日本数学会秋季総合分科会 トポロジー分科会 特別講演 於 山形大学 2017年9月11日 14:15 − 15:15

「写像類群, Goldman - Turaev Lie 双代数, 柏原 Vergne 問題」

"Mapping class groups, the Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem"

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Contents

- $\Bbbk$: field of characteristic 0

  $\mathfrak{A}$: associative $\Bbbk$-algebra $\Rightarrow$ Lie algebra $\left([a,b] := ab - ba,\ a, b \in \mathfrak{A} \right)$

  $|\mathfrak{A}| := \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ abelianization as a Lie algebra

  where $[\mathfrak{A}, \mathfrak{A}] \subset \mathfrak{A}$: $\Bbbk$-vector subspace generated by $\{[a,b] : a, b \in \mathfrak{A}\}$

  $|\cdot| : \mathfrak{A} \to |\mathfrak{A}|$, $a \mapsto |a|$, quotient map

  e.g. (1) V : finite dimensional $\Bbbk$-vector space, $\hat{T}(V) := \prod_{k=0}^{\infty} V^{\otimes k}$ completed tensor algebra over V

  $|\hat{T}(V)| = \prod_{k=0}^{\infty} (V^{\otimes k})_{\mathbb{Z}/k}$ cyclic coinvariants

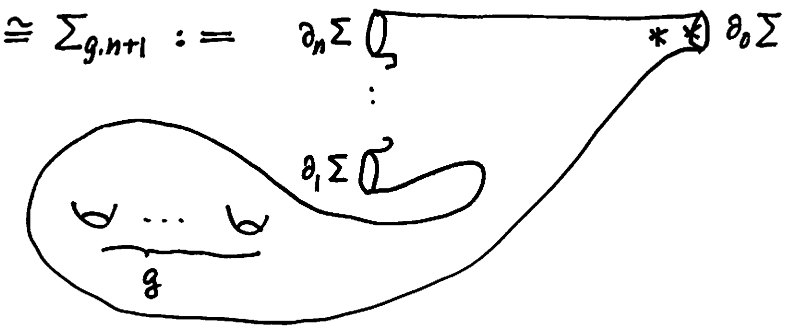
  (2) G : group. $\Bbbk G = \{ \sum_{\gamma \in G} a_\gamma \gamma \ ;\ a_\gamma \in \Bbbk,\ a_\gamma = 0$ for all but finite $\gamma \}$ group ring

  $|\Bbbk G| = \Bbbk(G/\text{conjugate})$

- Σ : compact connected oriented C^∞ surface with $\partial \Sigma \neq \phi$

$\Longrightarrow$ $\exists g, \exists n \geq 0$, $\Sigma \cong \Sigma_{g,n+1} :=$

Classification Theorem



$* \in \partial_0 \Sigma$

Outline of this talk

$\mathcal{M}(\Sigma) := \pi_0 \mathrm{Diff}(\Sigma, \mathrm{id.\ on}\ \partial\Sigma)$ mapping class group

subset $\bigcup$

$\forall$ Dehn twists $\in \mathcal{M}(\Sigma)^{\circ}$ $\xrightarrow[\tau]{\text{Johnson homom.}}$ (completed Goldman-Turaev Lie bialgebra $\widehat{\mathbb{k}\hat{\pi}}$) $\circlearrowleft$ $\xrightarrow[\text{(special/ symplectic expansion)}]{\cong}$ $\prod_{k=0}^{\infty} (H_1(\Sigma : \mathbb{k})^{\otimes k})_{\mathbb{Z}/k}$

($\bigcup$ $\Sigma = \Sigma_{g,1}$ $\mathcal{I}_{g,1}$ Torelli group)

$\bigcup$

(Ker(the Turaev cobracket (framed)) $\overline{\times}$ $H^1(\Sigma : \mathbb{k})$)

(special/symplectic expansion) $\Rightarrow$ (formal description of the Goldman bracket)

$\Downarrow\!\!\!\Uparrow$ $\boxed{\text{solution of the Kashiwara-Vergne problem}}$ $\Leftrightarrow$ (formal description of (the Goldman bracket and the Turaev cobracket (framed)))

$\Downarrow\!\!\!\Uparrow$

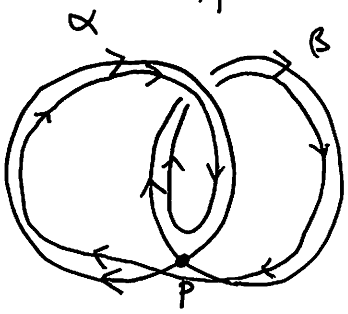
Cor — equivalent: the Enomoto-Satoh trace $\leftarrow$ Thm

