Counterexample to variational Torelli for some surfaces of geometric genus 2

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Motivation

Our purpose is to examine *Torelli type problems* for surfaces with $p_g = 2$, $K^2 = 1$, $q = 0$, where p_g is the geometric genus, K^2 is self-intersection number of the canonical bundle, and q is the irregularity.

It is known in [4] that they are represented as hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.

Torelli type problems ask in general whether a period map is injective. They are *infinitesimal* Torelli, *generic* Torelli and *variational* Torelli.

It is known that **infinitesimal Torelli theorem** holds for such surfaces([6]).

Griffiths introduced *IVHS* for the interplay between Hodge structure and geometry.

Variational Torelli asks in general whether a variety is recovered by its **IVHS**. It is known that **variational Torelli** leads *generic Torelli* ([2]).

Moreover, **variational Torelli** theorem holds for most hypersurfaces ([3],[5]). But our surfaces are exceptional case in [5].

Notation

$X := \mathrm{Proj}(\mathbb{C}[x,y,z,w]/(F))$,

$F := w^2 - z^5 - \Phi_4 z^3 - \Phi_6 z^2 - \Phi_8 z - \Phi_{10}$,

x, y, z, w : variables of degree $1, 1, 2, 5$ respectively,

$\Phi_i = \Phi_i(x,y)$: degree i for $i = 4, 6, 8, 10$.

$S := \mathbb{C}[x,y]$,

S^a : the degree a part of S,

$G := \mathrm{GL}(2,\mathbb{C})$ acts on S canonically in x, y,

$$A := \begin{bmatrix} \Phi_{4,x} & \Phi_{4,y} & \Phi_{6,x} & \Phi_{6,y} \\ \Phi_{6,x} & \Phi_{6,y} & (\Phi_{8,x} - \frac{3}{5}\Phi_{4,x}\Phi_4) & (\Phi_{8,y} - \frac{3}{5}\Phi_{4,y}\Phi_4) \\ \Phi_{8,x} & \Phi_{8,y} & (\Phi_{10,x} - \frac{2}{5}\Phi_{4,x}\Phi_6) & (\Phi_{10,y} - \frac{2}{5}\Phi_{4,y}\Phi_6) \\ \Phi_{10,x} & \Phi_{10,y} & -\frac{1}{5}\Phi_{4,x}\Phi_8 & -\frac{1}{5}\Phi_{4,y}\Phi_8 \end{bmatrix}.$$

Main theorem

Variational Torelli problem does *NOT* hold for hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.

Remark.1.

It is shown that variational Torelli theorem holds for most hypersurfaces([3],[5]).

Theorem

The prolonged period map from parameter space to IVHS for "*general*" X *is identified with* the following map:

$$v : (S^4 \oplus S^6 \oplus S^8 \oplus S^{10})/G \to S^{28}/G,$$
$$(\Phi_4, \Phi_6, \Phi_8, \Phi_{10}) \mapsto \det(A).$$

Remark.2.

We can show that the invariant A is related to the rank of the map determined by the differential of the period map for the *fiber* of our surface.

More precisely, when we regard $y \in T_t\mathbb{P}^1$ as the element of $\mathrm{Hom}(\mathbf{H^0}(\mathbf{\Omega^1_{C_t}}), \mathbf{H^1}(\mathcal{O}_{C_t}))$,

rank$(y) = \mathbf{2} \Leftrightarrow \det(\mathbf{A}) \neq \mathbf{0}$,
and **rank**$(y) = \mathbf{1} \Leftrightarrow \mathrm{rank}(\mathbf{A'}) = \mathbf{1}$,

while **rank**$(y) \leq \mathbf{2}$.

Here, $T_t\mathbb{P}^1$ is the tangent space of $\mathbb{P}^1$ at t for $t \in \mathbb{P}^1$, and

$$A' := \begin{bmatrix} \Phi_{4,x} & \Phi_{4,y} \\ \Phi_{6,x} & \Phi_{6,y} \\ \Phi_{8,x} & \Phi_{8,y} \\ \Phi_{10,x} & \Phi_{10,y} \end{bmatrix},$$

i.e., A' is the submatrix of A. □

Further problems

- Relate A with *Noether-Lefschetz* locus of X.
- Examine variational Torelli for hypersurfaces of degree $10n$ in $\mathbb{P}[1,1,2n,5n]$ for $n \geq 2$.
- Main theorem shows that IVHS is *not enough* to identify our surfaces. Thus we must find any other information.

IVHS and proof of Main theorem

IVHS is defined by Griffiths et. al. as the differential of the period map.

Definition

IVHS (infinitesimal variation of Hodge structure) consists of:

- a polarised Hodge structure H with a bilinear form b;
- a vector space T;
- a complex linear map $\delta : T \to \mathrm{End}(H_{\mathbb{C}}, b)$.

We can get the main theorem by using well-known theorem by Griffiths:

Theorem(Griffiths)

Let R be the Jacobian ring of X. The *IVHS* for X is identified with the following multiplication maps:

$T := P^1(X, T_X) = \{\eta \in H^1(X, T_X) \mid c_1(\mathcal{O}_X(1)) \wedge \eta = 0\}$

δ = the differential of the period map

$R^{11} \otimes R^{11} \to R^{22} \cong \mathbb{C}$,

$R^{10} \otimes R^1 \to R^{11}$.

Theorem $\Longrightarrow$ Main Theorem

$\mathrm{rank}(\mathrm{Ker}(v)) \geq 3$ from the theorem.

□

Sketch of the proof of Theorem

Consider the following objects:

$$K := \mathrm{Ker}(R^{10} \otimes R^1 \to R^{11}),$$
$$\Lambda := \{\lambda \in \mathbb{C} \mid v \otimes (x - \lambda y) \in K \text{ for some } v \in R^{10}\},$$
$$M := \begin{cases} \prod_{\lambda \in \Lambda}(x - \lambda y) & (\Lambda : \text{finite}), \\ 0 & (\Lambda : \text{infinite}). \end{cases}$$

Then, we can show $M = c \cdot \det(A)$ for some $c \in \mathbb{C}^{\times}$.

□

References

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