Séminaire Algèbre et topologie, IRMA, Université de Strasbourg

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"Differential forms and functions on the moduli space of Riemann surfaces"

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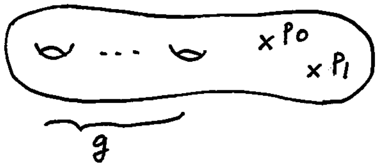
$g \geqq 2$ $\quad \Sigma_g \ni p_0 \neq p_1 =$ 

C^∞ closed oriented surface of genus g

<u>moduli spaces</u> $\longleftrightarrow$ <u>mapping class groups</u>

$$\left.\begin{array}{l} [C, p_0] \in \mathbb{C}_g := \{(C, p_0) ;\ C : \text{compact Riemann surface of genus } g,\ p_0 \in C\}/\text{biholo} \\ \quad \downarrow \qquad \downarrow \pi \\ [C] \in \mathbb{M}_g := \{C ;\ C : \text{compact Riemann surface of genus } g\}/\text{biholo} \end{array}\right\} \text{moduli spaces}$$

$$\left.\begin{array}{l} \pi_1^V(\mathbb{M}_g) = \mathcal{M}_g := \pi_0 \mathrm{Diff}^+(\Sigma_g) \\ \pi \uparrow \\ \pi_1^V(\mathbb{C}_g) = \mathcal{M}_g^1 := \pi_0 \mathrm{Diff}^+(\Sigma_g, p_0) \leftarrow \pi_1^V(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g) = \mathcal{M}_g^1 \times_{\mathcal{M}_g} \mathcal{M}_g^1 \\ \quad \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nearrow \\ \pi_1^V(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal}) = \mathcal{M}_g^2 := \pi_0 \mathrm{Diff}^+(\Sigma_g, p_0, p_1) \end{array}\right\} \text{mapping class groups}$$

$$H := \coprod_{[C] \in \mathbb{M}_g} H^1(C : \mathbb{C}) \overset{\text{Poincaré duality}}{\cong} \coprod_{[C]\in\mathbb{M}_g} H_1(C:\mathbb{C}) \to \mathbb{M}_g$$ flat vector bundle

$$H^*(\mathbb{M}_g : \Lambda^* H) = H^*(\mathcal{M}_g : \Lambda^* H^1(\Sigma_g : \mathbb{C}))$$

$$H^*(\mathbb{C}_g : \Lambda^* H) = H^*(\mathcal{M}'_g : \Lambda^* H^1(\Sigma_g : \mathbb{C}))$$ ($\Leftarrow$ Teichmüller space $\simeq *$)

twisted de Rham cohomology — group cohomology

$$e := c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) = \mathrm{Euler}(\mathbb{Z} \to \pi_0(\mathrm{Diff}^+(\Sigma_g, v_0)) \to \mathcal{M}'_g) \in H^2(\mathbb{C}_g : \mathbb{Q}) = H^2(\mathcal{M}'_g : \mathbb{Q})$$

(group cohomology; $0 \neq v_0 \in T_{P_0}\Sigma_g$)

Mumford-Morita-Miller (MMM, tautological) classes $i \geq 0$

$$e_i = (-1)^{i+1}\kappa_i = \int_{\text{fiber}} e^{i+1} \in H^{2i}(\mathbb{M}_g : \mathbb{Q}) = H^{2i}(\mathcal{M}_g : \mathbb{Q}) \qquad (e_0 = 2-2g \in \mathbb{Z})$$

Madsen-Weiss $H^*(\mathbb{M}_g : \mathbb{Q}) = \mathbb{Q}[e_i : i \geq 1]$ for $* \ll g$ (stable range)

Want to have explicit differential forms representing e_i's

cf) Wolpert uses the hyperbolic metric to obtain them. If $i=1$, $e_1 \leadsto \omega_{WP}$ Weil-Petersson Kähler form

Our strategy using twisted MMM classes to get them

$\leadsto$ vanishing of e_i's on the hyperelliptic locus

$\leadsto$ physics (string theory)

<u>twisted MMM classes</u> (K.), $m_{i,j} \in H^{2i+j-2}(\mathbb{C}_g : \Lambda^j H)$, $(i,j \geq 0,\ 2i+j \geq 2)$

⇑ <u>extension</u> of $\mathcal{M}_g^2$-modules (i.e., extension of flat vector bundles $/(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal})$)

$H^*(-) = H^*(- : \mathbb{C})$

$$0 \to H^1(\Sigma_g) \xrightarrow{\text{incl}^*} H^1(\Sigma_g \setminus \{p_0, p_1\}) \to \underset{\overset{\|\wr}{\mathbb{C}}}{H^2(\Sigma_g, \Sigma_g \setminus \{p_0, p_1\})} \to 0 \quad \text{(exact)}$$

$$\Rightarrow \quad \begin{array}{ccccc} H^0(\mathcal{M}_g^2 : \mathbb{C}) & \xrightarrow{\delta^*} & H^1(\mathcal{M}_g^2 : H^1(\Sigma_g)) \cong H^1(\mathcal{M}_g^1 \times_{\mathcal{M}_g} \mathcal{M}_g^1 : H^1(\Sigma_g)) & = & H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H) \\ \Psi & & \Psi & & \Psi \\ 1 & \longmapsto & \delta^*(1) & \underset{\text{extension class}}{\longmapsto} & \exists!\ k_0 \end{array}$$

$$m_{i,j} \overset{\text{def}}{=} \int_{\text{fiber}} e^i k_0^j \in H^{2i+j-2}(\mathbb{C}_g : \Lambda^j H) \cong H^{2i+j-2}(\mathcal{M}_g^1 : \Lambda^j H^1(\Sigma_g))$$

the $(i,j)^{th}$ twisted MMM class (K.)

