ALGEBRAIC STRUCTURES RELATED TO SURFACE RIBBONS IN 3-SPACE

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1. Introduction

Compact orientable surfaces such that each component has a non-empy boundary, embedded in 3-space in the form of ribbons, are called *surface ribbons*. This is an overview of recent work with Emanuele Zappala on algebraic structures related to surface ribbon diagrams, for defining their invariants. Details can be found in [11–15].

Surface ribbons are represented by diagrams of thin ribbons in the plane with ribbon crossings as in Figure 1 (A) and fattened trivalent vertices as in (C) as building blocks. For simplicity we also represent surface ribbons by trivalent graphs as in (B) and (D) in the figure. Labels on strings are used later.

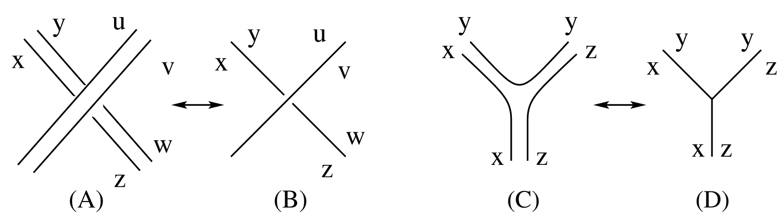

Figure 1. Building blocks

In [10], it was shown that the isotopy class of a compact orientable surface with boundary embedded in 3-space is determined diagrammatically by the moves given in Figure 2. Moves RII, RIII and CL are the framed Reidemeister moves for framed links. Moves IY, YI and IH appear also in the study of handlebody knots in 3-space, see for instance [9].

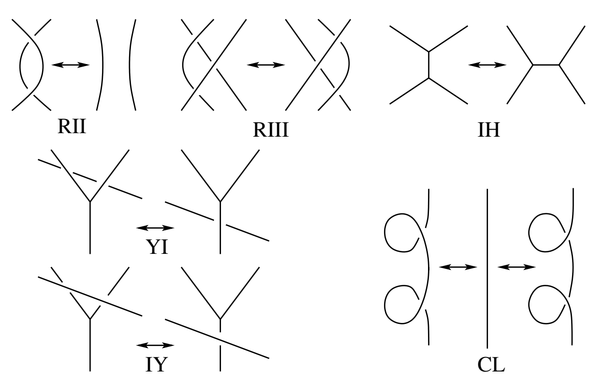

Figure 2. Moves

2. Fundamental heap

In this section we provide an overview of results in [6].

2.1. **Definitions.** We recall the definition and basic properties of heaps. Given a set X with a ternary operation $[-]$, the set of equalities

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, [x_4, x_3, x_2], x_5] = [x_1, x_2, [x_3, x_4, x_5]]$$

is called *para-associativity.* The equations $[x, x, y] = y$ and $[x, y, y] = x$ are called the *degeneracy* conditions. A *heap* is a non-empty set with a ternary operation satisfying the para-associativity and the degeneracy conditions [6].

A typical example of a heap is a group G where the ternary operation is given by $[x, y, z] = xy^{-1}z$, which we call a *group heap.* Conversely, given a heap X with a fixed element e, one defines a binary operation on X by $x * y = [x, e, y]$ which makes $(X, *)$ into a group with e as the identity, and the inverse of x is $[e, x, e]$ for any $x \in X$. Moreover, the associated group heap coincides with the initial heap structure.

Let X be a set with a ternary operation $(x, y, z) \mapsto T(x, y, z)$. The condition $T((x, y, z), u, v) = T(T(x, u, v), T(y, u, v)T(z, u, v))$ for all $x, y, z, u, v \in X$, is called *ternary self-distributivity*, TSD for short. It is known and easily checked that the heap operation $(x, y, z) \mapsto [x, y, z] = T(x, y, z)$ is ternary self-distributive. In this paper we focus on the TSD structures of group heaps.

Definition 2.1 ([14])**.** The *fundamental heap* $h(S)$ of a surface ribbon S is defined as follows. Let D be a diagram of S with double arcs of ribbons with building blocks as in Figure 1 (A) at crossings and (C) at trivalent vertices. We define $h(D)$ by a presentation using D. Let $\mathcal{A}$ be the set of arcs. Two arcs of a ribbon segment (doubled arcs) are listed as separate (distinct) elements of $\mathcal{A}$. Each arc is assigned a generator. In Figure 1, generators are represented by letters (labels) x, y, u, v, z, w. Letters assigned to arcs are identified with (the names of) the arcs themselves, and regarded as elements of $\mathcal{A}$. Then the set of generators of $h(D)$ is $\mathcal{A}$.

