102$^e$ RCP: "Combinatorics, topology and biology" for our friend R.C. Penner

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# "Johnson homomorphisms and the Morita-Penner cocycle — a survey"

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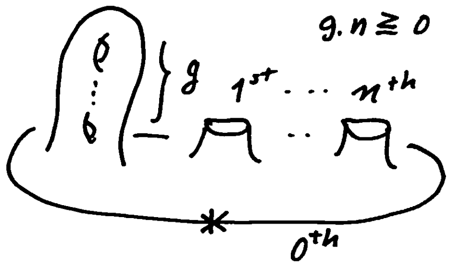
$\Sigma_g :=$ 

, $\Sigma_g' := \Sigma_g \setminus \{p_0\}$

$\Sigma_{g,n+1} :=$ 


$g, n \geqq 0$

$\mathcal{M}_g := \pi_0 \mathrm{Diff}^+(\Sigma_g)$, $\mathcal{M}_{g,*} := \pi_0 \mathrm{Diff}^+(\Sigma_g, p_0)$, $\mathcal{M}_{g,n+1} := \pi_0 \mathrm{Diff}^+(\Sigma_{g,n+1}, \mathrm{id.\ on}\ \partial\Sigma_{g,n+1})$

mapping class groups

$$H_{\mathbb{Z}} := H_1(\Sigma_g; \mathbb{Z}) = H_1(\Sigma_g'; \mathbb{Z}) = H_1(\Sigma_{g,1}; \mathbb{Z})$$
$$\underset{\text{Poincaré duality}}{=} H^1(\Sigma_g; \mathbb{Z}) = H^1(\Sigma_g'; \mathbb{Z}) = H^1(\Sigma_{g,1}; \mathbb{Z})$$

## § 1. The Johnson homomorphisms

$g \geq 1$, $\Sigma = \Sigma_{g,1} =$  $* \in \partial\Sigma$

$\zeta$ negative boundary loop

$\pi := \pi_1(\Sigma, *) \cong F_{2g}$ free group of rank $2g$

$\{\Gamma_k \pi\}_{k=1}^{\infty}$ : lower central series of $\pi$

$$\Gamma_1 \pi := \pi, \quad \Gamma_{k+1}\pi := [\Gamma_k \pi, \pi], \quad k \geq 1$$

<u>Theorem</u> (Magnus, Witt) $\forall k \geq 1$

$$\Gamma_k \pi / \Gamma_{k+1} \pi = \mathcal{L}_k(H_{\mathbb{Z}})$$ the degree $k$ component of the free Lie algebra over $H_{\mathbb{Z}} = \pi^{abel}$

<u>e.g.,</u> $\Gamma_1 \pi / \Gamma_2 \pi = \pi / [\pi, \pi] = \pi^{abel} = H_{\mathbb{Z}} = H_1(\Sigma_{g,1}; \mathbb{Z})$

$$\Gamma_2 \pi / \Gamma_3 \pi = \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}} = (H_{\mathbb{Z}} \otimes H_{\mathbb{Z}})^{sgn, \mathfrak{S}_2}$$

$$\rotatebox{90}{$\in$}$$

$$\zeta \bmod \Gamma_3 \pi \longmapsto \omega_0 := \sum_{i=1}^{g} x_i y_i - y_i x_i$$

symplectic form.

$\{x_i, y_i\}_{i=1}^{g} \subset H_{\mathbb{Z}}$ symplectic basis

$$\mathcal{L}(H_{\mathbb{Z}}) = \bigoplus_{k=1}^{\infty} \mathcal{L}_k(H_{\mathbb{Z}})$$ the free Lie algebra over $H_{\mathbb{Z}}$

mapping class group

**Theorem** (Dehn - Nielsen)

neg. boundary loop

$$\mathcal{M}_{g,1} \underset{\text{identify}}{\cong} \{\varphi \in \mathrm{Aut}(\pi) : \varphi(\zeta) = \zeta\} < \mathrm{Aut}(\pi)$$

**Andreadakis - Johnson filtration** $k \geq 0$

$$A(k) \overset{\text{def}}{=} \mathrm{Ker}(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi))$$

$$\mathcal{M}(k) \overset{\text{def}}{=} \mathcal{M}_{g,1} \cap A(k)$$

e.g., $\mathcal{M}(0) = \mathcal{M}_{g,1}$, $\mathcal{M}(1) = \mathcal{I}_{g,1}$ : the Torelli group, $\mathcal{M}(2) = \mathcal{K}_{g,1}$ : the Johnson kernel

**Description of** $gr\,\mathcal{M}$

**classical** $\mathcal{M}(0)/\mathcal{M}(1) \cong Sp_{2g}(\mathbb{Z}) < \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_1(H_{\mathbb{Z}}))$

**$k \geq 1$**: the $k^{th}$ Johnson homomorphism (D. Johnson)

$$\begin{array}{ccc}
\varphi & \longmapsto & (\gamma \mapsto \gamma^{-1}\varphi(\gamma)) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\mathcal{M}(k)/\mathcal{M}(k+1) & \hookrightarrow & \mathrm{Hom}(\pi, \Gamma_{k+1}\pi/\Gamma_{k+2}\pi) \\
& \circlearrowleft & \| \\
& \overset{\tau_k}{\searrow} & \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}(H_{\mathbb{Z}}))
\end{array}$$

easy to check

the $k^{th}$ Johnson homomorphism

**D. Johnson**

$\mathcal{M}(1)/\mathcal{M}(2) \cong \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}$

$\mathrm{Ker}(\tau_1 : \mathcal{I}_{g,1}^{abel} \to \mathcal{M}(1)/\mathcal{M}(2))$ : 2-torsion

$\mathcal{M}(2) = \mathcal{K}_{g,1}$ is generated by "BSCC maps"

S. Morita

(1) $\tau : \bigoplus_{k=0}^{\infty} m(k)/m(k+1) \hookrightarrow \mathrm{Hom}(H_{\mathbb{Z}}, \bigoplus_{k=0}^{\infty} \mathcal{L}_{k+1}(H_{\mathbb{Z}})) = \mathrm{Der}(\mathcal{L}(H_{\mathbb{Z}}))$

is a Lie algebra homomorphism

(2) $\mathrm{Im}\,\tau \subset \mathrm{Der}_{\omega_0}(\mathcal{L}(H_{\mathbb{Z}})) := \{ D \in \mathrm{Der}(\mathcal{L}(H_{\mathbb{Z}})) \,;\, D\omega_0 = 0 \}$ symplectic derivations

(3) $\subsetneqq$ $\Leftarrow$ Morita trace $\overset{\text{refinement}}{\rightsquigarrow}$ Enomoto-Satoh trace.

