{a}} \mathfrak{p} & (\mathbf{a} \in \tau \cap S(\mathfrak{p})) \\ G_{\mathbf{a}} & (\text{otherwise}). \end{cases}$$

Proposition 4.6. The $\mathbb{Z}^d$ -graded ideal $\mathfrak{P}(\mathfrak{p}, \tau)$ is prime.

Theorem 4.7 ([17]). Let $\mathfrak{P}$ be a $\mathbb{Z}^d$ -graded prime ideal with $\mathfrak{P}_0 = \mathfrak{p}$. Then there exists $\tau \in \Sigma(\mathfrak{p})$ such that $\mathfrak{P} = \mathfrak{P}(\mathfrak{p}, \tau)$.

Proposition 4.8. $\mathfrak{P}(\mathfrak{p}, \tau) \subseteq \mathfrak{P}(\mathfrak{p}', \tau')$ if and only if $\mathfrak{p} \subseteq \mathfrak{p}'$ and $\tau \supseteq \tau'$.

Proposition 4.9. $\dim G/\mathfrak{P}(\mathfrak{p}, \tau) = \dim \mathbb{C}[s]/\mathfrak{p} + \dim \tau$.

The Running Example-5.

$$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}. \quad \text{Let } a \in \mathbb{Z}.$$

- $\mathfrak{P}((s), \mathbb{R}_{\geq 0})_a = \begin{cases} G_a s & (a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$
 - $\mathfrak{P}((s), \{0\})_a = \begin{cases} G_a s & (a = 0) \\ G_a & (a \neq 0). \end{cases}$
 - $\mathfrak{P}((s), \mathbb{R}_{\leq 0})_a = \begin{cases} G_a s & (-a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$
- $$\mathfrak{P}((s), \mathbb{R}_{\geq 0}) \subseteq \mathfrak{P}((s), \{0\}) \supseteq \mathfrak{P}((s), \mathbb{R}_{\leq 0}).$$
- $\mathfrak{P}((s - \beta), \mathbb{R})_a = G_a(s - \beta) \quad (\forall a \in \mathbb{Z}) \quad \text{for } \beta \neq 0.$

5. CHARACTERISTIC VARIETY

5.1. Critical Modules. We denote by $\text{Kdim } M$ the **Krull dimension** for the lattice of $\mathbb{Z}^d$ -graded $D(R_A)$ -submodules in the sense of Rentschler and Gabriel ([2], [13]).

The Krull dimension is defined inductively:

- $\text{Kdim } M = 0 \stackrel{\text{def.}}{\iff} M$: Artinian
- $\text{Kdim } M = \delta$ if
 - $\text{Kdim } M \neq \delta'$ for $\delta' < \delta$.
 - for every descending chain of $\mathbb{Z}^d$ -graded $D(R_A)$ -submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

$$\text{Kdim } M_i/M_{i+1} < \delta \text{ for all but finitely many } i.$$

Definition 5.1. For a $\mathbb{Z}^d$ -graded $D(R_A)$ -module M of Krull dimension δ ,

$$M \text{ is } \delta\text{-critical} \stackrel{\text{def.}}{\iff} \text{Kdim}(M/N) < \delta \text{ for all nonzero } \mathbb{Z}^d\text{-graded } D(R_A)\text{-submodules } N \text{ of } M.$$

Then the 0-critical modules are exactly the simple modules. In this sense, a critical module is a generalization of a simple module. The notions of Krull dimension and critical modules enable us to use Artinian type method to Noetherian rings (see e.g. [5], [9], [10]).

Remark 5.2. Let R be a commutative Noetherian ring, and M a finitely generated R -module. Then

$$M \text{ is critical} \iff \exists \mathfrak{p} \in \text{Spec}(R) \text{ s.t. } \text{Ass}(M) = \{\mathfrak{p}\}, \text{ and } \text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 1.$$

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{C}[s]$, and let $\delta = \dim \mathbb{C}[s]/\mathfrak{p}$. Define a $\mathbb{Z}^d$ -graded $D(R_A)$ -module $L(\mathfrak{p}) := D(R_A)/I(\mathfrak{p})$ by

$$I(\mathfrak{p})_{\mathbf{a}} := \begin{cases} D(R_A)_{\mathbf{a}}\mathfrak{p} & (\mathfrak{p} \sim \mathfrak{p} + \mathbf{a}) \\ D(R_A)_{\mathbf{a}} & (\text{otherwise}), \end{cases}$$

where

- $\mathfrak{p} \sim \mathfrak{p} + \mathbf{a} \stackrel{\text{def.}}{\iff} \mathbb{I}(\Omega(\mathbf{a})) \not\subseteq \mathfrak{p} \text{ and } \mathbb{I}(\Omega(-\mathbf{a})) \not\subseteq \mathfrak{p} + \mathbf{a},$
- $\mathfrak{p} + \mathbf{a} = \{f(s - \mathbf{a}) : f(s) \in \mathfrak{p}\}.$

Then $L(\mathfrak{p})$ is δ -critical [16].

