Berkeley - Tokyo Winter School "Geometry, Topology and Representation Theory"
Common Room 1015, Evans Hall, Department of Mathematics, University of California, Berkeley

"Mapping class groups and the Johnson homomorphisms"

Nariya Kawazumi (University of Tokyo)

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I. Mapping Class Groups

§ 1. 1. Surfaces

a surface = a connected oriented 2-dim. C^∞ manifold

Classification of closed surfaces

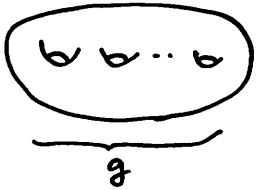
S : closed surface.

$\Longrightarrow \exists g \geq 0, \ H_1(S:\mathbb{Z}) = \pi_1(S)^{abel} \cong \mathbb{Z}^{2g}$ ($\because$ Intersection form on $H_1(S:\mathbb{Z})$ is anti-symmetric and non-degenerate)

$g(S) \overset{\text{def}}{=} g = \frac{1}{2} \operatorname{rank} H_1(S:\mathbb{Z})$

genus of the surface S

Theorem (Classical ; Classification Theorem)

$S \underset{C^\infty \text{diffeo.}}{\cong} \Sigma_g :=$ 

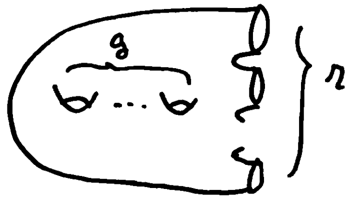
Classification of compact surfaces : by the 2 data

① $r := \# \pi_0(\partial S)$, the number of boundary components

$\Rightarrow \partial S \underset{C^\infty}{\cong} \underbrace{S^1 \amalg \cdots \amalg S^1}_{r}$ disjoint union of r copies of the circle S^1

$\Rightarrow \overline{S} := S \underset{\partial S}{\cup} (r \text{ copies of } D^2)$: closed surface

② $g := \text{genus}(\overline{S})$

Theorem $S \underset{C^\infty}{\cong} \Sigma_{g,r} :=$ 

($\Leftarrow$ the previous theorem $\oplus$ Disk Theorem)

Kerékjártó (1920's) : Classification of non-compact surfaces

◎ topological classification of a single surface

$\Downarrow$

topological classification of a family of surfaces

What is a family of surfaces ?

M, B : oriented C^∞ manifolds, $\dim M - \dim B = 2$,

$\varpi : M \to B$. a surjective proper C^∞ map, $\partial M = \varpi^{-1}(\partial B)$

($\Updownarrow$ $\forall K \subset B$ compact, $\varpi^{-1}(K)$ compact)

$\mathrm{Crit}(\varpi) \overset{\text{def}}{=} \{$ critical values of $\varpi \} \subset B$ measure zero (Sard's Thm)

$= \{ t \in B : {}^{\exists} p \in \varpi^{-1}(t), (d\varpi)_p : T_p M \to T_t B$ not surjective $\}$

$B_0 := \underbrace{B \setminus \mathrm{Crit}(\varpi)}_{\text{regular values}}, \quad M_0 := \varpi^{-1}(B_0)$

$\pi := \varpi|_{M_0} : M_0 \to B_0$, C^∞ proper submersion, $\partial M_0 = \varpi^{-1}(\partial B_0)$

$\Longrightarrow$ C^∞ fiber bundle (Ehresmann Thm)

$$\left(\text{i.e., } \forall t_1 \in B_0 \quad t_1 \in {}^{\exists} U \overset{\text{open}}{\subset} B_0 \qquad \begin{array}{ccc} \pi^{-1}(U) & \overset{C^\infty}{\cong} & U \times \pi^{-1}(t_1) \\ & {\scriptstyle \pi} \searrow \circlearrowleft \swarrow & \\ & U & \end{array} \quad \begin{array}{c} (t,p) \\ \swarrow \\ t \end{array} \right)$$