$m_{i+1,0} = e_i$, $\quad m_{0,2} = 2 \times (\text{symplectic form}) \in \Lambda^2 H^1(\Sigma_g)$

[<u>Morita-K.</u> $\frac{1}{6} m_{0,3} = (\text{the extended } 1^{st} \text{ Johnson homomorphism})$

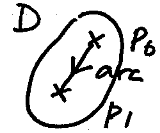
$\in H^1(\mathbb{C}_g : \Lambda^3 H) = H^1(\mathcal{M}_g^1 : \Lambda^3 H^1(\Sigma_g))$

<u>How to compute</u> $k_0 \leftarrow\!\shortmid \delta^*(1) \in H^1(\mathcal{M}_g^2 : H^1(\Sigma_g)) = H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal} : H)$

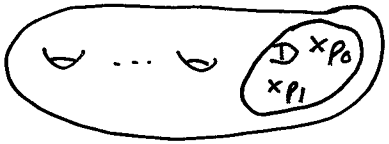
$$\begin{array}{ccccccc} 0 \to H^1(\Sigma_g) & \xrightarrow{\text{incl}^*} & H^1(\Sigma_g \setminus \{p_0, p_1\}) & \xrightarrow{\delta^*} & H^2(\Sigma_g, \Sigma_g \setminus \{p_0, p_1\}) & \to 0 \\ & & \Psi & & \|\wr & \\ & \text{Choose some} & u_0 & \longmapsto & 1 \in \mathbb{C} & \end{array}$$

Then, the cocycle $\delta u_0 : \varphi \in \mathcal{M}_g^2 \mapsto \varphi u_0 - u_0 \in H^1(\Sigma_g)$ represents $\delta^*(1)\ (\mapsto k_0)$

2 kinds of "explicit" choices of $u_0 \in H^1(\Sigma_g \setminus \{p_0, p_1\})$



(1) <u>topological choice</u> Choose an embedded disk $D \subset \Sigma_g$ s.t. $p_0, p_1 \in \mathring{D}$



$$\begin{tikzcd} 0 \arrow[r] & H^1(\Sigma_g) \arrow[r, "incl^*"] \arrow[dr, "\cong"'] & H^1(\Sigma_g \setminus \{p_0, p_1\}) \arrow[r] \arrow[d, "incl^*"] & \mathbb{C} \arrow[r] & 0 \\ & & H^1(\Sigma_g \setminus \mathring{D}) & & \end{tikzcd}$$

$\rightsquigarrow$ <u>topological choice</u>

$u_0 = \left(\begin{array}{l}\text{Poincaré dual of an arc} \\ \text{from } p_0 \text{ to } p_1 \text{ inside } \mathring{D}\end{array}\right)$

(cf) gluing of a symplectic expansion for $\Sigma_{g,1}$ and a special expansion for $\Sigma_{0,3}$ along the boundary ∂D)

(2) <u>analytic choice</u>

<u>Recall</u> $H^2_{de Rham}(\Sigma_g, \Sigma_g \setminus \{p_0, p_1\}) = \mathbb{C}(\delta_{p_1} - \delta_{p_0}) \cong \mathbb{C}$

<u>where</u> $\delta_P = d\left(\frac{1}{2\pi} d(\arg z)\right)$ for z : complex coordinate with $z(P) = 0$

delta current at $P \in \Sigma_g$.

Choose a complex structure on Σ_g (review on classical harmonic integrals)

i.e., consider C : compact Riemann surface $\approx \Sigma_g$

$\{\psi_i\}_{i=1}^g \subset H^0(C; \Omega^1_C) = \{\text{holomorphic 1-forms}\}$ orthonormal basis

$$\frac{\sqrt{-1}}{2} \int_C \psi_i \wedge \overline{\psi_j} = \delta_{ij} \quad (1 \le i, j \le g)$$

$B := \frac{\sqrt{-1}}{2g} \sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$ volume form on C, indep of the choice of $\{\psi_i\}_{i=1}^g$

$$\int_C B = \int_C \delta_P = 1. \quad (\forall P \in C)$$

$A^q(C) := \{\mathbb{C}\text{-valued } q\text{-currents (forms) on } C\}$, $q = 0, 1, 2$. (ex). $B, \delta_P \in A^2(C)$)

$* : A^1(C) \to A^1(C)$, Hodge $*$-operator (local z: complex coordinate $*dz = -\sqrt{-1}\,dz$, $*d\bar{z} = \sqrt{-1}\,d\bar{z}$)

$\exists!\ \hat{\Phi} : A^2(C) \to A^0(C)$ the Green operator associated with B

s.t. $\begin{cases} \text{(i)} & d*d\,\hat{\Phi}(\Omega) = \Omega - (\int_C \Omega) B \\ \text{(ii)} & \int_C \hat{\Phi}(\Omega) B = 0 \end{cases}$ $\quad (\forall \Omega \in A^2(C))$