The Running Example-6.

$$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}. \quad \text{Let } \beta \in \mathbb{C}, a \in \mathbb{Z}.$$

- $\mathfrak{p} = (s - \beta), \quad \delta = 0.$
- Let $\beta \notin \mathbb{Z}$. Then $(s - \beta) \sim (s - \beta) + a$ for $\forall a \in \mathbb{Z}$.
 $I((s - \beta))_{\mathbf{a}} = D(R_A)_{\mathbf{a}}(s - \beta)$ for $\forall a \in \mathbb{Z}$.
- Let $\beta \in \mathbb{N}A$. Then $(s - \beta) \sim (s - \beta) + a \iff \beta + a \in \mathbb{N}A$.
 $I((s - \beta))_{\mathbf{a}} = \begin{cases} D(R_A)_{\mathbf{a}}(s - \beta) & (\beta + a \in \mathbb{N}A) \\ D(R_A)_{\mathbf{a}} & (\beta + a \notin \mathbb{N}A). \end{cases}$
- Let $\beta \in \mathbb{Z} \setminus \mathbb{N}A$. Then $(s - \beta) \sim (s - \beta) + a \iff \beta + a \notin \mathbb{N}A$.
 $I((s - \beta))_{\mathbf{a}} = \begin{cases} D(R_A)_{\mathbf{a}}(s - \beta) & (\beta + a \notin \mathbb{N}A) \\ D(R_A)_{\mathbf{a}} & (\beta + a \in \mathbb{N}A). \end{cases}$

Theorem 5.3 ([16]). *Let M be a δ -critical $\mathbb{Z}^d$ -graded left $D(R_A)$ -module singly generated by $v \in M_0$ with $\text{Ann}_{\mathbb{C}[s]}(v) = \mathfrak{p}$. Then M is isomorphic to $L(\mathfrak{p})$.*

5.2. Characteristic Varieties. For a cyclic $D(R_A)$ -module $D(R_A)/I$, the support of the G -module $G/\text{Gr}I$, where $\text{Gr}I = \bigoplus_{k=0}^{\infty} I \cap D_k(R_A)/I \cap D_{k-1}(R_A)$, is called the *characteristic variety* $\text{Ch}(D(R_A)/I)$ of $D(R_A)/I$. For details about characteristic varieties, see any textbook of the theory of D -modules, for example, [7], [8], etc.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{C}[s]$ **homogeneous with respect to** $s_1, \dots, s_d$.

Define $\tau(\mathfrak{p} + \beta)$ by

$$\tau(\mathfrak{p} + \beta) := \bigcap_{\sigma \in \mathcal{F}(\mathfrak{p}); F_{\sigma}(\beta) \in F_{\sigma}(\mathbb{N}A)} (F_{\sigma} \geq 0) \cap \bigcap_{\sigma \in \mathcal{F}(\mathfrak{p}); F_{\sigma}(\beta) \in \mathbb{Z} \setminus F_{\sigma}(\mathbb{N}A)} (F_{\sigma} \leq 0).$$

Then $\tau(\mathfrak{p} + \beta)$ is a union of cones in $\Sigma(\mathfrak{p})$.

Theorem 5.4 ([17]). (1) $\sqrt{\text{Gr} I(\mathfrak{p} + \beta)} = \bigcap_{\tau \in \Sigma(\mathfrak{p}), \tau \subseteq \tau(\mathfrak{p} + \beta)} \mathfrak{P}(\mathfrak{p}, \tau)$. Hence

$$\text{Ch}(L(\mathfrak{p} + \beta)) = \bigcup_{\tau \in \Sigma(\mathfrak{p}), \tau \subseteq \tau(\mathfrak{p} + \beta)} \text{Supp}(G/\mathfrak{P}(\mathfrak{p}, \tau)).$$

(2) The characteristic variety of $L(\mathfrak{p} + \beta)$ is irreducible if and only if $\tau(\mathfrak{p} + \beta) \in \Sigma(\mathfrak{p})$. In this case,

$$\sqrt{\text{Gr} I(\mathfrak{p} + \beta)} = \mathfrak{P}(\mathfrak{p}, \tau(\mathfrak{p} + \beta)).$$

(3) If $F_{\sigma}(\beta) \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}(\mathfrak{p})$, then the characteristic variety of $L(\mathfrak{p} + \beta)$ is irreducible.

(4) $\text{Gr} I(\mathfrak{p}) = \mathfrak{P}(\mathfrak{p}, \tau(\mathfrak{p}))$.