$\forall t_1 \in B_0$, $\pi^{-1}(t_1)$: closed oriented 2-dim. C^∞ mfd

may assume $\pi^{-1}(t_1)$: connected $(\Longrightarrow {}^{\exists} g \geq 0,\ \pi^{-1}(t_1) \cong \Sigma_g)$

($\because$) If necessary, take a finite covering of B_0)

$\Longrightarrow$ $\pi : M_0 \to B_0$ "family of surfaces"

"The 1st step" of topological classification of families of surfaces :

monodromy of $\pi : M_0 \to B_0$

Fix a point $t_0 \in B_0$, $S := \pi^{-1}(t_0)$ $(\cong \Sigma_g,\ {}^{\exists} g \geq 0)$

$\forall \ell : ([0,1], \{0,1\}) \to (B_0, t_0)$ C^∞ loop

$\ell^* M_0 := [0,1] \times_{B_0} M_0 = \{(s,p) \in [0,1] \times M_0 : \ell(s) = \pi(p)\}$ pullback of $\pi : M_0 \to B_0$

$$\exists\ \Phi_\ell : [0,1] \times S \xrightarrow{\cong} \ell^* M_0$$

isomorphism of C^∞ fiber bundles $/[0,1]$

(Feldbau Thm)

$$\begin{tikzcd}[0,1] \times S \arrow[rr, "\Phi_\ell", "\cong"'] \arrow[dr] & \circlearrowleft & \ell^* M_0 \arrow[dl] \\ & {[0,1]} & \end{tikzcd}$$

$$(s,q) \mapsto s \qquad (s,p) \mapsto s$$

$(\text{monodromy along } \ell) := \Phi_\ell(1,\cdot)^{-1} \circ \Phi_\ell(0,\cdot) : S \xrightarrow{\cong} S$

orientation-preserving diffeomorphism

$\in \mathrm{Diff}^+(S) := \{\varphi : S \to S \text{ orientation-preserving diffeomorphisms}\}$

Want to construct : $\pi_1(B_0, t_0) \to \mathrm{Diff}^+(S)$, $[\ell] \mapsto \Phi_\ell(1,\cdot)^{-1} \circ \Phi_\ell(0,\cdot)$

- NOT well-defined, i.e., depends on the choice of ℓ and Φ_ℓ
- we need to take some quotient of $\mathrm{Diff}^+(S)$

$\Downarrow$

mapping class group

§ 1.2. Definition of the mapping class group

S : <u>compact</u> surface (may have non-empty ∂S)

$\mathrm{Diff}^+(S,\partial S) := \{\varphi : S \to S$; orientation-preserving diffeomorphism ; $\varphi|_{\partial S} = 1_{\partial S}\}$

$\mathrm{Diff}_0(S,\partial S) := \{\varphi \in \mathrm{Diff}^+(S,\partial S)$; $\exists$ isotopy connecting φ to 1_S fixing ∂S <u>pointwise</u> $\}$

<u>normal</u> subgroup in $\mathrm{Diff}^+(S,\partial S)$

$\mathcal{M}(S) \overset{\text{def}}{=} \mathrm{Diff}^+(S,\partial S)/\mathrm{Diff}_0(S,\partial S)$ the <u>mapping class group</u> of the surface S

<u>examples</u> $\mathcal{M}(\Sigma_0) = \mathcal{M}(\Sigma_{0,1}) = \{1\}$ $(\Sigma_0 = S^2)$

(<u>Smale</u> $\mathrm{Diff}^+(\Sigma_0) \simeq SO_3$, $\mathrm{Diff}^+(\Sigma_{0,1}) \simeq *$)

exceptional $\mathcal{M}(\Sigma_1) = SL_2(\mathbb{Z})$ $(\Sigma_1 = T^2 = S^1 \times S^1)$