$\fallingdotseq$ the Alekseev-Torossian divergence cocycle in the Kashiwara-Vergne problem.

R. Hain : "Infinitesimal presentations of the Torelli groups" J. Amer. Math. Soc., 10 (1997) 597-651

in particular, it is generated by $\mathrm{Im}\,\tau_1 = \Lambda^3 H_{\mathbb{Q}}$

Open Problem : Find a complete list of "defining equations" of $\mathrm{Im}(\tau \otimes_{\mathbb{Z}} \mathbb{Q})$ in $\mathrm{Der}(\mathcal{L}(H_{\mathbb{Q}}))$

$\Downarrow$

Morita trace, Enomoto-Satoh trace

explicit computations of $(\mathrm{Im}\,\tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ up to degree 6

deg 6 : Morita-Sakasai-Suzuki. Adv. Math. 282 (2015), p. 297

## §2. The Morita-Penner cocycle

--- an explicit cocycle $j$ on the dual fatgraph complex ② $\hat{\mathcal{G}}_T$ of $\Sigma_g^1$

representing the extended $1^{st}$ Johnson homomorphism ① $\tilde{k} \in H^1(\mathcal{M}_g^1; \frac{1}{2}\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})$

① the extended $1^{st}$ Johnson homomorphism (Morita)

$$
\begin{tikzcd}
\mathcal{I}_{g,1} = \mathcal{M}(1) \arrow[r, "\tau_1"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} = \mathcal{M}(1)/\mathcal{M}(2) \arrow[dd, phantom, "\cap" rotate=90] \\
\mathcal{I}_{g,*} = \mathrm{Ker}(\mathcal{M}_{g,*} \to Sp_{2g}(\mathbb{Z})) \arrow[ur] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \\
\mathcal{M}_{g,*} \arrow[r, "\text{Morita}"] & \frac{1}{2}\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}
\end{tikzcd}
$$

$\exists \tilde{k}$ : 1-cocycle (not homom ; $\mathcal{M}_{g,*}^{abel} = \mathcal{M}_{g,1}^{abel} = 0$ if $g \geq 3$)

unique up to coboundary

**Theorem** (Morita, Invent. math. 111 (1993)) $g \geq 3$

$$H^1(\mathcal{M}_{g,*} ; \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}) = \mathbb{Z}(2\tilde{k}) \oplus \mathbb{Z}(\omega_0 \wedge k) \cong \mathbb{Z}^2$$

where $\omega_0 \in \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}}$ symplectic form

$k := C \circ (2\tilde{k}) : \mathcal{M}_{g,*} \to H_{\mathbb{Z}}$ Earle class (will be explained in §4)

$C : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \to H_{\mathbb{Z}}$, $x \wedge y \wedge z \mapsto (x\cdot y)z + (y \cdot z)x + (z \cdot x)y$ intersection number

<u>Remark</u> $H^1(\mathcal{M}_{g,*} : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}) = H^1(\mathcal{M}_{g,1} : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})$

$H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}}) = H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$

(∵) $0 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1$ (central extension)

$\quad 1 \mapsto \begin{pmatrix}\text{Dehn twist} \\ \text{along } \partial\Sigma_{g,1}\end{pmatrix}$ , $(\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})^{\mathcal{M}_g} = (H_{\mathbb{Z}})^{\mathcal{M}_g} = 0$ //

<u>/$\mathbb{Q}$</u> $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q} = H_1(\Sigma_g : \mathbb{Q})$

$\tilde{k}_* : (\Lambda^*_{\mathbb{Q}}(\Lambda^3 H_{\mathbb{Q}}))^{Sp_{2g}(\mathbb{Q})} \longrightarrow H^*(\mathcal{M}_{g,*} : \mathbb{Q})$ algebra homom

$\alpha \in (\Lambda^m_{\mathbb{Q}}(\Lambda^3 H_{\mathbb{Q}}))^{Sp_{2g}(\mathbb{Q})} \longmapsto \alpha_*(\tilde{k}^m)$

$\mathbb{C}_g \ni [X, p]$

$\downarrow \qquad \mapsto$

$\mathbb{M}_g \ni [X]$

universal family of compact Riemann surfaces of genus $g$

<u>Morita</u> Image $\tilde{k}_* \supset \mathbb{Q}[e, e_i : i \geq 1]$ (deg 2 : <u>Morita's recipe</u> for $e$ and $e_1$)

where

$e := \mathrm{Euler}(\mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*}) = c_1(T\mathbb{C}_g/\mathbb{M}_g) \in H^2(\mathcal{M}_{g,*} : \mathbb{Q})$

$e_i := \int_{fiber} e^{i+1} = (-1)^{i+1} \kappa_i \in H^{2i}(\mathcal{M}_g : \mathbb{Q})$ the $i^{th}$ Mumford-Morita-Miller class $i \geq 1$.

<u>Morita-K.</u> Image $\hat{k}_* = \mathbb{Q}[e, e_i : i \geq 1]$ even in the <u>unstable</u> range

<u>in theory,</u>

explicit description of $\tilde{k} \overset{\text{induces}}{\Longrightarrow}$ explicit description of $e_i$'s.

but too complicated to realize ... ??

② <u>the dual fatgraph complex</u> $\widehat{\mathcal{G}}_T$ (Penner)

( original for $\Sigma'_g$, but we consider $\Sigma_{g,1}$ in order to explain later developments )

$\widehat{\mathcal{G}}_T$ = the CW complex dual to a canonical ideal simplicial decomposition of the Teichmüller space $\mathcal{T}_{g,1}$ for $\Sigma_{g,1}$.