I. Goldman bracket

$\hat{\pi} = \hat{\pi}(\Sigma) := [S^1, \Sigma]$ free homotopy classes of free loops on Σ

$= \pi_1(\Sigma)/\text{conjugate}$, $\quad |\cdot| : \pi_1(\Sigma) \to \hat{\pi}(\Sigma)$, $\gamma \mapsto |\gamma|$, $\quad$ quotient map $=$ forgetful map of the basepoint

$\alpha, \beta \in \hat{\pi}$ represented by generic immersions $(\Rightarrow \alpha \cap \beta$: finite, transverse $)$

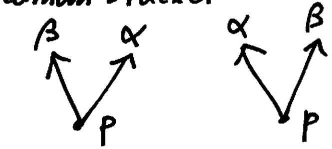


$\varepsilon_p = -1$

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi} \qquad \text{Goldman bracket}$$

where $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ local intersection number

$\alpha_p, \beta_p \in \pi_1(\Sigma, p)$



$\varepsilon_p = +1 \qquad \varepsilon_p = -1$

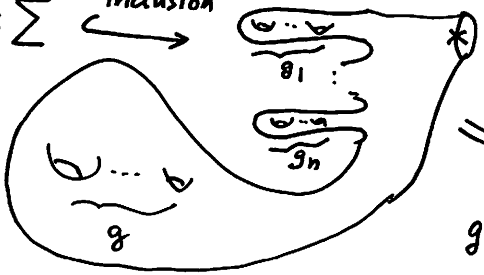
Theorem (Goldman)

(1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$ Lie algebra $\rightsquigarrow$ Goldman Lie algebra of Σ

Completion of $\mathbb{k}\hat{\pi}$, $\quad \pi := \pi_1(\Sigma, *)$, $* \in \partial_0 \Sigma$, $\quad \mathbb{Z}\hat{\pi} < \mathbb{k}\hat{\pi} = |\mathbb{k}\pi|$

$\iota : \Sigma \xrightarrow{\text{inclusion}}$



$* \in \partial_0 \Sigma$

$=: \tilde{\Sigma}$

$g_1, \cdots, g_n \geq 1$

$$\begin{tikzcd} \iota_* : \mathbb{k}\pi \arrow[r, "\text{injective}"] & \mathbb{k}\pi_1(\tilde{\Sigma}, *) \\ F^p(\mathbb{k}\pi) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[ur, phantom, "\urcorner"] & (\mathrm{Ker}(\mathrm{aug}))^p \arrow[u, phantom, "\cup"] \end{tikzcd} \quad , \quad p \geq 0$$

(indep of the choice of $\tilde{\Sigma}$)

$$\left.\begin{aligned} \widehat{\mathbb{k}\pi} &:= \varprojlim_{p\to\infty} \mathbb{k}\pi / F^p(\mathbb{k}\pi) \\ gr(\mathbb{k}\pi) &:= \prod_{p=0}^{\infty} F^p(\mathbb{k}\pi)/F^{p+1}(\mathbb{k}\pi) \end{aligned}\right\} \text{complete Hopf algebras}$$

$$\left.\begin{aligned} \widehat{\mathbb{k}\hat{\pi}} &:= \varprojlim_{p\to\infty} \mathbb{k}\hat{\pi} / |F^p(\mathbb{k}\pi)| \\ gr(\mathbb{k}\hat{\pi}) &:= \prod_{p=0}^{\infty} |F^p(\mathbb{k}\pi)| / |F^{p+1}(\mathbb{k}\pi)| \end{aligned}\right\} \text{complete Lie bialgebras} \Leftarrow \left\{\begin{array}{l}\text{Goldman bracket} \\ \text{(framed) Turaev cobracket}\end{array}\right.$$

<u>Formality Problem</u>: $\widehat{\mathbb{k}\hat{\pi}} \overset{?}{\cong} gr(\mathbb{k}\hat{\pi})$ Lie bialgebra isomorphism ?

<u>Description of $gr(\mathbb{k}\pi)$ and $gr(\mathbb{k}\hat{\pi})$</u>

$H := H_1(\Sigma ; \mathbb{k})$, $z_j := [\partial_j \Sigma] \in H$, $0 \le j \le n$, $z_0 = -\sum_{j=1}^{n} z_j$

$H^{(1)} := H \supset H^{(2)} := \mathbb{k}\langle z_1, \cdots, z_n \rangle$

$\hat{T} = \hat{T}(H) := \prod_{k=0}^{\infty} H^{\otimes k}$

complete Hopf algebra — complete tensor product

$\Delta : \hat{T} \to \hat{T} \hat{\otimes} \hat{T}$ continuous $\mathbb{k}$-algebra homom.