($\mathrm{Diff}^+(\Sigma_1) \simeq T^2 \rtimes SL_2(\mathbb{Z})$)

$\mathcal{M}(\Sigma_{1,1}) = B_3$ Artin braid group of 3-strands

$\pi : M_0 \to B_0$: C^∞ proper submersion , $\dim M_0 - \dim B_0 = 2$

<u>Assume</u> $\partial M_0 = \pi^{-1}(\partial B_0)$

M_0, B_0 : oriented

$\forall t \in B_0$ $\pi^{-1}(t)$: connected

B_0 : <u>connected</u> ← new assumption

$t_0 \in B_0$ basepoint, $\quad S := \pi^{-1}(t_0)$ closed surface

$\ell : ([0,1], \{0,1\}) \longrightarrow (B_0, t_0)$ C^∞ loop, $\quad \exists\, \Phi_\ell : [0,1] \times S \xrightarrow{\cong} \ell^* M_0$. isom. of C^∞ fiber bundles

$\rho([\ell]) := [\Phi_\ell(1,\cdot) \circ \Phi_\ell(0,\cdot)^{-1}] \in \mathcal{M}(S)$ monodromy of π along $[\ell]$

$\rho : \pi_1(B_0, t_0) \longrightarrow \mathcal{M}(S)$, $[\ell] \longmapsto \rho([\ell])$ monodromy of π

well-defined anti-homomorphism of groups

i.e., $\quad \rho([\ell' \cdot \ell'']) = \rho([\ell'']) \circ \rho([\ell']) \in \mathcal{M}(S)$

where $\quad (\ell' \cdot \ell'')(s) \overset{\text{def}}{=} \begin{cases} \ell'(2s) & \text{if } 0 \le s \le 1/2 \\ \ell''(2s-1) & \text{if } 1/2 \le s \le 1 \end{cases}$

Change of the basepoint $t_0 \in B_0$

$t_0' \in B_0$, $\quad S' := \pi^{-1}(t_0')$

$\exists\, \ell_0 : ([0,1], 0, 1) \to (B_0, t_0, t_0')$ C^∞ path $\quad (\because\ B_0 : \text{connected})$

$\pi_1(B_0, t_0) \cong \pi_1(B_0, t_0')$, $\quad [\ell] \longmapsto [\ell_0]^{-1}[\ell][\ell_0]$ $\quad$ isom of groups

$\exists\, \Phi_{\ell_0} : [0,1] \times S \xrightarrow{\cong} \ell_0^* M_0$ isom. of C^∞ fiber bundles, $\quad \rho_0 := \Phi_{\ell_0}(1,\cdot) \circ \Phi_{\ell_0}(0,\cdot)^{-1} : S \to \pi^{-1}(t_0') = S'$
ori. pres. diffeo

$$\begin{array}{ccccc}
[\ell] & \pi_1(B_0, t_0) & \xrightarrow{\ \rho\ } & \mathcal{M}(S) & [\varphi] \\
\downarrow & \wr\| & \circlearrowleft & \wr\| & \downarrow \\
[\ell_0]^{-1}[\ell][\ell_0] & \pi_1(B_0, t_0') & \xrightarrow[\ \rho\ ]{} & \mathcal{M}(S') & [\rho_0 \varphi \rho_0^{-1}]
\end{array}$$

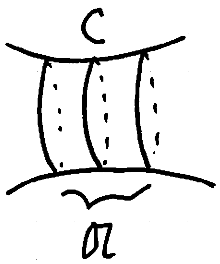
In particular, these 2 monodromy are conjugate to each other.