- $\overline{\hat{\Phi}} = \hat{\Phi}$: real operator
- $\hat{\Phi}|_{\mathrm{Ker}(\int_C)} : \mathrm{Ker}(\int_C : A^2(C) \to \mathbb{C}) \to A^0(C)/\underset{\text{const. function}}{\mathbb{C}}$ depends only on the complex structure.
- $\forall \Omega, \forall \Omega' \in A^2(C)$, $\int_C \hat{\Phi}(\Omega)\Omega' = \int_C \Omega\, \hat{\Phi}(\Omega')$

$$\int_C \hat{\Phi}(\Omega)\Omega' = \frac{1}{4\pi} \iint_{(P_0, P_1) \in C \times C} (\Omega)_{P_1} (\log G(P_1, P_2)) (\Omega')_{P_2} = \int_C \Omega\, \hat{\Phi}(\Omega')$$

(where $G(\cdot,\cdot)$ the Arakelov-Green function)

<u>analytic choice</u>

$$u_0 = \underbrace{[*d\hat{\Phi}(\delta_{P_1} - \delta_{P_0})]}_{\text{de Rham class on } C \setminus \{P_0, P_1\}} \in H^1(C \setminus \{P_0, P_1\}) \xmapsto{\delta^*} \delta_{P_1} - \delta_{P_0} \in H^2(C, C \setminus \{P_0, P_1\})$$

(<u>classically</u> $*d\hat{\Phi}(\delta_{P_1} - \delta_{P_0}) = \mathrm{Re}$ (the normalized Abelian differential of the 3rd kind))

$\hat{k}_0 := \left(\text{the } 1^{st} \text{ variation of } [*d\hat{\Phi}(\delta_{P_1}-\delta_{P_0})]\right) \in \Omega^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal} : H)$

$$H^1(\mathcal{M}_g^2 : H^1(\Sigma_g)) = H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal} : H)$$

extension class $\delta^*(1) \longmapsto [\hat{k}_0]$ ⟸ "explicit" computation of $\hat{k}_0$ using $\hat{\Phi}|_{\mathrm{Ker}\int_C}$

$\hat{k}_0$ naturally extends to the diagonal. $\Rightarrow \hat{k}_0 \in \Omega^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H)$

$k_0 = [\hat{k}_0] \in H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H)$

<u>canonical differential form representing</u> $\frac{1}{6} m_{0,3}$, <u>the extended 1^{st} Johnson homomorphism</u>

$$\tilde{k} := \frac{1}{6} \int_{fiber} \hat{k}_0{}^3 \in \Omega^1(\mathbb{C}_g : \Lambda^3 H)$$

<u>explicit description of $\tilde{k}$ in terms of the Green operator $\hat{\Phi}$</u>

$\varphi_a = \varphi_a' + \varphi_a'' \in H^1(C : \mathbb{C}) \xlongequal{\text{identify}} \{\text{harmonic 1-forms on } C\}$, $a = 1, 2, 3$

$\varphi_a', \overline{\varphi_a''} \in H^0(C : \Omega^1_C)$ holomorphic 1-forms

$$Q(\varphi_1, \varphi_2, \varphi_3) \overset{\text{def}}{=} -\sqrt{-1}\,\varphi_1' \partial\hat{\Phi}\left(\varphi_2 \wedge \varphi_3 - \left(\int_C \varphi_2 \wedge \varphi_3\right)\delta_{P_0}\right) - \sqrt{-1}\,\varphi_2' \partial\hat{\Phi}\left(\varphi_3 \wedge \varphi_1 - \left(\int_C \varphi_3 \wedge \varphi_1\right)\delta_{P_0}\right)$$
$$- \sqrt{-1}\,\varphi_3' \partial\hat{\Phi}\left(\varphi_1 \wedge \varphi_2 - \left(\int_C \varphi_1 \wedge \varphi_2\right)\delta_{P_0}\right)$$

(the same as the 1^{st} var. of the pointed harmonic volume (B. Harris, M. Pulte))

$$\in T^*_{[C,P_0]}\mathbb{C}_g = \left\{\text{mero. quadratic differential on } C ;\ \begin{matrix}\text{holomorphic on } C \setminus \{P_0\} \\ \text{order at } P_0 \geqq -1\end{matrix}\right\}$$

<u>Theorem</u> (K.)

$\langle \tilde{k}, [\varphi_1] \otimes [\varphi_2] \otimes [\varphi_3] \rangle$ at $[C, P_0] \in \mathbb{C}_g$

$$= -Q(\varphi_1, \varphi_2, \varphi_3) - \overline{Q(\varphi_1, \varphi_2, \varphi_3)} \in T^*_{[C,P_0]}\mathbb{C}_g \oplus \overline{T^*_{[C,P_0]}\mathbb{C}_g}$$

<u>Corollary</u> $\tilde{k}$ = the 1^{st} variation of the pointed harmonic volume (B. Harris, M. Pulte)

Hyperelliptic locus

$\mathcal{H}_g := \{[C] \in \mathbb{M}_g ;$ hyperelliptic curves, i.e., $C \overset{\exists}{\xrightarrow{2:1}} \mathbb{P}^1\} \subset \mathbb{M}_g$ $\longrightarrow$ covering transformation $\iota : C \to C$ hyperelliptic involution

$\mathcal{H}_2 = \mathbb{M}_2$, $\mathcal{H}_g \subsetneqq \mathbb{M}_g$ if $g \geqq 3$, $\dim_{\mathbb{C}} \mathcal{H}_g = 2g-1$, $\dim_{\mathbb{C}} \mathbb{M}_g = 3g-3$

$\mathcal{H}_g^1 := \{[C, P_0] \in \mathbb{C}_g : [C] \in \mathcal{H}_g$, $P_0 \in C$ is a ramification point of $C \xrightarrow{2:1} \mathbb{P}^1\} \subset \mathbb{C}_g$