$\forall X \in H$, $\Delta(X) = X \otimes 1 + 1 \otimes X$

$\iota : \hat{T} \to \hat{T}$ continuous $\mathbb{k}$-algebra anti-homom

$\forall X \in H$, $\iota(X) = -X$, $\iota(X_1 X_2 \cdots X_m) = (-1)^m X_m \cdots X_2 X_1$

$|\hat{T}| = \prod_{k=0}^{\infty} (H^{\otimes k})_{\mathbb{Z}/k}$ cyclic coinvariants

$\{F^p(\hat{T})\}_{p=0}^{\infty}$: filtration on $\hat{T}$ induced by $H^{(1)} \supset H^{(2)}$

$$A = A^{(g.n+1)} := \mathrm{gr}(\hat{T}) = \prod_{p\geq 0}^{\infty} F^p(\hat{T})/F^{p+1}(\hat{T}) = \mathrm{gr}(\Bbbk\pi) \cong \hat{T}\left((H^{(1)}/H^{(2)}) \oplus H^{(2)}\right)$$

p. 20, l. 9.

$$F^p A := \prod_{m\geq p} F^m(\hat{T})/F^{m+1}(\hat{T}), \quad p \geq 0$$

A : complete Hopf algebra

$\mathrm{Grp}(A) := \{a \in A \setminus \{0\} ;\ \Delta a = a \otimes a\}$ group-like elements, group.

exp $\uparrow \ \wr\!\wr \ \downarrow$ log

$L = L^{(g.n+1)} := \{u \in A :\ \Delta u = u \otimes 1 + 1 \otimes u\}$ Lie-like elements, Lie algebra

$$\exp(u) := \sum_{k=0}^{\infty} \frac{1}{k!} u^k, \quad \log(a) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (a-1)^k$$

<u>Definition</u>

$\theta : \pi \to \mathrm{Grp}(A)$ group-like expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) θ : group-homomorphism

2) $\theta : \Bbbk\pi \to A$, $\sum a_\gamma \gamma \mapsto \sum a_\gamma \theta(\gamma)$ preserves the filtrations, and

$\mathrm{gr}(\theta) = 1_A : \mathrm{gr}(\Bbbk\pi) = A \to A = \mathrm{gr}(A)$

$\Longrightarrow$ $\theta : \widehat{\Bbbk\pi} \xrightarrow{\cong} A$ isom of complete Hopf algebras

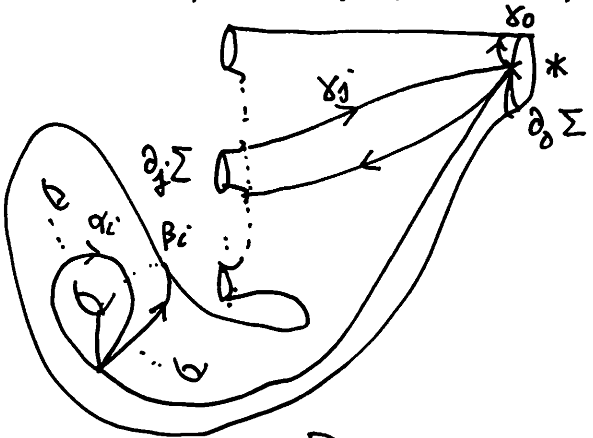
$\theta : |\widehat{\Bbbk\hat{\pi}}| \to |A|$ isom of $\Bbbk$-vector spaces

$gr|\theta| : gr(\mathbb{k}\hat{\pi}) \to gr(|A|) = |A|$. isom. indep of the choice of θ

$\Rightarrow$ $|A|$: complete Lie bialgebra $=$ necklace Lie bialgebra with Schedler's cobracket

$\left(\underset{\overline{\overline{}}}{\text{as Lie algebras}} \left\{ \begin{array}{ll} \text{Kontsevich's associative wor}(l)\text{d} & (\Sigma = \Sigma_{g,1}) \\ \text{special derivation Lie algebra} & (\Sigma = \Sigma_{0,n+1}) \end{array} \right. \right.$

<u>Special / symplectic expansions</u>



$\pi = \pi_1(\Sigma, *) = \langle \alpha_i, \beta_i\ (1 \le i \le g),\ \gamma_j\ (1 \le j \le n) \rangle \cong F_{2g+n}$

free generators

$$\gamma_0^{-1} = \prod_{i=1}^{g} (\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}) \prod_{j=1}^{n} \gamma_j \in \pi$$