For each crossing as depicted in Figure 1 (A), the relations are given by $\{z = xu^{-1}v, w = yu^{-1}v\}$. Specifically, when the arc x goes under the arcs (u, v), in this order, to the arc z, then the relation is defined as $z = xu^{-1}v$, and similar from y to w. The set of union of the two relations over all crossings is denoted by $\mathcal{T}$ and constitutes the set of relations of $h(D)$. For each trivalent vertex as in Figure 1 (C), each connected arc receives the same letter, and no relation is imposed.

The fundamental heap $h(D)$ is the group heap of the group whose presentation is given by a set of generators corresponding to double arcs, and the set of relations assigned to all crossings: $\langle \mathcal{A} \mid \mathcal{T} \rangle$. In the next lemma, it is proved that $h(D)$ does not depend on the choice of D and, therefore that it is well defined for S, and it is denoted by $h(S)$. For a connected disk B^2, it is defined as $h(B^2) = \mathbb{Z}$.

Lemma 2.2. *The fundamental heap $h(S)$ is well defined, that is, the isomorphism class of the group heap $h(D)$ is independent on the choice of D.*

2.2. **Properties.** In [14], the following properties were shown.

- For a surface ribbon with connected components $S = S_1 \cup \cdots \cup S_\nu$, it holds that $h(S) \cong F_\nu * \hat{h}(S)$ for some group $\hat{h}(S)$ where F_ν denotes the free group of rank μ.
- There exists an epimorphism $\Gamma : \hat{h}(S_1 \natural S_2) \longrightarrow \hat{h}(S_1) * \hat{h}(S_2)$, where $S_1 \natural S_2$ denotes the boundary connected sum.
- Any finitely presented group G is realized as a free product factor of $h(S)$ for some surface ribbon S $(h(S) \cong F_k * G)$.
- There exists an epimorphism $\lambda : \pi_1(S^3 \setminus S) \to \hat{h}(S)$.
- For any surface ribbon S, there exists another surface ribbon S' such that S' is obtained from S by a sequence of stabilizations (1-handle additions) and $h(S')$ is a free group.

- For any S with $b(S) > 1$ and any $\chi' \leq \chi(S)$, there exists S' such that $h(S') \cong h(S)$ and $\chi(S') = \chi'$.

2.3. **Colorings and cocycle invariant.** Colorings of diagrams by group heaps are defined similarly to racks and quandles. Ternary self-distributive homology was defined, and 2-ocycles can be used for defining cocycle invariants by a state-sum formula similar to [3].

2.4. **Questions/problems.** The following questions arise.

- Find a geometric interpretation of the fundamental heap, similar to the fundamental group.
- Find a more explicit relation to the fundamental group.
- Can a similar invariant defined for non-orientable surfaces?
- Find the maximal Euler characteristic of surfaces with fundamental heap being a given group.
- Find a relation of the cocycle invariant to group (heap) extensions.

3. Braided Frobenius algebras

This section is a summary of [12]. For an algebra V, an invertible homomorphism $\beta : V \otimes V \to V \times V$ satisfying the *Yang-Baxter* (YB) equation $(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \beta)(\beta \otimes \mathbb{1}) = (\mathbb{1} \otimes \beta)(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \beta)$ is called the YB operator. A *braided Frobenius algebra* was defined in [12] as a Frobenius algebra with a Yang-Baxter (YB) operator that satisfy compatibility condition diagramatized in Figure 2 YI and IY moves, as follows.