§ 1.3. Dehn twists

the most basic degeneration of Riemann surfaces $\Rightarrow$ Dehn twist

S : surface endowed with a complex structure i.e., Riemann surface

$C \subset S \setminus \partial S$: simple closed curve (i.e., closed submfd diffeo to S^1)



$C \subset {}^{\exists}\Omega \subset S$, $0 < {}^{\exists} r < 1$ s.t. $(\Omega, C) \underset{\text{holomorphic}}{\overset{\alpha}{\cong}} (A_r, \{|z| = \sqrt{r}\})$

where $A_r := \{z \in \mathbb{C} ; r \leq |z| \leq 1\}$ annulus $\beta(u,v) := e^{2\pi\sqrt{-1}u - v \log r}$

$$\underset{C^\infty}{\overset{\beta}{\cong}} (\mathbb{R}/\mathbb{Z}) \times [0,1] \ni (u,v)$$

$\Delta := \{t \in \mathbb{C} ; |t| \leq r\}$

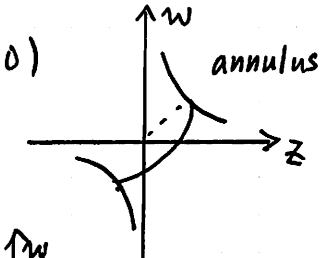
$\mathcal{R} := \{(z,w,t) \in \mathbb{C}^3 ; zw = t, \max\{|z|, |w|\} \leq 1, t \in \Delta\}$ 2-dim$_{\mathbb{C}}$ complex manifold

$$\begin{array}{ccc} \mathcal{R} & \ni & (z,w,t) \\ \varpi \downarrow & & \downarrow \\ \Delta & \ni & t \end{array}$$

$R_t := \varpi^{-1}(t)$, $t \in \Delta$.

$(d\varpi)_{(z,w,t)} = 0 \Leftrightarrow (z,w,t) = (0,0,0)$

R_t $(t \neq 0)$

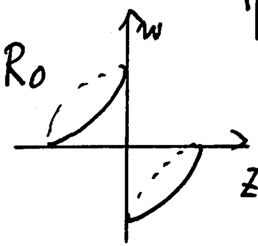


annulus

$\Phi : (\Omega \setminus C) \times \Delta \longrightarrow \mathcal{R}$ holomorphic embedding

$$(\alpha(z), t) \longmapsto \begin{cases} (z, tz^{-1}) & \text{if } \sqrt{r} < |z| \leq 1 \\ (\frac{t}{r} z, r z^{-1}) & \text{if } r \leq |z| < \sqrt{r} \end{cases}$$

R_0



2 disks intersecting at 1 point

at $t = r$.

$\Phi : (\Omega \setminus C) \times \{r\} \underset{\cong}{\rightarrow} R_r$ biholomorphic

$\subset \Omega \cong$

$$M \overset{\text{def}}{=} R \underset{\Phi}{\cup} ((S \setminus C) \times \Delta)$$

$\downarrow \varpi$

Δ

$\mathrm{Crit}(\varpi) = \{0\}$, C degenerates into a 1 point $(z, w, t) = (0, 0, 0)$ at $t = 0$

$\Delta_0 := \Delta \setminus \{0\} = \{$regular values of $\varpi\}$ $\rightsquigarrow$ C : "vanishing cycle"

$M_0 := \varpi^{-1}(\Delta_0)$ $\quad \| \{t \in \mathbb{C} : 0 < t \leq r\}$

$\pi := \varpi|_{M_0} : M_0 \to \Delta_0$ $\quad C^\infty$ proper submersion

<u>monodromy of</u> $\pi : M_0 \to \Delta_0$ $\Leftarrow$ <u>monodromy of</u> $\pi := \varpi|_{R_0} : R_0 := \varpi^{-1}(\Delta_0) \to \Delta_0$

$\ell : [0,1] \to \Delta_0$, $s \mapsto r e^{2\pi\sqrt{-1}s}$, C^∞ loop, generator of $\pi_1(\Delta_0, r) \cong \mathbb{Z}$