$x_i := [\alpha_i]$, $y_i := [\beta_i] \in H$, $1 \le i \le g$

$z_j = [\gamma_j] \in H$, $0 \le j \le n$

$$\omega := \sum_{i=1}^{g} (x_i y_i - y_i x_i) + \sum_{j=1}^{n} z_j \in F^2(A)$$

<u>Definition</u> $\theta : \pi \to Grp(A)$ special / symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) θ : group-like expansion

2) $1 \le {}^\forall j \le n$, ${}^\exists c_j \in Grp(A)$, $\theta(\gamma_j) = c_j^{-1} \exp(z_j) c_j$

] "tangential expansion"

3) $\theta(\gamma_0^{-1}) = \exp(\omega)$

(<u>e.g.</u> $\theta^{exp} : \pi \to Grp$ $\quad \alpha_i \mapsto \exp x_i$, $\beta_i \mapsto \exp y_i$ $(1 \le i \le g)$, $\gamma_j \mapsto \exp z_j$ $(1 \le j \le n)$ tangential but <u>not</u> special / symplectic)

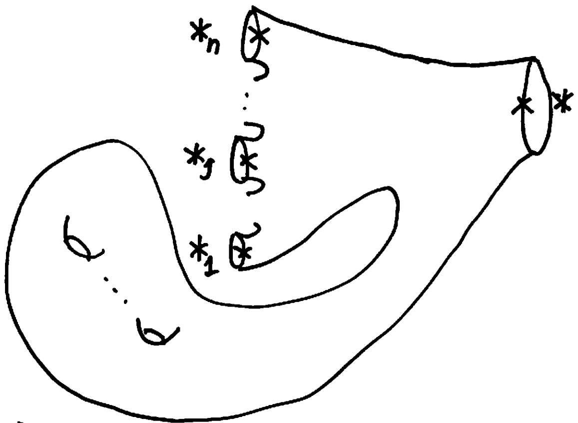
($\Sigma = \Sigma_{0,n+1}$: special expansion, Habegger-Masbaum, Alekseev-Enriquez-Torossian, ...
$\Sigma = \Sigma_{g,1}$: symplectic expansion Massuyeau)

<u>Theorem</u> (($\Sigma = \Sigma_{g,1}$) Kuno-K., (general), Massuyeau-Turaev, Kuno-K.)

$\theta : \pi \to \mathrm{Grp}|A|$. special / symplectic expansion.

$\Rightarrow \quad \theta : \widehat{\Bbbk \hat{\pi}} \xrightarrow{\cong} |A|$, Lie algebra isomorphism. $\rightsquigarrow$ formal description of the Goldman bracket

II. Mapping class groups and the Johnson homomorphism



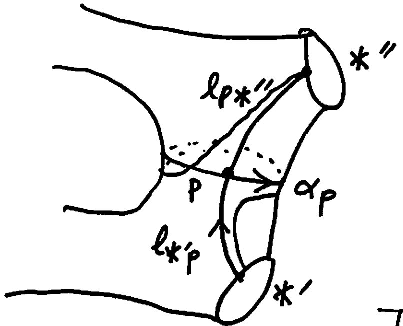
$* \in \partial_0 \Sigma$, $*_j \in \partial_j \Sigma$, $1 \leq j \leq n$. basepoints

$E := \{*\} \cup \{*_j : 1 \leq j \leq n\}$

$\Pi\Sigma|_E$: fundamental groupoid of Σ restricted to E

- **objects**: $*', *'' \in E$
- **morphisms**: $\Pi\Sigma(*', *'') := [([0,1], 0, 1), (\Sigma, *', *'')]$ homotopy classes of paths from $*'$ to $*''$

$\ell \in \Pi\Sigma(*', *'')$, $\alpha \in \hat{\pi}$ represented by generic immersions



$$\sigma(\alpha)(\ell) := \sum_{p \in \alpha \cap \ell} \varepsilon_p(\alpha, \ell)\, \ell_{*'p}\, \alpha_p\, \ell_{p*''} \in \mathbb{Z}\Pi\Sigma(*', *'')$$

$\left(\overset{\times}{\alpha \cup \ell} \Rightarrow \overset{\circ}{\alpha \cap \ell} \right)$ p.21. l.-4.

$\widehat{\Bbbk\Pi\Sigma|_E}$: completion of $\Bbbk\Pi\Sigma|_E$ with respect to $\{F^p \Bbbk\pi\}_{p=0}^{\infty}$

$\mathrm{Der}_\partial(\widehat{\Bbbk\Pi\Sigma|_E}) := \{ D : \text{continuous derivation of } \widehat{\Bbbk\Pi\Sigma|_E} : 0 \leq {}^\forall j \leq n,\ D(\partial_j\Sigma) = 0 \}$

Theorem (Kuno-K.)