Definition 3.1. A *braided Frobenius algebra* is a Frobenius algebra $X = (V, \mu, \eta, \Delta, \epsilon)$ (multiplication, unit, comultiplication, counit) over unital ring $\Bbbk$, endowed with a YB operator $\beta : V \otimes V \to V \otimes V$, such that the Frobenius operations commute with β as follows:

$$\begin{array}{cc} (\mu \otimes \mathbb{1})(\mathbb{1} \otimes \beta)(\beta \otimes \mathbb{1}) = \beta \otimes (\mathbb{1} \otimes \mu), & (\mathbb{1} \otimes \mu)(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \beta) = \beta \otimes (\mu \otimes \mathbb{1}), \\ (\Delta \otimes \mathbb{1})\beta = (\beta \otimes \mathbb{1})(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \Delta), & (\mathbb{1} \otimes \Delta)\beta = (\beta \otimes \mathbb{1})(\beta \otimes \mathbb{1})(\Delta \otimes \mathbb{1}), \\ (\mathbb{1} \otimes \eta)\beta = \eta \otimes \mathbb{1}, & \beta(\eta \otimes \mathbb{1}) = \mathbb{1} \otimes \eta, \\ (\mathbb{1} \otimes \epsilon)\beta = \epsilon \otimes \mathbb{1}, & (\epsilon \otimes \mathbb{1})\beta = \mathbb{1} \otimes \epsilon. \end{array}$$

The first two conditions are related tp the YI and IY moves in Figure 2. Since they satisfy graph moves, braided Frobenius algebras are expected to be useful in constructing surface ribbon invariants, and constructions of concrete examples are desirable. In [12], braided Frobenius algebras are constructed from cocommutative Hopf algebras as follows.

Recall from the preceding section that a heap is a ternary operation exemplified by a group with the operation $(x, y, z) \mapsto xy^{-1}z$, that is ternary self-distributive. Hopf algebras can be endowed with the algebra version of the heap operation. Using this, in [12], we construct braided Frobenius algebras from a class of certain Hopf algebras that admit integrals and cointegrals. For these Hopf algebras we show that the heap operation induces a Yang-Baxter operator on the tensor product, which satisfies the required compatibility conditions. Diagrammatic methods are employed for proving commutativity conditions.

3.1. **Quantum heaps.** For a Hopf algebra H, define $T : H^{\otimes 3} \to H$ by $T(x \otimes y \otimes z) = \mu(\mu(x \otimes S(y)) \otimes z)$ on simple tensors, where μ denotes multiplication and S denotes the antipode. This is an analogue of group heap operation $T(x, y, z) = xy^{-1}z$ and is called a *quantum heap*. A diagrammatic representation is depicted in Figure 3. In the right hand side of the figure, a trivalent vertex represent multiplication of a Hopf algebra, and a circle represents an antipode.

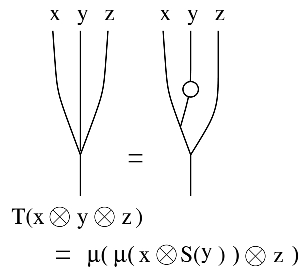

FIGURE 3. Quantum heap

3.2. **Defining R-matrix.** Using quatum heap operation, solutions (R-matrices) to the Yang-Baxter equation was constructed, and the construction is depicted in Figure 4. In the figure, the comultiplication is denoted for simple tensors by Sweedler's notation $\Delta(y) = y^{(1)} \otimes y^{(2)}$.

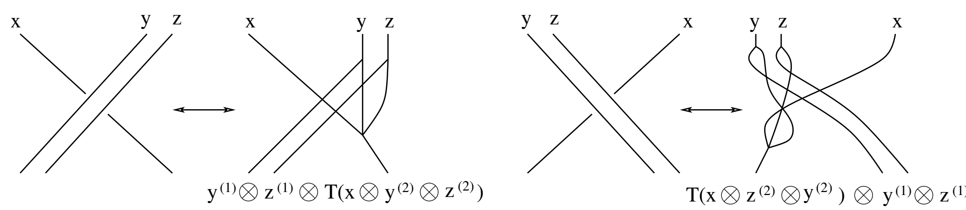

FIGURE 4. Crossings using quantum heap

In [12], the following was proved, which shows that quantum heaps produce Yang-Baxter operators.

Lemma 3.2 ([12])**.** *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a cocommutative Hopf algebra. Then the map $\beta : X^{\otimes 2} \to X^{\otimes 2}$ defined on simple tensors as*

$$x \otimes y \otimes z \otimes w \mapsto z^{(1)} \otimes w^{(1)} \otimes xS(z^{(2)})w^{(2)} \otimes yS(z^{(3)})w^{(3)}$$

is a Yang-Baxter operator.