$\Psi : A_r \times [0,1] \longrightarrow R$

$$(z, s) \longmapsto \left(z \exp\left(2\pi\sqrt{-1}s \frac{\log|z|}{\log r}\right), \; r z^{-1} \exp\left(2\pi\sqrt{-1}s \left(\frac{\log r - \log|z|}{\log r}\right)\right), \; r e^{2\pi\sqrt{-1}s} \right)$$

$\Psi|_{\{|z|=1\} \cup \{|z|=r\}} = id$, Ψ induces $A_r \times [0,1] \underset{C^\infty}{\cong} \ell^* R$ isom of C^∞ fiber bundles

$\Phi_\ell : \Psi \cup_\Phi (1_{S \setminus C} \times 1_{[0,1]}) : S \times [0,1] \xrightarrow{\cong} \ell^* M_0$ isom of C^0 fiber bundles

($\xrightarrow{\text{some smoothing}}$ isom of C^∞ fiber bundles)

$\Phi_\ell(z, 0) = (z, r z^{-1})$ $\quad (\forall z \in A_r)$

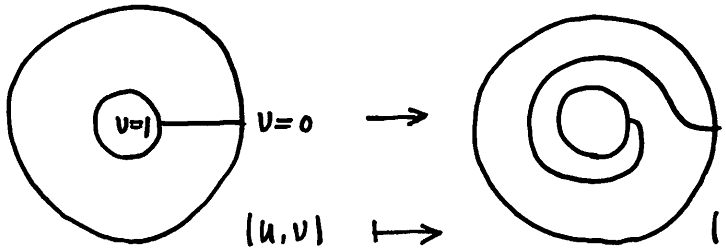
$$\Phi_\ell(z, 1) = \left(z \exp\left(2\pi\sqrt{-1} \frac{\log|z|}{\log r}\right), \; r z^{-1} \exp\left(2\pi\sqrt{-1} \left(\frac{\log r - \log|z|}{\log r}\right)\right) \right)$$

(monodromy along ℓ) :

$$\begin{array}{ccc} z \in A_r & \longmapsto & z \exp\left(2\pi\sqrt{-1} \frac{\log|z|}{\log r}\right) \in A_r \\ \beta \updownarrow \cong & \circlearrowleft & \beta \updownarrow \cong \\ (u, v) \in (\mathbb{R}/\mathbb{Z}) \times [0,1] & \longmapsto & (u+v, v) \in (\mathbb{R}/\mathbb{Z}) \times [0,1] \end{array}$$

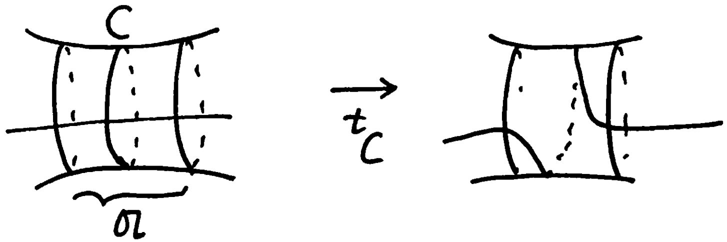
$\leftarrow$ Conclusion!!

$(\mathbb{R}/\mathbb{Z}) \times [0,1]$ $\qquad$ $(\mathbb{R}/\mathbb{Z}) \times [0,1]$



$(u, v) \longmapsto (u+v, v)$

right-handed Dehn twist



$t_C \in \mathcal{M}(S)$
right-handed Dehn twist along C
(depends only on the isotopy class of C)

(<u>Conclusion</u> monodromy of M_0 along ℓ $= t_C \in \mathcal{M}(S)$ Dehn twist along C