The Running Example-7.

$$A = (2, 3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}. \quad F_{\{0\}}(s) = s. \quad (s) + \beta = (s - \beta).$$

$$\tau((s - \beta)) = \begin{cases} \mathbb{R}_{\geq 0} & (\beta \in \mathbb{N}A) \\ \mathbb{R}_{\leq 0} & (\beta \in \mathbb{Z} \setminus \mathbb{N}A) \\ \mathbb{R} & (\beta \notin \mathbb{Z}). \end{cases}$$

- $\text{Gr} I((s - \beta)) = \mathfrak{P}((s), \mathbb{R}_{\geq 0}) \quad (\beta \in \mathbb{N}A).$
- $\sqrt{\text{Gr} I((s - \beta))} = \mathfrak{P}((s), \mathbb{R}_{\leq 0}) \quad (\beta \in \mathbb{Z} \setminus \mathbb{N}A).$
- $\sqrt{\text{Gr} I((s - \beta))} = \mathfrak{P}((s), \mathbb{R}_{\geq 0}) \cap \mathfrak{P}((s), \mathbb{R}_{\leq 0}) \quad (\beta \notin \mathbb{Z}).$

Example.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. There are four facets: σ_{14} , σ_{36} , σ_{123} , σ_{456} . The primitive integral support functions are $F_{14}(s) = s_2$, $F_{36}(s) = 3s_1 - s_2$, $F_{123}(s) = s_3$, $F_{456}(s) = s_1 - s_3$.

$$\begin{aligned} F_{14}(\mathbb{N}A) &= \mathbb{N} \setminus \{1\}, & F_{36}(\mathbb{N}A) &= \mathbb{N}, \\ F_{123}(\mathbb{N}A) &= \mathbb{N}, & F_{456}(\mathbb{N}A) &= \mathbb{N}. \end{aligned}$$

$$\begin{aligned} \tau(\mathfrak{m} + {}^t[0, 1, 0]) &= (F_{14} \leq 0) \cap (F_{36} \geq 0) \cap (F_{123} \geq 0) \cap (F_{456} \geq 0) \\ &= \{\mathbf{0}\}, \quad \text{where } \mathfrak{m} = (s_1, s_2, s_3). \\ \sqrt{\text{Gr} I(\mathfrak{m} + {}^t[0, 1, 0])} &= \mathfrak{P}(\mathfrak{m}, \{\mathbf{0}\}). \end{aligned}$$

Hence $\dim \text{Ch}(L(\mathfrak{m} + {}^t[0, 1, 0])) = 0$.

Theorem 5.5. *Suppose that $\mathbb{N}A$ is scored. Then the following are equivalent:*

- (1) $\dim \text{Ch}(M) \geq d$ for all nonzero finitely generated $\mathbb{Z}^d$ -graded $D(R_A)$ -modules M .
- (2) $D(R_A)$ is simple.
- (3) For all $\beta \in \mathbb{C}^d$,

$$(5.1) \quad \left\{ \gamma \in \mathbb{R}^d \mid \begin{array}{ll} F_\sigma(\gamma) > 0 & (\forall \sigma \in \mathcal{F}_+(\beta)) \\ F_\sigma(\gamma) < 0 & (\forall \sigma \in \mathcal{F}_-(\beta)) \end{array} \right\} \neq \emptyset,$$

where

$$\begin{aligned} \mathcal{F}_+(\beta) &= \{ \sigma \in \mathcal{F} \mid F_\sigma(\beta) \in F_\sigma(\mathbb{N}A) \}, \\ \mathcal{F}_-(\beta) &= \{ \sigma \in \mathcal{F} \mid F_\sigma(\beta) \in \mathbb{Z} \setminus F_\sigma(\mathbb{N}A) \}. \end{aligned}$$

Proof. (2) $\Leftrightarrow$ (3). This is [15, Theorem 7.25].

(1) $\Leftrightarrow$ (3). The condition (1) is equivalent to the condition:

$$(5.2) \quad \dim \text{Ch}(L) \geq d \text{ for all simple } \mathbb{Z}^d\text{-graded } D(R_A)\text{-modules } L.$$

Any simple $\mathbb{Z}^d$ -graded $D(R_A)$ -module is of the form $L(\mathbf{m} + \beta)$ [12, Proposition 3.17], where $\mathbf{m} = (s_1, s_2, \dots, s_d)$ and $\beta \in \mathbb{C}^d$. By Proposition 4.9 and Theorem 5.4 (1), $\dim \text{Ch}(L(\mathbf{m} + \beta)) = \dim \tau(\mathbf{m} + \beta)$. Clearly $\dim \tau(\mathbf{m} + \beta) = d$ if and only if (5.1) is satisfied. $\square$

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