$\sigma : \widehat{\Bbbk\hat{\pi}} / \Bbbk 1 \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\Bbbk\Pi\Sigma|_E})$ Lie algebra isomorphism

$\mathcal{M}(\Sigma) = \pi_0 \mathrm{Diff}(\Sigma, \mathrm{id\ on\ } \partial\Sigma)$ the mapping class group

$\cup$ subset

$\mathcal{M}(\Sigma)^0 := \{ \varphi \in \mathcal{M}(\Sigma) \,;\, \log \varphi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\varphi - 1)^k$ converges on $\widehat{\Bbbk\Pi\Sigma}|_E \}$

$\ni$ ${}^{\forall}$Dehn twists $\qquad \cup_{\Sigma = \Sigma_{g,1}}$ $\mathcal{I}_{g,1}$ Torelli group

$\tau := \sigma^{-1} \circ \log : \mathcal{M}(\Sigma)^0 \to \mathrm{Der}_\partial(\widehat{\Bbbk\Pi\Sigma}|_E) \xrightarrow{\cong} \widehat{\Bbbk\pi}/\Bbbk 1$

geometric version of the Johnson homomorphism

($(\Sigma = \Sigma_{g,1})$ $\mathrm{gr}(\tau) : \mathrm{gr}(\mathcal{I}_{g,1}) \to \mathrm{gr}(\widehat{\Bbbk\pi}/\Bbbk 1)$ the classical Johnson homomorphism

constraints of the image of $\mathrm{gr}(\tau)$

$\cap$ # Morita trace
Enomoto - Satoh trace) 1^{st} term is related to a framing of $\Sigma_{g,1}$ (Furuta))

<u>Theorem</u> (Kuno-K.) $C \subset \Sigma$ simple closed curve. $C = |\gamma|$, $\gamma \in \pi$

$t_C \in \mathcal{M}(\Sigma)$ right-handed Dehn twist along C

$\Rightarrow \tau(t_C) = \frac{1}{2} |(\log \gamma)^2| \in \widehat{\Bbbk\pi}/\Bbbk 1$

<u>S. Tsuji</u> Skein-ization

(Kauffman skein algebra
HOMFLY-PT skein algebra of $\Sigma \times [0,1]$) $\Rightarrow$ polynomial $\left(\begin{array}{l}\text{in 1 variable}\\ \text{in 2 variables}\end{array}\right)$ invariants of $\mathbb{Z}HS^3$

III. Turaev cobracket (a framed version)

<u>framing of Σ</u>. $T\Sigma$: trivial $(\because \partial\Sigma \neq \phi)$

$\mathcal{F}(\Sigma) := \{ T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$ orientation preserving isom$/_\Sigma \}$ / homotopy

$\quad\Psi$

$f : T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2 \xrightarrow{pr_2} \mathbb{R}^2$

$\ell : S^1 \to \Sigma$: C^∞ immersion, $\dot{\ell} : S^1 \to T\Sigma \setminus (0\text{-section})$, velocity vector

$\mathrm{rot}_f(\ell) := \deg(f \circ \dot{\ell} : S^1 \to \mathbb{R}^2 \setminus \{0\})$ rotation number

$\mathcal{F}(\Sigma)_{\ni f} \curvearrowleft \mathcal{M}(\Sigma)_{\ni \varphi}$ $\quad \mathrm{rot}_{f\varphi}(\ell) = \mathrm{rot}_f(\varphi \circ \ell)$

$f^{adp} \in \mathcal{F}(\Sigma)$ <u>adapted</u> w.r.to $\{\alpha_i, \beta_i\}$

$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \mathrm{rot}_{f^{adp}}(\alpha_i) = \mathrm{rot}_{f^{adp}}(\beta_i) = 0, & 1 \le {}^\forall i \le g \\ \mathrm{rot}_{f^{adp}}(\partial_j \Sigma) = -1, & 1 \le {}^\forall j \le n \end{cases}$

$\left(\xRightarrow[\text{Poincaré-Hopf formula}]{} \mathrm{rot}_{f^{adp}}(\partial_0\Sigma) = 1 - 2g \right)$

$\mathcal{F}(\Sigma)$: affine set modeled by $H^1(\Sigma; \mathbb{Z})$

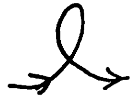
$\quad\Psi \qquad\qquad\qquad\qquad\qquad \Psi$