- If $[C, P_0] \in \mathcal{H}_g^1$,

$$(Q|_{[C,P_0]})(\Lambda^3 H^1(C:\mathbb{C})) \subset ((-1)\text{-eigenspace of } \iota_* : T^*_{[C,P_0]}\mathbb{C}_g \circlearrowleft)$$

$(\because)$ (B. Harris) $\iota(P_0) = P_0$, $\iota_* = -1$ on $\Lambda^3 H^1(C:\mathbb{C})$. Q is natural w.r.to any biholo. map //)

in other words, $\tilde{k}|_{\mathcal{H}_g^1} = 0 \in \Omega^1(\mathcal{H}_g^1 : \Lambda^3 H)$

$(\because)$ $T^*_{[C,P_0]}\mathcal{H}_g^1 = (+1)\text{-eigen space of } \iota_* : T^*_{[C,P_0]}\mathbb{C}_g \circlearrowleft$ //)

- In general, $U := \mathrm{Ker}\,(\Lambda^3 H^1(C:\mathbb{C}) \xrightarrow[\text{by the intersection number}]{\text{contraction}} H^1(C:\mathbb{C}))$

$$\begin{array}{ccc} \Lambda^3 H^1(C:\mathbb{C}) & \xrightarrow{Q} & T^*_{[C,P_0]}\mathbb{C}_g \\ \cup & \circlearrowleft & \uparrow \pi^* \\ U & \underset{Q}{\overset{\exists !}{\dashrightarrow}} & T^*_{[C]}\mathbb{M}_g \end{array}$$

This map depends only on $[C] \in \mathbb{M}_g$

(e.g. the set of branched points of ι is stable under the cpx conj on $\mathbb{P}^1$)

Theorem (B. Harris) $\forall g \geqq 3$, $\exists [C_0] \in \mathcal{H}_g$, hyperelliptic curve s.t. $Q : U \to ((-1)\text{-eigen space of } \iota_* : T^*_{[C_0]}\mathbb{M}_g)$ is <u>surjective</u>.

i.e., $\tilde{k}|_{[C_0]}$ is <u>non-degenerate</u> in the normal direction to $\mathcal{H}_g$

Morita's construction

$[\tilde{k}] = \frac{1}{6} m_{0,3} \in H^1(\mathcal{C}_g : \Lambda^3 H)$, the extended 1st Johnson homomorphism

$N \geq 0$, $f \in \mathrm{Hom}(\Lambda^N(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})}$ invariant linear form $\leftrightarrow$ trivalent graph

$\Longrightarrow$ $f_* : H^N(\mathcal{C}_g : \Lambda^N(\Lambda^3 H)) \to H^N(\mathcal{C}_g : \mathbb{C})$

$[\tilde{k}]^N \longmapsto f_*([\tilde{k}]^N)$

$\Longrightarrow$ $\alpha : \mathrm{Hom}(\Lambda^*(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})} \longrightarrow H^*(\mathcal{C}_g : \mathbb{C})$ algebra homomorphism

$f \longmapsto f_*([\hat{k}]^{\otimes})$

$(\tilde{k}|_{\partial\mathcal{H}_g} = 0)$ Alternative proof of $\forall n \geq 1$ $e_n = 0 \in H^*(\partial\mathcal{H}_g : \mathbb{C})$

$\mathrm{Im}\,\alpha \supset \mathbb{C}[e, e_i : i \geq 1]$, $e = c_1(T\mathcal{C}_g/\mathcal{M}_g)$. (Morita)

$\mathrm{Im}\,\alpha = \mathbb{C}[e, e_i : i \geq 1]/\sim$ (even in the unstable range) (Morita-K.)

in degree 2

$\alpha : \mathrm{Hom}(\Lambda^2(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})} \xrightarrow{\cong} H^2(\mathcal{C}_g : \mathbb{C}) \overset{\text{Harer}}{=\!=} \mathbb{C}e \oplus \mathbb{C}e_1$ (if $g \geq 3$)

$\exists!\, f_0$, $\exists!\, f_1$, $f_{0*}[\tilde{k}]^2 = e$, $f_{1*}[\tilde{k}]^2 = e_1$ (Ker $\alpha \neq 0$ for even degree ≥ 4)

differential forms

$e^J := f_{0*}\tilde{k}^2 \in \Omega^2(\mathcal{C}_g)$, $e_1^J := f_{1*}\tilde{k}^2 \in \mathrm{Im}(\Omega^2(\mathcal{M}_g) \xrightarrow{\pi^*} \Omega^2(\mathcal{C}_g))$ $\xrightarrow{(\tilde{k}|_{\partial\mathcal{H}_g} = 0)}$ $e_1^J|_{\partial\mathcal{H}_g} = 0 \in \Omega^2(\partial\mathcal{H}_g)$

$e_1^F := \int_{fiber}(e^J)^2 \in \Omega^2(\mathcal{M}_g)$

$e_1 = [e_1^F] = [e_1^J] \in H^2(\mathcal{M}_g : \mathbb{C})$ but $e_1^F \neq e_1^J \in \Omega^2(\mathcal{M}_g)$

$\sqrt{-1}\,\partial\bar{\partial}(\exists\ \mathbb{R}\text{-valued function}) = e_1^F - e_1^J$

(unique up to const. if $g \geq 3$)

Kawazumi - Zhang (KZ) invariant (K, 0801.4218, Zhang 0812.0371) $[C] \in \mathbb{M}_g$

$$a_g(C) \overset{\text{def}}{=} \sum_{i,j=1}^{g} \int_C \psi_i \wedge \overline{\psi_j}\, \hat{\Phi}(\psi_j \wedge \overline{\psi_i}) \in \mathbb{R} \quad (\because\ \overline{\hat{\Phi}} = \hat{\Phi})$$