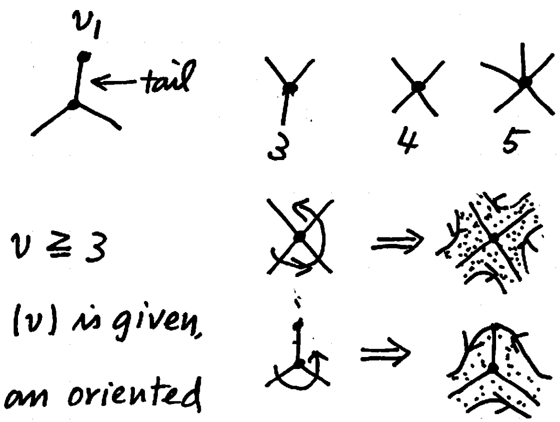
$\widehat{\mathcal{G}}_T \simeq *$, $\quad \mathcal{M}_{g,1} \curvearrowright \widehat{\mathcal{G}}_T$.

$\widehat{\mathcal{G}}_M := \widehat{\mathcal{G}}_T / \mathcal{M}_{g,1} \simeq T\mathbb{C}_g/\mathbf{M}_g \setminus (0\text{-section}) \simeq K(\mathcal{M}_{g,1}, 1)$

each cell in $\widehat{\mathcal{G}}_T \longleftrightarrow (G, [f : \Sigma(G) \to \Sigma_{g,1}])$ marked bordered fatgraph

<u>Definition</u>. $G$ : bordered fatgraph (for $\Sigma_{g,1}$)

$\overset{\text{def}}{\Longleftrightarrow}$
- (0) $G$ : a finite graph.
- (1) $\exists!\, v_1 \in \mathrm{Vertex}(G)$, $\deg v_1 = 1$
  $\forall\, v \in V^{int}(G) := \mathrm{Vertex}(G) \setminus \{v_1\}$ $\quad \deg v \geqq 3$
- (2) $\forall v \in V^{int}(G)$, a cyclic order on Half-edge $(v)$ is given.
- (3) Flattening $G$ by the cyclic orders, we get an oriented surface $\Sigma(G)$ which is diffeomorphic to $\Sigma_{g,1}$




- $[f : \Sigma(G) \to \Sigma_{g,1}]$ : marking

$\overset{\text{def}}{=}$ [ isotopy class of an orientation-preserving diffeomorphism $f : \Sigma(G) \overset{\cong}{\Rightarrow} \Sigma_{g,1}$ ]

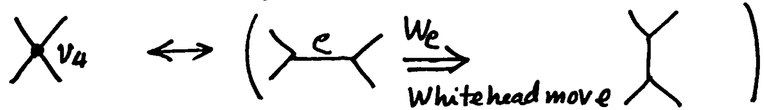
<u>mapping class group action</u>

$\varphi \in \mathcal{M}_{g,1}$ : $[f : \Sigma|G| \to \Sigma_{g,1}] \longmapsto [\varphi \circ f : \Sigma|G| \to \Sigma_{g,1}]$

$(G, [f : \Sigma|G| \to \Sigma_{g,1}])$ corresponds to

a 0-cell in $\hat{\mathcal{G}}_T \iff {}^\forall v \in V^{int}(G)$ $\deg v = 3$

a 1-cell in $\hat{\mathcal{G}}_T \iff {}^\exists v_4 \in V^{int}(G)$ $\deg v_4 = 4$, ${}^\forall v \in V^{int}(G) \setminus \{v_4\}$ $\deg v = 3$



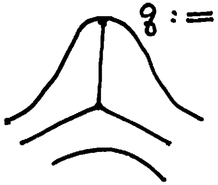

a 2-cell in $\hat{\mathcal{G}}_T \iff$

- ${}^\exists v_4' \neq {}^\exists v_4'' \in V^{int}(G)$
  $\deg v_4' = \deg v_4'' = 4$, ${}^\forall v \in V^{int}(G) \setminus \{v_4', v_4''\}$ $\deg v = 3$
  (commutativity relation)

$\boxed{\text{OR}}$

- ${}^\exists v_5 \in V^{int}(G)$, $\deg v_5 = 5$, ${}^\forall v \in V^{int}(G) \setminus \{v_5\}$, $\deg v = 3$
  (pentagon relation)

<u>two basepoints</u>



$q := f(v_1) \in \partial \Sigma_{g,1}$

$\Downarrow$

$(f(G) \cap \partial \Sigma_{g,1} = \{q\})$

Take another basepoint

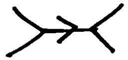
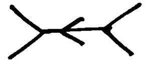
$p \in \partial \Sigma_{g,1} \setminus \{q\}$

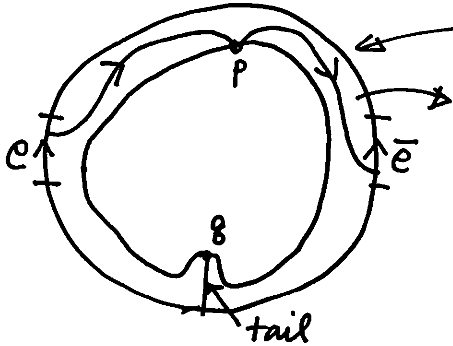
( Identify $\Sigma(G)$ and $\Sigma_{g,1}$ through $f : \Sigma(G) \xrightarrow{\cong} \Sigma_{g,1}$ )

$\pi$ - <u>marking</u> $\quad \pi = \pi_1(\Sigma_{g,1}, p)$

$\mathcal{E}_{or}(G) := \{$ oriented edges of $G \}$

$\Psi$

$e$  , $\bar{e}$  , $\bar{\bar{e}} = e$




cut and open $\Sigma(G)$ along the edges except the tail

$\to \gamma_e \in \pi_1(\Sigma_{g,1}, p) = \pi$

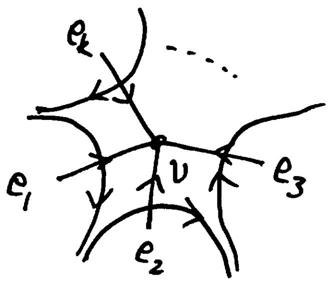
We obtain a canonical map

$$\gamma : \mathcal{E}_{or}(G) \longrightarrow \pi \quad : \pi\text{-marking}$$
$$e \longmapsto \gamma_e$$