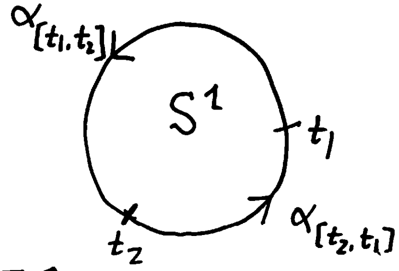
$\quad f \qquad\qquad\qquad\qquad\qquad\quad \chi$

$$\mathrm{rot}_{f+\chi}(\ell) = \mathrm{rot}_f(\ell) + \chi([\ell])$$

<u>Turaev cobracket</u> Fix $f \in \mathcal{F}(\Sigma)$

monogon





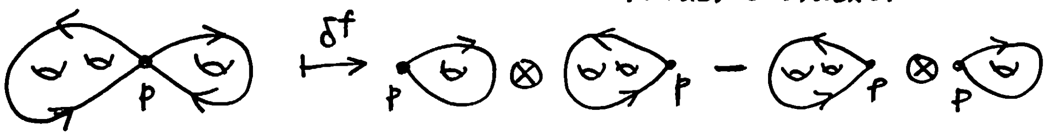
$\alpha \in \hat{\pi}$ generic immersion with $\mathrm{rot}_f\, \alpha = 0$

$D_\alpha := \{(t_1,t_2) \in S^1 \times S^1 \,;\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

$$\delta^f(\alpha) \overset{\mathrm{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{[t_1,t_2]}| \otimes |\alpha_{[t_2,t_1]}| \in \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$$

Turaev cobracket

(ex)



)

(<u>original definition by Turaev</u> without using a framing

$$\delta : \mathbb{Z}\hat{\pi}/\mathbb{Z}(1) \to (\mathbb{Z}\hat{\pi}/\mathbb{Z}(1))^{\otimes 2}$$
)

<u>Theorem</u> (Turaev) (modified)

(1) δ^f : well-defined

(2) $(\mathbb{Z}\hat{\pi}, [\,,\,], \delta^f)$: Lie bialgebra

(<u>Theorem</u> (Chas) (modified) involutivity

$$0 = [\,,\,] \circ \delta^f : \mathbb{Z}\hat{\pi} \xrightarrow{\delta^f} \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi} \xrightarrow{[\,,\,]} \mathbb{Z}\hat{\pi}$$

δ^f extends to $\widehat{\Bbbk\hat{\pi}}$ (completion)

$\mathcal{M}(\Sigma, f) := \{\varphi \in \mathcal{M}(\Sigma) ;\ f\varphi = f\}$

Theorem (Kuno-K.)

$$(\delta^f \circ \tau)(\mathcal{M}(\Sigma, f) \cap \mathcal{M}(\Sigma)^0) = 0$$

$(\Sigma = \Sigma_{g,1})$ the Enomoto-Satoh trace

$(\Sigma = \Sigma_{0,n+1})$ the Alekseev-Torossian divergence

} appears in $gr(\delta^f) =$ Schedler's cobracket

formal description of the Turaev cobracket δ^f ?

Definition $\theta : \pi \to Grp(A)$ homomorphic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to Grp(A)$ special / symplectic expansion

2) $\theta : \widehat{\Bbbk\hat{\pi}} \xrightarrow{\cong} |A|$ Lie bialgebra isomorphism

IV. Kashiwara-Vergne problem

(various Lie groups / algebras) $A = gr(\hat{T}(H_1(\Sigma : \Bbbk)))$, $L \subset A$ Lie-like elements

$\cup$ $\mathrm{Aut}(A) := \{U : A \to A : \text{topological algebra autom.}\}$.

$\mathrm{Aut}(L) := \{U : L \to L : \text{topological Lie alg. autom}\} = \{U \in \mathrm{Aut}(A) ; (U \otimes U) \circ \Delta = \Delta \circ U\}$

$\mathrm{IAut}(A) := \{U \in \mathrm{Aut}(A) : \forall p \geq 1, U(F^pA) = F^pA, \ gr(U) = 1\}$, $\mathrm{IAut}(L) := \mathrm{IAut}(A) \cap \mathrm{Aut}(L)$

$\mathrm{der}^+(A) := \mathrm{Lie}\, \mathrm{IAut}(A) = \{u \in \mathrm{der}(A) \text{ conti. derivation}, \ \forall p \geq 0, \ u(F^pA) \subset F^{p+1}(A)\}$