(Fact: $\forall g \geq 2$, $\forall [C] \in \mathbb{M}_g$, $a_g(C) > 0$)

- Zhang : Arakelov geometry ⋯ a decomposition of the Faltings delta invariant δ
- (Theorem (K) $\dfrac{-2\sqrt{-1}(2g-2)^2}{2g(2g+1)} \partial\bar{\partial} a_g = e_1^F - e_1^J \in \Omega^2(\mathbb{M}_g ; \mathbb{C})$
- de Jong - explicit relation with the invariant δ and the Hain-Reed function β
  - asymptotic behavior of a_g
- d'Hoker - Green (1308.4597) - application of a_2 $(g=2)$ to physics
- Pioline - application of a_2 to physics
  - explicit formula of a_2 in terms of the theta function

(Theorem (d'Hoker-Green-Pioline-Russo) (1405.6226)
$\Delta a_2 = 5 a_2$ on $\mathbb{M}_2$

twisted version of the KZ invariant (K,)

$$A_{i\bar{j}k\bar{l}} \overset{\text{def}}{=} \int_C \psi_i \wedge \overline{\psi_j}\, \hat{\Phi}(\psi_k \wedge \overline{\psi_l}) \in \mathbb{C}, \quad 1 \leq i, j, k, l \leq g$$

Question (Torelli type) Is the map induced by $(A_{i\bar{j}k\bar{l}})$

(the Torelli space) $\to \mathbb{C}^{g^4}$

injective ?

where (the Torelli space) $= \left\{ (C, \{A_i, B_i\}_{i=1}^g) : \begin{array}{l} C : \text{compact Riemann surface of genus } g, \\ \{A_i, B_i\}_{i=1}^g \subset H_1(C; \mathbb{Z}) \text{ sympl. basis} \end{array} \right\} \Big/ \text{biholo}$

elementary properties

(0) $\sum_{i,j=1}^{g} A_{i\bar{j}j\bar{i}} = a_g(C)$

(1) $\overline{A_{i\bar{j}k\bar{\ell}}} = \int_C \overline{\psi_i} \wedge \psi_j \hat{\Phi}(\overline{\psi_k} \wedge \psi_\ell) = \int_C \psi_j \wedge \overline{\psi_i} \hat{\Phi}(\psi_\ell \wedge \overline{\psi_k}) = A_{j\bar{i}\ell\bar{k}}$

(2) $A_{i\bar{j}k\bar{\ell}} = \int_C \psi_i \wedge \overline{\psi_j} \hat{\Phi}(\psi_k \wedge \overline{\psi_\ell}) = \int_C \hat{\Phi}(\psi_i \wedge \overline{\psi_j}) \psi_k \wedge \overline{\psi_\ell} = A_{k\bar{\ell}i\bar{j}}$

(3) $\sum_{i=1}^{g} A_{i\bar{i}k\bar{\ell}} = 0 \quad (\because \hat{\Phi}(B) = 0)$

$(A_{i\bar{j}k\bar{\ell}}) \in C^\infty(\mathbb{M}_g : \mathrm{Sym}^2(\Lambda^2 H))$ global section

$\longrightarrow$ differential forms representing $e_1 = \kappa_1$

holomorphic cotangent space of $\mathbb{M}_g$ at $[C] \in \mathbb{M}_g$

$$T^*_{[C]}\mathbb{M}_g = H^0(C;(\Omega^1_C)^{\otimes 2}) = \{\text{holomorphic quadratic differentials on } C\}$$

$$(\text{ex}) \ \psi_\ell \psi_j = \psi_j \psi_\ell \in T^*_{[C]}\mathbb{M}_g, \ \overline{\psi_i}\overline{\psi_k} = \overline{\psi_k}\overline{\psi_i} \in \overline{T^*_{[C]}\mathbb{M}_g}$$

Theorem (K.; discovered around 20 years ago, and written in arXiv 0801.4218, eq (5.5))

$$e_1^P|_{[C]} := 48\sqrt{-1} \sum_{i,j,k,\ell} (\psi_\ell \psi_j A_{i\bar{j}k\bar{\ell}} \overline{\psi_i}\overline{\psi_k} - \psi_\ell \psi_j A_{i\bar{j}k\bar{i}} \overline{\psi_k}\overline{\psi_\ell}) \in T^*_{[C]}\mathbb{M}_g \otimes \overline{T^*_{[C]}\mathbb{M}_g}$$

$([C] \in \mathbb{M}_g)$

$$\Longrightarrow \frac{g^2}{(2g-2)^2} e_1^F - \frac{2g^2+2g-1}{3(2g-2)^2} e_1^J = \frac{1}{12} e_1^P \in \Omega^2(\mathbb{M}_g)$$

$$\frac{1}{12} e_1^P = \frac{g^2}{(2g-2)^2} e_1^F - \frac{2g^2+2g-1}{3(2g-2)^2} e_1^J = \frac{g^2}{(2g-2)^2}(e_1^F - e_1^J) + \frac{1}{12} e_1^J = \frac{-2\sqrt{-1}g}{2(2g+1)}(\partial\bar{\partial} a_g) + \frac{1}{12} e_1^J$$

$$[e_1^P] = e_1 \in H^2(\mathbb{M}_g ; \mathbb{C})$$

$$\boxed{\frac{1}{12} e_1^P = \frac{-2\sqrt{-1}g}{2(2g+1)}(\partial\bar{\partial} a_g) + \frac{1}{12} e_1^J} \quad \cdots (*)$$