<u>Penner's presentation of</u> $\pi$

<u>generators</u> : $\gamma_e$ , $e \in \mathcal{E}_{or}(G)$

<u>relations</u> : (i) $\gamma_{\bar{e}}\, \gamma_e = 1$

(ii) $\gamma_{e_1} \gamma_{e_2} \cdots \gamma_{e_k} = 1$ for  $\forall v \in V^{int}(G)$



$\xrightarrow{\hspace{2em}}$ read

<u>homology-marking</u> $\quad \mathrm{abel}\circ\gamma : \mathcal{E}_{or}(G) \xrightarrow{\gamma} \pi \xrightarrow{\text{abelianization}} \pi^{abel} = H_{\mathbb{Z}}$

$$\begin{array}{ccccc} \mathcal{E}_{or}(G) & \xrightarrow{\gamma} & \pi & \xrightarrow{\text{abelianization}} & \pi^{abel} = H_{\mathbb{Z}} \\ \Psi & & \Psi & & \Psi \\ e & \longmapsto & \gamma_e & \longmapsto & [\gamma_e] =: e \end{array}$$ by abuse of notation

can be defined on the Torelli groupoid $\widehat{\mathcal{G}}_I := \widehat{\mathcal{G}}_T / \mathcal{I}_{g,1}$


$e = c+d = -a-b$

$f = a+d = -b-c$

<u>Morita-Penner cocycle</u> $\quad j \in C^1(\widehat{\mathcal{G}}_T : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})^{\mathcal{M}_{g,1}} = C^1(\widehat{\mathcal{G}}_I : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})^{Sp_{2g}(\mathbb{Z})}$

$$j(W_e) \overset{\text{def}}{=} a\wedge b\wedge c \in \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}$$

<u>Remark</u> $\quad a\wedge b\wedge c = c\wedge d\wedge a$

$\Big( \because)\ d = e-c = -a-b-c$

$\quad c\wedge d\wedge a = -c\wedge(a+b+c)\wedge a = -c\wedge b\wedge a = a\wedge b\wedge c \ /\!/ \Big)$

<u>Theorem</u> (Morita-Penner, Math. Proc. Camb. Phil. Soc <u>144</u> (2008))

(1) $j$ is a cocycle i.e. $j \in Z^1(\widehat{\mathcal{G}}_T : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})^{\mathcal{M}_{g,*}}$

(2) $[j] = 6\tilde{k} \in H^1(\mathcal{M}_{g,*} : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}) = H^1(\mathcal{M}_{g,1} : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})$

<u>Pf</u> : explicit computations

(1) $j$ vanishes on both of $\left\{ \begin{array}{l} \text{- commutativity relations} \\ \text{- pentagon relations} \end{array} \right.$

(2) uses computations of $H_1(\mathcal{I}_{g,*} : \mathbb{Z})$ (Johnson) and $H^1(\mathcal{M}_{g,*} : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})$ (Morita) $/\!/$

Using $j$ and Morita's recipe, one can obtain 2-cocycles on $\hat{\mathfrak{g}}_T / \mathcal{M}_{g,*}$ (for $\Sigma_g^1$)

representing $\quad e = c_1(T\mathcal{C}_g/\mathcal{M}_g) \in H^1(\mathcal{M}_{g,*} : \mathbb{Q})$ and

$\qquad\qquad e_1 = \kappa_1 = [\text{Weil-Petersson Kähler form}] \in H^1(\mathcal{M}_{g,*} : \mathbb{R})$

<u>Open Problem</u>: Describe them explicitly !!

# §3. fatgraph Magnus expansions

Questions 1) intrinsic construction of the Morita-Penner cocycle ?

2) generalization to higher degree Johnson homomorphisms ?

$\Uparrow$

another description of the lower central series $\{\Gamma_k \pi\}_{k=1}^{\infty}$ of $\pi = \pi_1(\Sigma_{g,1}, *)$, a free group.

$\mathbb{K}$ : field of characteristic $0$

$\mathbb{K}\pi := \{ \sum_{\gamma \in \pi} a_\gamma \gamma \ : a_\gamma \in \mathbb{K},\ a_\gamma = 0$ except for finite $\gamma$'s $\}$ group ring of $\pi$

$I\pi := \mathrm{Ker}(\mathrm{aug} : \mathbb{K}\pi \to \mathbb{K},\ \sum a_\gamma \gamma \mapsto \sum a_\gamma)$ augmentation ideal.

Magnus $\forall k \geq 1$, $\Gamma_k \pi = \{ \gamma \in \pi ;\ \gamma - 1 \in (I\pi)^k \}$

$$\widehat{gr}_{(I\pi)^\bullet}(\mathbb{K}\pi) \overset{\mathrm{def}}{=} \prod_{k=0}^{\infty} (I\pi)^k/(I\pi)^{k+1} \cong \prod_{k=0}^{\infty} H_{\mathbb{K}}^{\otimes k} =: \widehat{T}(H_{\mathbb{K}})$$ completed tensor algebra

where $H_{\mathbb{K}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K} = H_1(\Sigma ; \mathbb{K})$

Definition $\theta : \pi \to \widehat{T}(H_{\mathbb{K}})$ generalized Magnus expansion.