The multiplication of a Frobenius algebra was defined on $V = X \times X$ using left integrals, and shown to provide braided Frobenius algebra together with the YB operator constructed above. Specifically, let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a finitely generated projective Hopf algebra over a (unital) ring $\Bbbk$. Then it is known that X has an integral and a cointegral. Let us indicate them by λ and γ, respectively. We define a cup on X by $\cup := \lambda\mu(\mathbb{1} \otimes S)$ and $\cap := \Delta\gamma$, as depicted in Figure 5. Integrals are represented by triangles in the figure. A product $\mu_{\otimes 2} : X^{\otimes 2} \otimes X^{\otimes 2} \to X^{\otimes 2}$ is defined by means of $\cup$ as $\mu_{\otimes 2} := \mathbb{1} \otimes \cup \otimes \mathbb{1}$. The coproduct $\Delta_{\otimes 2} : X^{\otimes 2} \to X^{\otimes 2} \otimes X^{\otimes 2}$ is obtained from $\cap$ by the definition $\Delta_{\otimes 2} := \mathbb{1} \otimes \cap \otimes \mathbb{1}$. Then the YB operator together with this (co)multiplication provides a braided Frobenius algebra.

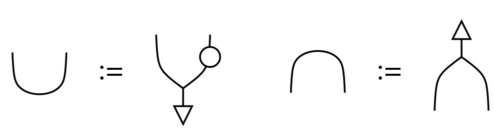

FIGURE 5. Cups and caps

Theorem 3.3 ([12]). *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a commutative and cocommutative Hopf algebra. Then $V = X \otimes X$ has a braided Frobenius algebra structure.*

Defining and studying invariants using these algebraic structures for framed links and surface ribbons are desirable.

4. Yang-Baxter Hochschild cohomology

Braided algebras are associative algebras endowed with a Yang-Baxter operator that satisfies certain compatibility conditions involving the multiplication. Along with Hochschild cohomology of algebras, there is also a notion of Yang-Baxter cohomology, which is associated to any Yang-Baxter operator. In [15], a cohomology theory for braided algebras in dimensions 2 and 3 that unifies Hochschild and Yang-Baxter cohomology theories was defined and studied. This section is an overview of this paper. It was shown that its second cohomology group classifies infinitesimal deformations of braided algebras. Infinite families of examples of braided algebras were provided, including Hopf algebras, tensorized multiple conjugation quandles, and braided Frobenius algebras. Moreover, the obstructions to quadratic deformations were derived, and shown that these obstructions lie in the third cohomology group. Relations to Hopf algebra cohomology were also discussed.

4.1. **Deformation 2-cocycles and graph moves.** Let (V, μ) be an associative algebra with coefficient unital ring $\Bbbk$. The cochain groups of Hochschild cohomology are defined by $C^0_{\rm H}(V, V) = 0$ and $C^n_{\rm H}(V, V) = \mathrm{Hom}(V^{\otimes n}, V)$ for $n \geq 1$. For $f \in C^1_{\rm H}(V, V)$ and $\psi \in C^2_{\rm H}(V, V)$, differentials are defined by

$$\begin{aligned} \delta^1_{\rm H}(f) &= \mu(f \otimes \mathbf{1}) + \mu(\mathbf{1} \otimes f) - f\mu, \\ \delta^2_{\rm H}(\psi) &= \mu(\psi \otimes \mathbf{1}) + \psi(\mu \otimes \mathbf{1}) - \mu(\mathbf{1} \otimes \psi) - \psi(\mathbf{1} \otimes \mu). \end{aligned}$$

Diagrammatic representations of these maps are depicted in Figure 6. These diagrammatics for deformation were used before [2].