<u>Theorem</u> ([DGPR] D'Hoker-Green-Pioline-Russo, arXiv:1405.6226v2)

$$\Delta a_2 = 5 a_2$$

where Δ = the Laplacian for the canonical Kähler metric on the Siegel upper half space $\mathfrak{H}_g$

($\Longrightarrow$ <u>Pioline</u> (arXiv:1504.04182v2) explicit formula for a_2 using theta function

<u>Higher genus case ?</u>

[DGPR] obtained the followings for $\forall g \geq 2$:

- <u>(B,4) + (C,1)</u> $\quad \delta_{\bar{u}\bar{u}}\, \delta_{ww}\, \varphi = \Psi_A^1 + \Psi_B + \Psi_A^2 + \Psi_C$ $\quad$ (<u>Rmk</u>: $\varphi = a_g$, $\delta_{\bar{u}\bar{u}} = \bar{\partial}$, $\delta_{ww} = \partial$)
- <u>(C,3)</u> $\quad \Psi_A^1 + \Psi_B$ = (explicit formula using $\hat{\Phi}$) (<u>Rmk</u>: (RHS) = (const) (e_1^P in my notation))
- <u>(C,5)</u> $\quad \Psi_A^2 + \Psi_C$ = (explicit formula using $\hat{\Phi}$) (<u>Rmk</u>: (RHS) = (const) (e_1^J in my notation))

$\Leftrightarrow$ (*bis) $\quad \frac{1}{12} e_1^D = \frac{-2\sqrt{-1}\, g}{2(2g+1)} (\partial\bar{\partial} a_g) + \frac{1}{12} e_1^J$

- <u>(C,9)</u> $\quad \Psi_A^2 + \Psi_C = 0$ if $g = 2$ $\quad$ (<u>Rmk</u>: $e_1^J|_{\mathcal{H}_g} = 0 \in \Omega^2(\mathcal{H}_g)$ $\quad \forall g \geq 2$)
- <u>(C,4)</u> $\quad \underbrace{\Delta \varphi \,|_{\Psi_A + \Psi_B}}_{\text{"the part of the Laplacian due to } \Psi_1^A + \Psi_B\text{"}} = (2g+1)\, \varphi$

$\Rightarrow$ <u>Thm</u> ([DGPR]) $\quad \Delta a_2 = 5 a_2$

<u>Rmk</u>: $\dim_{\mathbb{C}} \mathcal{M}_g = \dim_{\mathbb{C}} \mathfrak{H}_g \Leftrightarrow g \leq 3$

If $g \geq 4$, then $\Delta\varphi = \Delta a_g$ is <u>meaning less</u> !!

To understand the meaning of (C.4) $\boxed{\Delta\varphi|_{\tau_A'+\tau_B} = (2g+1)\varphi}$,

recall the Siegel upper half space $\mathfrak{H}_g$

$\mathfrak{H}_g := \{$complex structure J on the standard symplectic vector space $\mathbb{R}^{2g}$; some positivity condition$\}$

$\uparrow \qquad \cong Sp_{2g}(\mathbb{R})/U_g$

$E_g := \coprod_{J\in\mathfrak{H}_g} (\mathbb{R}^{2g}, J)$. $Sp_{2g}(\mathbb{R})$-equivariant holomorphic vector bundle $\Downarrow$ canonical Hermitian metric on E_g

$\mathbb{A}_g := \mathfrak{H}_g/Sp_{2g}(\mathbb{Z})$ the moduli space of principally polarized Abelian varieties

$\uparrow$

E_g (descends to $\mathbb{A}_g$)

$T\mathbb{A}_g \overset{\text{canonical isomorphism}}{=\!=\!=} Sym^2(E_g) \subset E_g^{\otimes 2}$, canonical Kähler metric on $\mathbb{A}_g$ $\Rightarrow$ Laplacian on $\mathbb{A}_g$ Δ

$T^*\mathbb{A}_g = Sym^2(E_g^*) \twoheadleftarrow (E_g^*)^{\otimes 2}$

Jac : $\mathbb{M}_g \to \mathbb{A}_g$, $C \mapsto Jac(C)$ (period matrix), period map, injective (Torelli theorem)

($\mathbb{M}_2 \underset{Jac}{\overset{open}{\subset}} \mathbb{A}_2$, $\mathbb{M}_3 \underset{Jac}{\overset{open}{\subset}} \mathbb{A}_3$ $\Rightarrow$ Δ makes sense for functions on $\mathbb{M}_2$ and $\mathbb{M}_3$)

$(Jac^* E_g)|_{[C]} = H^0(C;\Omega^1_C) = \{$holomorphic 1-forms on $C\}$

$Jac^*(T^*\mathbb{A}_g)|_{[C]} \twoheadleftarrow Jac^*(E^*)^{\otimes 2}|_{[C]} = H^0(C;\Omega^1_C)^{\otimes 2}$

(In general, X: Kähler manifold, $\Lambda : T^*X \otimes \overline{T^*X} \longrightarrow X \times \mathbb{C}$, Kähler contraction

$\Longrightarrow \quad \Delta f = 2\sqrt{-1}\,\Lambda \partial \bar{\partial} f \qquad \forall f \in C^\infty(X : \mathbb{C})$)

Since $\psi_A' + \psi_B = (\mathrm{const})\, e_1^P$, we interprete $\Delta \varphi |_{\psi_A' + \psi_B}$ as $(\mathrm{const})\, \Lambda \hat{e}_1^P$, where