$\overset{\mathrm{def}}{\Longleftrightarrow}$
1) $\forall \gamma, \forall \delta \in \pi \quad \theta(\gamma\delta) = \theta(\gamma)\theta(\delta)$
2) $\forall \gamma \in \pi, \quad \theta(\gamma) \equiv 1 + [\gamma] \mod \prod_{k \geq 2} H_{\mathbb{K}}^{\otimes 2}$

2) $\Rightarrow$ $\widehat{gr}(\theta) = 1_{\widehat{T}(H_{\mathbb{K}})}$

$$\theta : \widehat{\mathbb{K}\pi} := \varprojlim_{p \to \infty} \mathbb{K}\pi/(I\pi)^p \xrightarrow{\cong} \widehat{T}(H_{\mathbb{K}})$$ algebra isom.

$$\sum a_\gamma \gamma \mapsto \sum a_\gamma \theta(\gamma)$$

$\forall \varphi \in \mathcal{M}_{g,1}$
$$\begin{tikzcd} \widehat{\mathbb{K}\pi} \arrow[r, "\theta", "\cong"'] \arrow[d, "\varphi"', "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \widehat{T}(H_{\mathbb{K}}) \arrow[d, "\exists ! T^\theta(\varphi)\ :\ \text{alg. isom}"] \\ \widehat{\mathbb{K}\pi} \arrow[r, "\theta", "\cong"'] & \widehat{T}(H_{\mathbb{K}}) \end{tikzcd}$$

$|\varphi| : H_{\mathbb{K}} \circlearrowleft$ induces $|\varphi| : \widehat{T}(H_{\mathbb{K}}) \circlearrowleft$

$$\tau^\theta : \mathcal{M}_{g,1} \longrightarrow \operatorname{Hom}\Big(H_{\mathbb{K}}, \prod_{k \geq 1} H_{\mathbb{K}}^{\otimes(k+1)}\Big)$$
$$\varphi \mapsto \tau^\theta(\varphi) := T^\theta(\varphi) \circ |\varphi|^{-1}|_{H_{\mathbb{K}}}$$
$$\operatorname{Hom}\Big(H_{\mathbb{K}}, \prod_{k \geq 1} H_{\mathbb{K}}^{\otimes(k+1)}\Big) = \prod_{k=1}^{\infty} \operatorname{Hom}(H_{\mathbb{K}}, H_{\mathbb{K}}^{\otimes(k+1)}) \ni (\tau_k^\theta(\varphi))$$

<u>Proposition</u> (classical Magnus expansion: Kitano, generalized: K.)
$$\tau_k^\theta|_{\mathcal{M}(k)} = \tau_k \qquad (\forall k \geq 1)$$
$\Leftarrow$ Magnus's Thm stated above

<u>fatgraph Magnus expansions</u> (Bene – K. – Penner, Adv. Math. <u>221</u> (2009))

generalized Magnus expansion canonically associated with each 0-cell in $\widehat{\mathcal{G}}_T$ (trivalent marked bordered fatgraph)

$\Longleftarrow$ $\begin{cases} \text{BCH series (Baker – Campbell – Hausdorff series)} \\ \text{idea of iterated integrals} \end{cases}$

## BCH series

$\widehat{\mathcal{L}} := \prod_{k=1}^{\infty} \mathcal{L}_k(H_{\mathbb{K}}) \subset \prod_{k=1}^{\infty} H_{\mathbb{K}}^{\otimes k} \subset \widehat{T}(H_{\mathbb{K}})$ completed free Lie algebra

$\exp(\widehat{\mathcal{L}}) \subset$ multiplicative group $(\widehat{T}(H_{\mathbb{K}}))$ subgroup

(where
$$\exp(u) = \sum_{n=0}^{\infty} \frac{1}{n!} u^n \in \widehat{T}(H_{\mathbb{K}})$$
$$\log(1+w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} w^n \in \widehat{T}(H_{\mathbb{K}})$$
$u, w \in \prod_{k\geq 1} H_{\mathbb{K}}^{\otimes k}$

$$u * v = \mathrm{bch}(u,v) \overset{\mathrm{def}}{=} \log(\exp(u)\exp(v)) \quad (u, v \in \widehat{\mathcal{L}})$$
$$\in \widehat{\mathcal{L}} \quad (\text{BCH series})$$

$$h(u,v) \overset{\mathrm{def}}{=} u*v-u-v = \frac{1}{2}[u,v] + \frac{1}{12}([u,[u,v]]+[v,[v,u]]) + \cdots$$
$$= \sum_{n=1}^{\infty} h(u,v)_{(n)}, \quad h(u,v)_{(n)} \in \mathcal{L}_n(H_{\mathbb{K}}), \quad \left(\text{e.g., } h(u,v)_{(2)} = \frac{1}{2}[u,v]\right)$$

## Iterated integrals

$(G, [f : \Sigma(G) \to \Sigma_{g,1}])$ : 0-cell in $\widehat{\mathcal{G}}_T$ i.e., $\forall v \in V^{int}(G)$, $\deg v = 3$

Construct $\theta = \theta^G : \pi_1(\Sigma(G), v_1) (\cong \pi) \longrightarrow \widehat{T}(H_{\mathbb{K}})$, or equivalently,

$$l = \sum_{m=1}^{\infty} l_m := \log \theta : \mathcal{E}_{or}(G) \to \widehat{\mathcal{L}} = \prod_{m=1}^{\infty} \mathcal{L}_m(H_{\mathbb{K}})$$

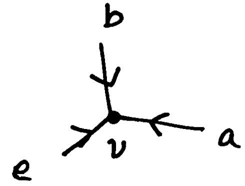
by induction on $m$ (= degree).

<u>Condition 2)</u>: $\ell_1(e) := [\gamma_e] = e \in H_{\mathbb{K}}$ $\quad \forall e \in \mathcal{E}_{or}(G)$ .