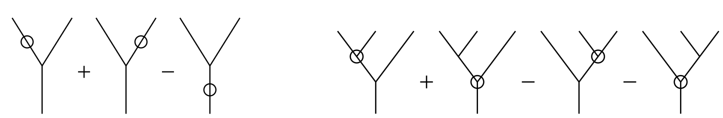

Figure 6. Hochschild differentials

The 2-cocycle condition is related to the deformation as follows. Let $\tilde{V} = (V \otimes_{\mathbf{k}} \mathbf{k}[[\hbar]])/(\hbar^2) \cong V \oplus \hbar V$ and $\psi \in Z^2_{\rm H}(V, V)$. Set $\tilde{\mu} = \mu + \hbar\psi$, then $(\tilde{V}, \tilde{\mu})$ is an algebra if and only if $\delta^2_{\rm H}(\psi) = 0$. This can be seen by computing the associativity as follows.

$$\begin{aligned} \tilde{\mu}(\tilde{\mu} \otimes \mathbf{1}) &= (\mu + \hbar\psi)((\mu + \hbar\psi) \otimes \mathbf{1}) \\ &= \mu(\mu \otimes \mathbf{1}) + \hbar[\mu(\psi \otimes \mathbf{1}) + \psi(\mu \otimes \mathbf{1})], \\ \tilde{\mu}(\mathbf{1} \otimes \tilde{\mu}) &= (\mu + \hbar\psi)(\mathbf{1} \otimes (\mu + \hbar\psi) \\ &= \mu(\mathbf{1} \otimes \mu) + \hbar[\mu(\mathbf{1} \otimes \psi) + \psi(\mathbf{1} \otimes \mu)] = 0 \end{aligned}$$

Then we see that 2-cocycle condition below gives associativity of $\tilde{\mu}$:

$$\delta^2_{\rm H}(\psi) = \mu(\psi \otimes \mathbf{1}) + \psi(\mu \otimes \mathbf{1}) - \mu(\mathbf{1} \otimes \psi) - \psi(\mathbf{1} \otimes \mu) = 0.$$

4.2. **Deformation of Yang-Baxter operators and differentials.** Let (V,R) be a $\Bbbk$-module with the YB operator $R : V^{\otimes 2} \to V^{\otimes 2}$. The cochain groups are defined by $C^0_{\rm YB}(V,V)=0$ and $C^n_{\rm YB}(V,V) = {\rm Hom}(V^{\otimes n}, V^{\otimes n})$ for $n>0$. We define the differentials for $f \in C^1_{\rm YB}(V,V)$ and $\phi \in C^2_{\rm YB}(V,V)$ by

$$\begin{aligned}
\delta^1_{\rm YB}(f) &= R(f\otimes \mathbb{1}) + R(\mathbb{1}\otimes f) - (f\otimes \mathbb{1})R - (\mathbb{1}\otimes f)R, \\
\delta^2_{\rm YB}(\phi) &= (R\otimes \mathbb{1})(\mathbb{1}\otimes R)(\phi\otimes \mathbb{1}) + (R\otimes \mathbb{1})(\mathbb{1}\otimes \phi)(R\otimes \mathbb{1}) + (\phi\otimes \mathbb{1})(\mathbb{1}\otimes R)(R\otimes \mathbb{1}) \\
&\quad -(\mathbb{1}\otimes R)(R\otimes \mathbb{1})(\mathbb{1}\otimes \phi) - (\mathbb{1}\otimes R)(\phi\otimes \mathbb{1})(\mathbb{1}\otimes R) - (\mathbb{1}\otimes \phi)(R\otimes \mathbb{1})(\mathbb{1}\otimes R).
\end{aligned}$$

The differentials are depicted in Figures 7. The cochains f and ϕ are represented by circles on an edge and a crossing, respectively. These are closely relates to Eisermann's YBE cohomology [5].

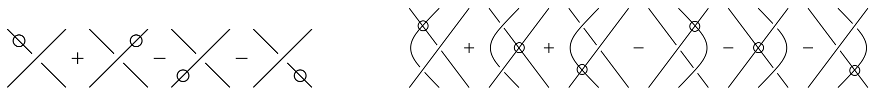

FIGURE 7. Yang-Baxter differentials

Let V be a braided algebra with coefficient unital ring $\Bbbk$. We define the cochain groups for a braided algebra V with coefficients in itself up to degree 3 as follows. We set $C^0_{\rm YBH}(V,V)=0$, and $C^{n,k}_{\rm YBH}(V,V) = {\rm Hom}(V^{\otimes n}, V^{\otimes k})$ for $n,k>0$. We also use different subscripts

$$C^{n,k}_{\rm YI}(V,V) = {\rm Hom}(V^{\otimes n}, V^{\otimes k}) = C^{n,k}_{\rm IY}(V,V)$$