$\hat{e}_1^P \in C^\infty(\mathbb{M}_g : \mathrm{Jac}^*(T^*\mathbb{A}_g \otimes \overline{T^*\mathbb{A}_g}))$ <u>lift of</u> e_1^P defined by

$$\hat{e}_1^P \Big|_{[C]} := 48\sqrt{-1} \sum_{i,j,k,l} \left((\psi_l \otimes \psi_j) A_{i\bar{j}k\bar{l}} (\overline{\psi_i} \otimes \overline{\psi_k}) - (\psi_l \otimes \psi_j) A_{i\bar{j}k\bar{i}} (\overline{\psi_k} \otimes \overline{\psi_l}) \right)$$

Then we have

$$\frac{1}{48} \Lambda \hat{e}_1^P = \sum_{s,t} (A_{s\bar{t}t\bar{s}} + A_{s\bar{s}t\bar{t}}) - \sum_{s,t} A_{s\bar{t}t\bar{s}} - g \sum_{s,t} A_{s\bar{t}t\bar{s}} = -g\, a_g$$

$$\therefore \quad \boxed{\frac{1}{48} \Lambda \hat{e}_1^P = -g\, a_g} \quad \cdots\cdots (**)$$

<u>Recall</u>: Rauch's formula for $d\,\mathrm{Jac}$

<u>Theorem</u> (Rauch) $\forall [C] \in \mathbb{M}_g$

$$\begin{array}{ccc} T^*_{[C]}\mathbb{M}_g & \xleftarrow{(d\,\mathrm{Jac})^*} & T^*_{[\mathrm{Jac}(C)]}\mathbb{A}_g \\ \| & \circlearrowright & \| \\ H^0(C : (\Omega_C^1)^{\otimes 2}) & \xleftarrow[\text{multiplication}]{} & \mathrm{Sym}^2 H^0(C : \Omega_C^1) \end{array}$$

$\Downarrow \Leftarrow$ M. Noether's Theorem

<u>Corollary</u> $(d\,\mathrm{Jac})_{[C]}$ is injective if and only if $\begin{cases} g = 2 \text{ or} \\ g \geqq 3 \text{ and } C : \text{non-hyperelliptic} \end{cases}$

<u>Today</u>: Consider the case $g=3$, $\mathbb{M}_3 \underset{\text{Jac}}{\overset{\text{open}}{\subset}} \mathbb{A}_3 \rightsquigarrow \begin{cases} Sp_6(\mathbb{R})\text{-inv Laplacian } \Delta \\ \text{Kähler contraction } \Lambda \end{cases}$

<u>Theorem</u> (K.; conjectured by physicists)

a_3 is <u>not</u> an eigenfunction of Δ

- $\mathrm{Jac}|_{\mathbb{M}_3 \setminus \mathcal{H}_3} : \mathbb{M}_3 \setminus \mathcal{H}_3 \to \mathbb{A}_3$ locally isomorphism
- $(*\text{bis})|_{g=3}$: $\frac{1}{12} e_1^P = -\frac{3}{7}\sqrt{-1}\,\partial\bar\partial a_3 + \frac{1}{12} e_1^J$

$$\Downarrow \ \Lambda \text{ on } \mathbb{M}_3 \setminus \mathcal{H}_3$$

$$-12\, a_3 = -\frac{12}{7}\Delta a_3 + \frac{1}{12}\Lambda e_1^J$$

$$\Delta a_3 = 7 a_3 + \frac{7}{144}\Lambda e_1^J, \qquad \Lambda e_1^J \in C^\infty(\mathbb{M}_3 \setminus \mathcal{H}_3; \mathbb{R})$$

<u>Claim</u> $[C_0] \in \mathcal{H}_3$ as in <u>Theorem</u> (Harris)

$\Rightarrow$ $|\Lambda e_1^J| \to +\infty$ as $[C] \to [C_0]$

$\Rightarrow$ Λe_1^J is not proportional to a_3 ($\because$ a_3 is locally bounded near $[C_0]$)

$\Rightarrow$ Theorem //

<u>proof of Claim</u> $H = H^1(C;\mathbb{C}) \underset{\text{duality}}{\overset{\text{Poincaré}}{=\!=}} H_1(C;\mathbb{C})$

$U := \mathrm{Ker}(\Lambda^3 H \xrightarrow{\text{contraction}} H) < \Lambda^3 H < H^{\otimes 3}$

$$\begin{tikzcd} \Lambda^3 H \arrow[r, "Q"] & T^*_{[C,P_0]}\mathbb{C}_g \\ U \arrow[u, phantom, "\cup"] \arrow[r, "\exists! Q"] \arrow[ur, phantom, "\circlearrowleft"] & T^*_{[C]}\mathbb{M}_g \arrow[u, "\pi^*"] \end{tikzcd} \qquad (Q \overset{\text{dual}}{\longleftrightarrow} \tilde{k})$$

indep. of the choice of P_0

$\langle\ ,\ \rangle : H^{\otimes 3} \times H^{\otimes 3} \to \mathbb{C}$, $\langle u_1^{(1)} \otimes u_2^{(1)} \otimes u_3^{(1)}, u_1^{(2)} \otimes u_2^{(2)} \otimes u_3^{(2)} \rangle := \prod_{i=1}^{3} (u_i^{(1)} \cdot u_i^{(2)})$

intersection number

$e_1^J = (\text{const})\langle \tilde{k}, \tilde{k} \rangle$ on $T_{[C]}\mathbb{M}_g \otimes \overline{T_{[C]}\mathbb{M}_g}$

$$H = H^1(C:\mathbb{C}) = \underset{\text{holo.}}{H^0(C:\Omega^1_C)} \oplus \underset{\text{anti-holo}}{\overline{H^0(C:\Omega^1_C)}}$$