$\mathrm{der}^+(L) := \mathrm{der}^+(A) \cap \mathrm{der}(L) = \mathrm{Lie}\, \mathrm{IAut}(L)$

$$\begin{array}{ccc}
F^{-1} \circ \theta^{\exp} & \longleftarrow\!\shortmid & F \\
 & & \rotatebox{90}{$\in$} \\
\left\{\begin{array}{c}\text{group-like expansions} \\ \theta : \pi \to \mathrm{Grp}(A)\end{array}\right\} & \xleftarrow{\cong} & \mathrm{IAut}(L) \\
\cup\!\!| & \circlearrowleft & \cup\!\!| \\
\left\{\begin{array}{c}\text{tangential expansions} \\ \theta : \pi \to \mathrm{Grp}(A)\end{array}\right\} & \xleftarrow{\cong} & \mathrm{TAut}(L)
\end{array}$$

$\mathrm{TAut}(A) := \{U \in \mathrm{IAut}(A) ; 1 \leq \forall j \leq n, \ \exists f_j \in 1 + F^1(A), \ U(z_j) = f_j^{-1} z_j f_j\}$

$\mathrm{tder}(A) := \mathrm{Lie}\, \mathrm{TAut}(A) = \{u \in \mathrm{der}^+(A) ; 1 \leq \forall j \leq n, \ \exists u_j \in F^1(A), \ u(z_j) = [z_j, u_j]\}$

$\mathrm{TAut}(L) := \{U \in \mathrm{IAut}(L) ; 1 \leq \forall j \leq n, \ \exists f_j \in \mathrm{Grp}(A) \ \ U(z_j) = f_j^{-1} z_j f_j\}$

$\mathrm{tder}(L) := \mathrm{Lie}\, \mathrm{TAut}(L) = \{u \in \mathrm{der}^+(L) : 1 \leq \forall j \leq n, \ \exists u_j \in L, \ u(z_j) = [z_j, u_j]\}$

$F \in \mathrm{TAut}(L)$ $\quad F^{-1} \circ \theta^{\exp}$: special/symplectic $\iff F(\omega) = \xi$ where $\xi := \log \theta^{\exp}(\gamma_0^{-1}) \in L$

p.25, ℓ.4

<u>divergence cocycle:</u> $\operatorname{div} : \operatorname{tder}(A) \to |A|$,

$$\operatorname{div}(u) := \sum_{i=1}^{g} \left| \partial_{x_i}(u(x_i)) + \partial_{y_i}(u(y_i)) \right| + \sum_{j=1}^{n} \left| z_j\, \partial_{z_j}(u_j) \right| , \quad u \in \operatorname{tder}(A) \quad \left(u(z_j) = [z_j, u_j] \right)$$

<u>where</u> $$a = a_0 + \sum_{i=1}^{g} \left((\partial_{x_i} a)\, x_i + (\partial_{y_i} a)\, y_i \right) + \sum_{j=1}^{n} (\partial_{z_j} a)\, z_j \qquad a \in A,\ a_0 \in \Bbbk$$

($\Sigma = \Sigma_{g,1}$) div = the Enomoto-Satoh trace

($\Sigma = \Sigma_{0,n+1}$) div = the Alekseev-Torossian divergence

<u>cocycle:</u> $$\operatorname{div}([u,v]) = u \cdot \operatorname{div}(v) - v \cdot \operatorname{div}(u) \quad (\forall u, \forall v \in \operatorname{tder}(A))$$

$\Downarrow$ integration

<u>Jacobian cocycle</u>: $j : \operatorname{TAut}(A) \to |A|$ group 1-cocycle = crossed homomorphism

$$j(\exp(u)) = \left(\frac{e^u - 1}{u} \right) \cdot \operatorname{div}(u) \in |A| . \quad (\forall u \in \operatorname{tder}(A))$$

$\Sigma = \Sigma_{g,n+1}$, $f \in \mathcal{F}(\Sigma)$ framing. $\quad$ (recall $\xi = \log \theta^{\exp}(\gamma_0^{-1}) \in L$)

$\Downarrow$

<u>Definition</u> ($KV_f^{(g,n+1)}$ problem) Find an element $F \in TAut(L)$ satisfying

(KV I) $\quad F(\omega) = \xi \quad$ ($\Leftrightarrow$ $F^{-1} \circ \theta^{\exp}$: special / symplectic)

(KV II) $\quad \exists\, h(s), h_1(s), \cdots, h_n(s) \in \mathbb{k}[[s]]$ s.t.

$$j(F) = \sum_{i=1}^{g} \left| \log\left(\frac{e^{x_i}-1}{x_i}\right) + \log\left(\frac{e^{y_i}-1}{y_i}\right) \right| - |h(\xi)| + \sum_{j=1}^{n} |h_j(z_j)|$$

$$+ \sum_{i=1}^{g} \left(\chi(y_i) + \sum_j \chi(z_j)\lambda_{ji}(F) \right) |x_i| + \left(-\chi(x_i) + \sum_j \chi(z_j)\mu_{ji}(F) \right) |y_i|$$

<u>where</u>

$f = f^{adp} + \chi$, $\quad \chi \in H^1(\Sigma : \mathbb{Z})$

$\lambda_{ji}(F), \mu_{ji}(F) \in \mathbb{k}$.