to distinguish different isomorphic direct summands. Define

$$\begin{aligned}
C^1_{\rm YBH}(V,V) &= C^{1,1}_{\rm YBH}(V,V) = {\rm Hom}(V,V), \\
C^2_{\rm YBH}(V,V) &= C^{2,2}_{\rm YBH}(V,V) \oplus C^{2,1}_{\rm YBH}(V,V), \quad \text{and} \\
C^3_{\rm YBH}(V,V) &= C^{3,3}_{\rm YBH}(V,V) \oplus C^{3,2}_{\rm YI}(V,V) \oplus C^{3,2}_{\rm IY}(V,V) \oplus C^{3,1}_{\rm YBH}(V,V).
\end{aligned}$$

We define differentials as follows. We set $\delta^1_{\rm YBH}$ to be the direct sum $\delta^1_{\rm YB} \oplus \delta^1_{\rm H}$. The second differential $\delta^2_{\rm YBH}$ is defined to be the direct sum of four terms $\delta^2_{\rm YB} \oplus \delta^2_{\rm YI} \oplus \delta^2_{\rm IY} \oplus \delta^2_{\rm H}$ where $\delta^2_{\rm YB}$ and $\delta^2_{\rm H}$ map in the first ($C^{3,3}_{\rm YBH}(V,V) = {\rm Hom}(V^{\otimes 3}, V^{\otimes 3})$) and last ($C^{3,1}_{\rm YBH}(V,V) = {\rm Hom}(V^{\otimes 3}, V)$) copies of $C^3_{\rm YBH}(V,V)$, respectively, while $\delta^2_{\rm YI}$ and $\delta^2_{\rm IY}$ map to the middle two factors $C^{3,2}_{\rm YI}(V,V)$ and $C^{3,2}_{\rm IY}(V,V)$, respectively. Each differential is defined as follows.

$$\begin{aligned}
\delta^2_{\rm YI}(\phi\oplus\psi) &= (\mathbb{1}\otimes\psi)(R\otimes \mathbb{1})(\mathbb{1}\otimes R) + (\mathbb{1}\otimes\mu)(\phi\otimes \mathbb{1})(\mathbb{1}\otimes R) \\
&\quad +(\mathbb{1}\otimes\mu)(R\otimes \mathbb{1})(\mathbb{1}\otimes \phi) - R(\psi\otimes \mathbb{1}) - \phi(\mu\otimes \mathbb{1}). \\
\delta^2_{\rm IY}(\phi\oplus\psi) &= (\psi\otimes \mathbb{1})(\mathbb{1}\otimes R)(R\otimes \mathbb{1}) + (\mu\otimes \mathbb{1})(\mathbb{1}\otimes \phi)(R\otimes \mathbb{1}) \\
&\quad +(\mu\otimes \mathbb{1})(\mathbb{1}\otimes R)(\phi\otimes \mathbb{1}) - R(\mathbb{1}\otimes\psi) - \phi(\mathbb{1}\otimes\mu).
\end{aligned}$$

The differential $\delta^2_{\rm IY}(\phi\oplus\psi)$ is represented diagrammatically in Figure 8, where 2-cochains $\phi \in C^{2,2}_{\rm YBH}(V,V)$ and $\psi \in C^{3,1}_{\rm YBH}(V,V)$ are represented by 4-valent (resp. 3-valent) vertices with circles. The YI and IY components of the cochain complex above are included to enforce the coherence axioms between deformed algebra structure, and deformed YB operator.

The following results were obtained in [15].

- Let (V,μ,R) be a braided algebra. Then the Yang-Baxter Hochschild second cohomology group classifies the infinitesimal deformations of (V,μ,R).
- Nontriviality of second YBH cohomology is proved using Hopf algebras.

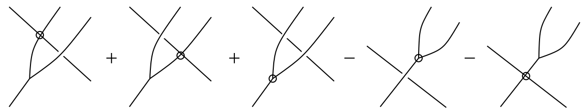

FIGURE 8. Yang-Baxter Hochschild 2-differential

- The second YBH cohomology group with multiplication μ, YB operator R injects to that of braided algebras with braided multiplication μR and YB operator R.
- The cohomology group in dimension 3 was defined, and it was shown that if $H^3_{\rm YBH}(V,V)=0$, then any infinitesimal deformation can be extended to a quadratic deformation.

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