<u>Recall</u> Penner's defining relations of $\pi$

(i) $\gamma_{\bar{e}}\gamma_e = 1$

(ii) $\gamma_e \gamma_a \gamma_b = 1$




$\forall\, v \in V^{int}(G)$

(i) $\Leftrightarrow$ $\ell(\bar{e}) = -\ell(e)$

Suffices to construct $g = \sum_{m=1}^{\infty} g_m : \mathcal{E}_{or}(G) \to \hat{\mathcal{L}} = \prod_{m=1}^{\infty} \mathcal{L}_m(H_{\mathbb{K}})$

such that $\ell(e) = g(e) - g(\bar{e})$

(ii) $\Leftrightarrow$ $\theta(\bar{e}) = \theta(a)\theta(b)$

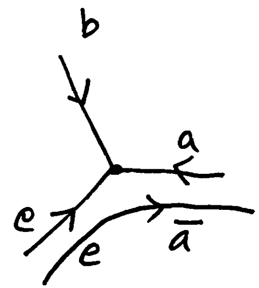
$\Leftrightarrow$ $-\ell(e) = \ell(a) + \ell(b) + h(\ell(a), \ell(b))$

$\Rightarrow$ $h(\ell(a), -\ell(\bar{b})) + h(\ell(b), -\ell(\bar{e})) + h(\ell(e), -\ell(\bar{a}))$

$= -3(\ell(a) + \ell(b) + \ell(e))$

$= -3(g(a) - g(\bar{b}) + g(b) - g(\bar{e}) + g(e) - g(\bar{a}))$

$\Leftarrow$ $g(\bar{a}) - g(e) = \frac{1}{3} h(\ell(e), -\ell(\bar{a}))$



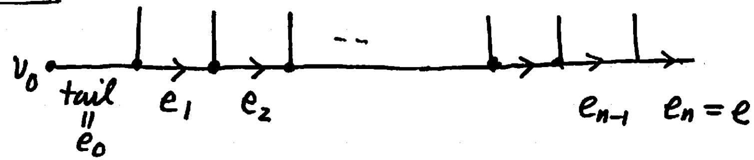
One can solve the equation by induction on $m$

$$g_m(e) \overset{\text{def}}{=} \frac{1}{3}\sum_{i=1}^{n} h(\ell(e_{i-1}), -\ell(e_i))_{(m)}$$

depends only on $\ell_1, \cdots, \ell_{m-1}$

- - - - - - - - iterated integrals

<u>where</u>

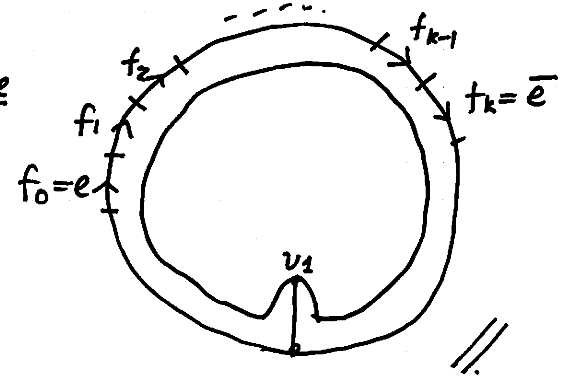



$\leftarrow$ boundary cycle

Hence $\quad \ell_m(e) = g_m(e) - g_m(\bar{e})$

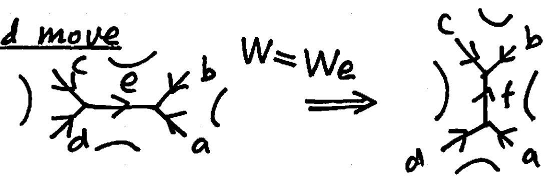
$$= -\frac{1}{3}\sum_{i=1}^{k} h(\ell(f_{i-1}), -\ell(f_i))_{(m)}$$

<u>where</u>




which satisfies the defining relations of $\pi$.

<u>Whitehead move</u>

$G = $  $= G'$


$$\tau^W := \theta^G \circ (\theta^{G'})^{-1} : \hat{T}(H_{\mathbb{K}}) \overset{\cong}{\Rightarrow} \widehat{\mathbb{K}\pi} \overset{\cong}{\Rightarrow} \hat{T}(H_{\mathbb{K}})$$

$$\tau^W|_H = 1_H + \sum_{m=1}^{\infty} \tau_m^W, \quad \tau_m^W \in \mathrm{Hom}(H, H^{\otimes(m+1)}) \overset{\text{Poincaré dual}}{=} H^{\otimes(m+2)}$$

<u>Theorem</u> (Bene - K. - Penner)

$$\tau_m^W = \frac{1}{3}\left[ a\, h(\ell(a), \ell(c)) + b\, h(\ell(b), \ell(c)) - b\, h(\ell(c), \ell(d)) + c\, h(\ell(a), \ell(b)) \right]_{(m+2)}$$

degree $m+2$ part

in particular, $\tau_1^W = \frac{1}{6}\, a \wedge b \wedge c = \frac{1}{6} j(W)$ (Morita-Penner cocycle) ($\because$ $h(\ell(b), \ell(c))_{(2)} = \frac{1}{2}[b, c]$ )

( <u>Remark</u> fatgraph Magnus expansions are <u>not</u> symplectic in the sense of Massuyeau )

## §4. The Earle class $k \in H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}}) = H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$

C. J. Earle, Ann. of Math. <u>107</u> (1978) 255–286

( <u>see also</u> Kuno, Math Proc. Camb. Phil. Soc. <u>140</u> (2009) 109–118 )

$g \geq 2$. Fix. $p_0 \in \Sigma_g$ and $\{A_i, B_i\}_{i=1}^{g} \subset \pi_1(\Sigma_g, p_0)$ symplectic generators

$\mathcal{T}_g^1$ = the Teichmüller space of once-punctured Riemann surfaces of genus $g$

$\ni [X, p, f]$. $X$ : compact Riemann surface of genus $g$, $p \in X$

$f : (X, p) \to (\Sigma_g, p_0)$ orientation-preserving diffeomorphism

$\{\omega_j\}_{j=1}^{g} \subset H^0(X : K_X)$ $\quad \int_{f_*^{-1} A_i} \omega_j = \delta_{ij}$ $\quad K_X$ : canonical bundle of $X$, $(T^*X)$

$\tau(X, f) = \tau = (\tau_{ij})_{i,j=1}^{g}$, $\quad \tau_{ij} \overset{\text{def}}{=} \int_{f_*^{-1} B_i} \omega_j$ $\quad$ period matrix

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_g \end{pmatrix} : \mathcal{T}_g^1 \to \mathbb{C}^g \qquad \eta_j([X, p, f]) \overset{\text{def}}{=} \frac{1}{1-g}\left( -\frac{1}{2}\tau_{jj} + \sum_{k=1}^{g} \int_{f_*^{-1} A_k} \omega_j \omega_k \right)$$