$F = \exp(v)$, $\quad v(z_j) = [z_j, v_j]$

$$v_j \equiv \sum_{i=1}^{g} \lambda_{ji}(F)\, x_i + \mu_{ji}(F)\, y_i \mod F^2(A)$$

<u>Remark</u> - If $g=0$, $KV_f^{(0,n+1)}$ does <u>not</u> depend on $f \in \mathcal{F}(\Sigma)$

- $KV_f^{(0,3)}$: the Kashiwara-Vergne problem in the formulation of Alekseev-Torossian

<u>Theorem</u> (Alekseev-K.-Kuno-Naef). Let $F \in \mathrm{TAut}(L)$ satisfy the condition (KVI). Then
$F^{-1} \circ \theta^{\exp}$: homomorphic $\Leftrightarrow$ F satisfies (KVII)

$\Uparrow$

- non-commutative Poisson geometry on $A = \mathrm{gr}(\hat{T}(H_1(\Sigma;\mathbb{K})))$ à la van den Bergh
- $\delta^{f^{\mathrm{adp}}}(\alpha) \text{``}=\text{''} t\mathrm{Div} \circ \sigma(\alpha) \quad (\forall \alpha \in \hat{\pi})$
-

$$\begin{array}{ccc} \mathrm{tder}(L) & \xrightarrow{\mathrm{div}} & |A| \\ \downarrow & \circlearrowleft & \downarrow (1\otimes\iota)\circ\Delta \\ \mathrm{tder}(A) & \xrightarrow[t\mathrm{Div}]{} & |A|^{\otimes 2} \end{array}$$

<u>double divergence</u>: $t\mathrm{Div} : \mathrm{tder}(A) \to |A|^{\otimes 2}$ ($\doteqdot \mathrm{gr}(\delta^f)$ Schedler's cobracket)

$$t\mathrm{Div}(u) := (|\,|\otimes|\,|)\left(\sum_{i=1}^{g}\left(D_{x_i}(u(x_i)) + D_{y_i}(u(y_i))\right) + \sum_{j=1}^{m}\left((z_j\otimes 1 + 1\otimes z_j)\right) D_{z_j}(u_j)\right)$$

<u>where</u> $D_{x_i}(w_1 w_2 \cdots w_m) := \sum_{w_k = x_i} w_1 \cdots w_{k-1} \otimes w_{k+1} \cdots w_m$, $w_k \in \{x_i, y_i, z_j\}$ p.26. l.7

<u>Theorem</u> (AKKN)

$\exists$ solutions to $KV_f^{(g,n+1)}$ if $\begin{cases} g \neq 1 & \text{or} \\ g = 1 & \text{and } \ rot_f(\alpha_1) = rot_f(\beta_1) = 0 \end{cases}$

$\Uparrow$ pants decomposition

gluing solutions of $KV_f^{(0,3)}$ and those of $KV_{f^{adp}}^{(1,1)}$

$\Uparrow$ $\begin{cases} \text{Alekseev-Meinrenken} \\ \text{Alekseev-Torossian} \end{cases}$ $\Uparrow$ $KV_f^{(0,3)} \oplus$ Enriquez' method

<u>Consequently</u> $gr(\mathbb{k}\hat{\pi}) \cong \widehat{\mathbb{k}\hat{\pi}}$

as Lie <u>bi</u>algebras

formality

<u>$g=0$</u>

Massuyeau

using the Kontsevich integral

<u>Corollary</u> (AKKN, $\Sigma = \Sigma_{g,1}$)

The constraint of the Johnson image coming from δ^f

is equivalent to that coming from the Enomoto-Satoh trace

- (genus 0 case)

Massuyeau ⇙ (Kontsevich integral) ⇐ Kontsevich (Drinfel'd associator)

(solution of $KV_f^{(0,n+1)}$) ⇐ Alekseev-Torossian (Conjecture ⇒)

AKKN ⇕

(formality of the Goldman-Turaev Lie bialgebra)

⇑ Drinfel'd

⇐ Alekseev-Naef (Knizhnik-Zamolodchikov connection)

harmonic Magnus expansion ??

- (Future research)

<u>Characterization of the Johnson image</u> (Morita)

- new operations of loops on Σ ?
- Conant's higher trace
- M. Felder's higher divergence ⇐ Severa-Willwacher ($\Sigma = \Sigma_{0,n+1}$) $(g=0)$
- subsurfaces bounded by self-intersection loops