$U = U^{3,0} \oplus U^{2,1} \oplus U^{1,2} \oplus U^{0,3}$, where $U^{p,q} := U \cap \left(\begin{array}{l}\text{contribution}\\ \text{of } (\text{holo})^{\otimes p} \otimes (\text{anti holo})^{\otimes q}\end{array} \right)$

$\tilde{k}(T_{[C]}\mathbb{M}_g) \subset (U^{2,1})^*$ (∵ definition of Q (dual of $\hat{k}$))

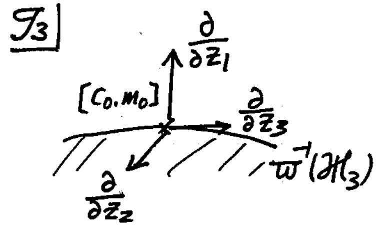
$0 \neq {}^{\forall}v \in U^{2,1}$, $\sqrt{-1}\langle v, \bar{v} \rangle > 0$

- $\mathcal{T}_g$: the Teichmüller space of genus g $\ni [C, m]$ ← marking of C

$$\begin{array}{ccc} \mathcal{T}_g & \ni & [C,m] \\ \varpi \downarrow & & \downarrow \\ \mathbb{M}_g & \ni & [C] \end{array}$$

$$\begin{array}{ccc} \mathcal{T}_g & \xrightarrow{\exists !\, Jac} & \mathfrak{H}_g \\ \varpi \downarrow & \circlearrowleft & \downarrow \text{quotient} \\ \mathbb{M}_g & \xrightarrow[Jac]{} & \mathbb{A}_g = \mathfrak{H}_g / Sp_{2g}(\mathbb{Z}) \end{array}$$

- $\underline{g=3}$ Choose a marking m_0 of C_0. Then



$\exists\ (z_1, z_2, z_3)$: complex coordinate centered at $[C_0, m_0] \in \mathcal{T}_3$

$\exists\ (w_1, w_2, w_3)$ at $Jac([C, m_0]) \in \mathfrak{H}_3$

s.t.
- $\{z_1 = 0\} \overset{loc.}{=} \varpi^{-1}(\mathcal{H}_3)$
- $Jac(z_1, z_2, z_3) = (z_1^2, z_2, z_3) = (w_1, w_2, w_3)$ (a 2 to 1 map in the normal direction of $\mathcal{H}_3$)

$$\longrightarrow \quad \frac{\partial}{\partial w_1} = \frac{1}{2z_1}\frac{\partial}{\partial z_1}$$

$\exists\ v_1, v_2, v_3$: $\mathbb{U}^{2,1}$-valued function defined around $[C_0, m_0] \in \mathcal{J}_3$

s.t. $\tilde{k} = v_1\, dz_1 + v_2\, dz_2 + v_3\, dz_3$ on $T\mathcal{J}_g$

Then. $v_1(0,*,*) \neq 0$ ($\Leftarrow$ Thm (Harris))

$v_2(0,*,*) = v_3(0,*,*) = 0$ ($\Leftarrow \tilde{k} = 0$ along $\mathcal{H}_3$)

$$e_1^J = (\text{const}) \sum_{i,j=1}^{3} \langle v_i, \overline{v_j} \rangle\, dz_i\, \overline{dz_j}$$

Since $\frac{\partial}{\partial w_1} = \frac{1}{2z_1} \frac{\partial}{\partial z_1}$, $|z_1| \cdot |\Lambda e_1^J|$ is bounded from below by a combination of

$\frac{1}{|z_1|} |\langle v_1, \overline{v_1} \rangle|$ and

$|\langle v_1, \overline{v_j} \rangle|$, $|\langle v_i, \overline{v_1} \rangle|$, $|z_1|\, |\langle v_i, \overline{v_j} \rangle|$, $(i, j = 2.3) \longrightarrow 0$ as $z_1 \to 0$

with coefficients in locally bounded functions near $[C_0]$ $(\because\ v_2, v_3 \to 0)$

$\left\{ \begin{array}{l} \text{the coefficient of } \frac{1}{|z_1|} |\langle v_1, \overline{v_1} \rangle| \text{ is positive } (\because \text{ definition of } \Lambda) \\ \sqrt{-1} \langle v_1, \overline{v_1} \rangle > 0 \ (\because\ v_1 \neq 0) \end{array} \right.$

$$\Longrightarrow \quad |z_1|\, |\Lambda e_1^J| \geqq \frac{(\text{positive const})}{|z_1|} |\langle v_1, \overline{v_1} \rangle| \longrightarrow +\infty \quad \text{as } z_1 \to 0 \quad \Rightarrow \text{Claim} \Rightarrow \text{Theorem} //$$

<u>Next problem</u> : $\forall g \geq 3$. $\mathcal{H}_g \overset{\text{Jac}}{\hookrightarrow} \mathbb{A}_g$ embedding,

canon. Kähler metric on $\mathbb{A}_g \underset{\text{induces}}{\overset{\text{Jac}}{\Longrightarrow}}$ Kähler metric on $\mathcal{H}_g \Rightarrow$ Laplacian $\Delta_{\mathcal{H}_g}$.

$\Delta_{\mathcal{H}_g}(a_g|_{\mathcal{H}_g}) = ?$