Chen's iterated integral

$\varphi \in \mathcal{M}_{g,*}$. $\varphi_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $H_{\mathbb{Z}}$ w.r.to $\{[A_i], [B_i]\}$, $a, b, c, d \in M_g(\mathbb{Z})$

$$w_\varphi([X, p, f]) \overset{\text{def}}{=} (a + \tau c)\,\eta(\varphi[X, p, f]) - \eta([X, p, f]) \in \mathbb{C}^g$$

<u>Theorem</u> (Earle) $\exists!\, \psi \in Z^1(\mathcal{M}_{g,*} : \frac{1}{2g-2} H_{\mathbb{Z}})$

s.t. $\forall \varphi \in \mathcal{M}_{g,*}$ $\quad w_\varphi([X, p, f]) = (I \ \ \tau)\,\psi(\varphi)$

moreover $\quad \psi|_{\pi_1(\Sigma_g, p_0)}$ = abelianization : $\pi_1(\Sigma_g, p_0) \to H_{\mathbb{Z}}$

$k := [(2-2g)\psi] \in H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}})$ Earle class

$(1-g)\eta$ = the vector of Riemann constants w.r.to $p$ and $\{f_*^{-1}A_i, f_*^{-1}B_i\}$

In particular, $(2g-2)\eta \bmod (\mathbb{Z}^g + \tau\mathbb{Z}^g)$ = Abel-Jacobi $(K_X - (2g-2)p) \in \mathrm{Jac}(X)$

<u>heuristically</u> on $\mathbb{C}_g = \mathcal{T}_g' / \mathcal{M}_{g,*} \ni [X, p]$ ( or $(E\mathcal{M}_{g*} \times \mathcal{T}_g')/\mathcal{M}_{g,*}$ )

$$\begin{array}{ccccc} \mathrm{Pic}^0(X) = H^1(X:\mathcal{O})/H^1(X:\mathbb{Z}) & \xrightarrow[\cong]{\text{Real part}} & H^1(X:\mathbb{R}/\mathbb{Z}) \\ \Psi & & \Psi \\ (K_X - (2g-2)p_0) & \longmapsto & K \end{array}$$

$$\begin{array}{ccc} H^0(\mathbb{C}_g : \coprod_{[X]} H^1(X:\mathbb{R}/\mathbb{Z})) & \underset{\text{"Bockstein"}}{\xrightarrow{\delta^*}} & H^1(\mathbb{C}_g : \coprod_{[X]} H^1(X:\mathbb{Z})) = H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}}) \\ \Psi & & \Psi \\ -K & \longmapsto & k \end{array}$$ Earle class

**Theorem** (Morita (1)(2): Ann. Inst. Fourier <u>39</u> (1989) (3) Invent. math <u>111</u> (1993))

$g \geq 2$

(1) $H^1(\mathcal{M}_{g,*} H_{\mathbb{Z}}) = H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}}) \cong \mathbb{Z}$

(2) $H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}}) \xrightarrow{\text{res}} H^1(\pi_1(\Sigma_g, p_0), \mathbb{Z})^{\mathcal{M}_g} \to \mathbb{Z}/_{2g-2} \to 0$

( in particular, $H^1(\mathcal{M}_{g,*} : \mathbb{Z}) = \mathbb{Z}k$ )

(3) $k = C \circ (2\tilde{k}) \in H^1(\mathcal{M}_{g,*} : H_{\mathbb{Z}})$, $C : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \to H_{\mathbb{Z}}$

intersection number

$x \wedge y \wedge z \mapsto (x \cdot y) z + (y \cdot z) x + (z \cdot x) y.$

$\Sigma$ : compact connected oriented surface with $\partial \Sigma \neq \phi$

$\mathcal{F}_r(\Sigma) := \{ f : T\Sigma \xrightarrow[\text{ori. pres}]{\cong} \Sigma \times \mathbb{R}^2 \} / \text{homotopy}$

framing

affine space modelled on $H^1(\Sigma : \mathbb{Z})$ i.e, $\forall f_0, \forall f_1 \in \mathcal{F}_r(\Sigma)$ $\quad f_1 - f_0 \in H^1(\Sigma : \mathbb{Z})$ well-defined.

<u>Observation</u> (Furuta (1997))

$f \in \mathcal{F}_r(\Sigma)$, $\quad k_f : \varphi \in \mathcal{M}_{g,1} \longmapsto f\circ\varphi^{-1} - f \in H^1(\Sigma_{g,1} : \mathbb{Z}) = H_{\mathbb{Z}}$ is a cocycle, and

$[k_f] = k \in H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$

- $k|_{\mathcal{I}_{g,1}}$ is already discovered by Chillingworth (Math. Ann. <u>196</u> (1972), <u>199</u> (1972))
  Chillingworth homomorphism

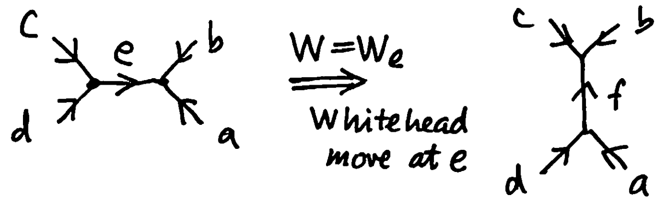
- $k$ = the $1^{st}$ Morita trace = the $1^{st}$ Enomoto-Satoh trace
  $\rightsquigarrow$ a relation between a framing and the Kashiwara-Vergne problem

- If $g \geqq 2$ or $g = 0$, then $\mathcal{F}_r(\Sigma_{g,n+1})/\mathcal{M}_{g,n+1}$ is described by $\partial$-data and associated spin structure
  (ess. due to Johnson, See also Randall-Williams J. Top. <u>7</u> (2014)).

  If $g = 1$, then $\mathcal{F}_r(\Sigma_{1,n+1})/\mathcal{M}_{1,n+1}$ is described by $\partial$-data and g.c.d {rotation # of non-sep. SCC's} (K.)

## §5. The Kuno-Penner-Turaev cocycle

$$m \in Z^1(\hat{\mathfrak{g}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}} \quad \text{representing} \quad 6k \in H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$$




$a+b+c+d=0$

<u>Recall</u>: <u>Morita-Penner cocycle</u>

$$j(W) = a \wedge b \wedge c \in \Lambda^3 H_{\mathbb{Z}}$$

$C \circ j \in Z^1(\hat{\mathfrak{g}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}}$ $\quad C : \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \to H_{\mathbb{Z}}$, $\; x \wedge y \wedge z \mapsto (x \cdot y) z + (y \cdot z) x + (z \cdot x) y$

$[C \circ j] = 3k \in H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$, $\quad (C \circ j)(W) = (a \cdot b) c + (b \cdot c) a + (c \cdot a) b$

<u>Kuno-Penner-Turaev cocycle</u> (Geom. Dedicata <u>167</u> (2013))

$$m(W) \overset{\text{def}}{=} a + c = -b - d \in H_{\mathbb{Z}}$$

<u>Theorem</u> (Kuno-Penner-Turaev)

(1) $m$ is a cocycle i.e., $m \in Z^1(\hat{\mathfrak{g}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}}$

(2) $[m] = 6k \in H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$

( (Pf) explicit computation similar to Morita-Penner )

<u>Kuno's secondary invariant</u> $\quad \xi \in C^0(\hat{\mathfrak{g}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}} \quad$ (Kuno, Alg. Geom. Top. 17 (2017))

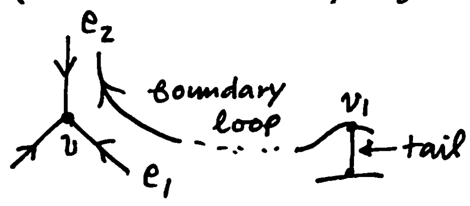
$[2(C\circ j) - m] = 6k - 6k = 0 \in H^1(\mathcal{M}_{g,1} : H_{\mathbb{Z}})$

1-coboundary ⟸ explicit description

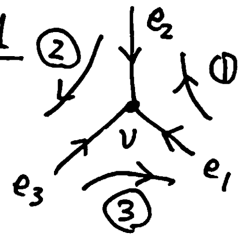
$G$ : 0-cell in $\hat{\mathfrak{g}}_T$ . i.e., $\forall v \in V^{int}(G)$ $(= \mathrm{Vertex}(G) \setminus \{v_1\})$ $\deg v = 3$

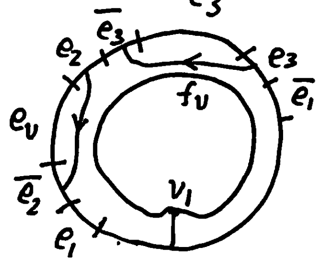
$V^{int}(G) = (\text{type } 1) \amalg (\text{type } 2)$

$$v \longmapsto \begin{cases} e_v \\ f_v \end{cases} \in H_{\mathbb{Z}}$$



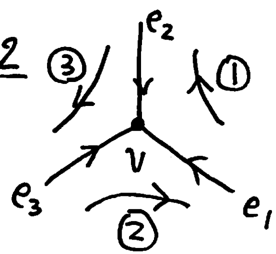

<u>type 1</u>

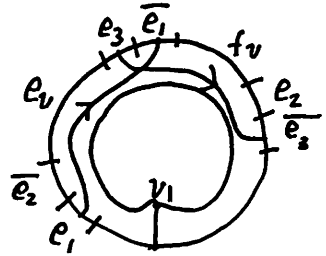



$$\begin{cases} e_v := e_2 \\ f_v := e_3 \end{cases} \in H_{\mathbb{Z}}$$

<u>type 2</u>




$$\begin{cases} e_v := e_1 \\ f_v := e_3 \end{cases} \in H_{\mathbb{Z}}$$

$$\xi_G \overset{\text{def}}{=} \sum_{v \in V^{int}(G)} e_v - f_v \in H_{\mathbb{Z}} \qquad \text{Kuno's secondary invariant}$$

**Theorem** (Kuno, ibid)

(1) $2(C\circ j) - m = \delta\xi \in C^1(\hat{\mathcal{G}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}}$

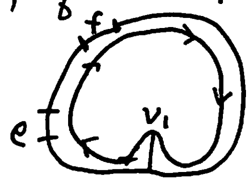
(2) If $\xi' \in C^0(\hat{\mathcal{G}}_T : H_{\mathbb{Z}})^{\mathcal{M}_{g,1}}$ satisfies $\delta\xi' = 2(C\circ j) - m$,
then $\xi' = \xi$

(3) $\forall G$ : 0-cell in $\hat{\mathcal{G}}_T$, $\xi_G \bmod 2 \neq 0 \in H_1(\Sigma_{g,1} : \mathbb{Z}/2) \overset{P.d}{=} H^1(\Sigma_{g,1} : \mathbb{Z}/2)$

Kuno's canonical pair of spin structures on $\Sigma(G)$

$q_G$ and $q_G'$ : $H_1(\Sigma(G) : \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$, quadratic forms

linear order on $\mathcal{E}_{or}(G) \ni e, f$. $\quad e < f \overset{def}{\Longleftrightarrow}$




$\mathrm{Edge}(G) \xrightarrow[\sigma]{\text{section}} \mathcal{E}_{or}(G)$

$e \longmapsto \sigma(e)$ such that $\sigma(e) < \overline{\sigma(e)}$

$e \in \mathrm{Edge}(G)$

$q_G(e) := \#\{f \in \mathrm{Edge}(G) ; \sigma(e) < \sigma(f) < \overline{\sigma(e)}\} \bmod 2 \in \mathbb{Z}/2$

$q_G'(e) := \#\{f \in \mathrm{Edge}(G) : \sigma(e) < \overline{\sigma(f)} < \overline{\sigma(e)}\} \bmod 2 \in \mathbb{Z}/2$

**Theorem** (Kuno, ibid)

(1) $q_G$ and $q_G'$ are well-defined spin structures on $H_1(\Sigma(G) : \mathbb{Z}/2)$

(2) $q_G' - q_G = \underset{\neq 0}{\xi_G \bmod 2} \in H^1(\Sigma(G) : \mathbb{Z}/2)$ //