<u>Action of t_C on the homology group</u> $H_1(S:\mathbb{Z})$

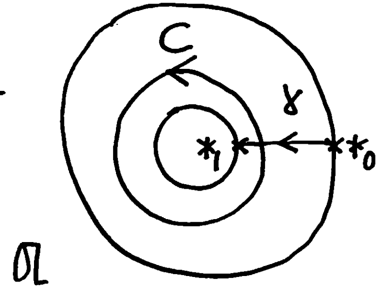
<u>Theorem</u> (Picard-Lefschetz formula) $\forall X \in H_1(S:\mathbb{Z})$

$$(t_C)_* X = X - (X \cdot [C])[C] \in H_1(S:\mathbb{Z}),$$

where $(\cdot) : H_1(S:\mathbb{Z}) \times H_1(S:\mathbb{Z}) \to \mathbb{Z}$ algebraic intersection number

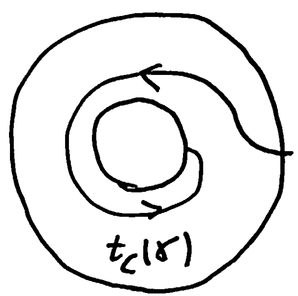
$[C] \in H_1(S:\mathbb{Z})$ homology class of C (with an orientation)

<u>proof</u>



It suffices to show the formula in $H_1(\Omega, \{*_0, *_1\}) = \mathbb{Z}[\gamma] \oplus \mathbb{Z}[C]$

$[\gamma]\cdot[C] = -1$



$$\begin{cases} (t_C)_*[C] = [C] = [C] - ([C]\cdot[C])[C] \\ (t_C)_*[\gamma] = [\gamma] + [C] = [\gamma] - ([\gamma]\cdot[C])[C] \end{cases}$$

//

(<u>tomorrow</u> : "explicit" description of the action of t_C on $\mathbb{Q}\pi_1(S)$)

In the case $S = \Sigma_{g,1}$, $H_{\mathbb{Z}} := H_1(\Sigma_{g,1} : \mathbb{Z})$, $H := H_1(\Sigma_{g,1} : \mathbb{Q})$

$Sp(H_{\mathbb{Z}}) := \{U \in GL(H_{\mathbb{Z}}) : \forall X, \forall Y \in H_{\mathbb{Z}}, (UX)\cdot(UY) = X\cdot Y\}$

$\cap$

$Sp(H) := \{U \in GL(H) : \forall X, \forall Y \in H, (UX)\cdot(UY) = X\cdot Y\}$

<u>Classical Result</u> $Sp(H_{\mathbb{Z}})$ is generated by $\{(t_C)_* : C \subset \Sigma_{g,1}$: simple closed curves$\}$

<u>Corollary</u> $\mathcal{M}(\Sigma_{g,1}) \to Sp(H_{\mathbb{Z}})$, $\varphi \mapsto \varphi_*$. surjective

<u>Theorem</u> (Dehn-Lickorish) $\forall S$: <u>compact</u> surface

$\mathcal{M}(S)$ is generated by $\{t_C : C \subset S \setminus \partial S$ simple closed curve$\}$

§ 1.4. Classifications of families of surfaces

Topologically the monodromy completely classifies the families of surfaces if $\chi(S) < 0$

1) degenerations of Riemann surfaces on $\Delta = \{z \in \mathbb{C} ; |z| \ll 1\}$ $(g \geq 2)$
2) fiber bundles with fiber Σ_g $=: \Sigma_g$-bundles $(g \geq 2)$

1)

Theorem (Matsumoto-Montesinos) $g \geq 2$

(1) $\left\{ \begin{array}{l} \varpi : M \to \Delta \text{ ; proper holomorphic map, } \dim_{\mathbb{C}} M = 2, \ \mathrm{Crit}(\varpi) = \{0\} \\ \text{exceptional divisor} \not\subset {}^{\forall}\text{fiber} \\ {}^{\forall} t \in \Delta \setminus \{0\},\ \varpi^{-1}(t) \underset{C^\infty}{\cong} \Sigma_g \end{array} \right\} \Big/ \text{topological isom}$

$\xrightarrow[\text{monodromy}]{} \mathcal{M}(\Sigma_g)/\text{conjugate}$ <u>injective</u>

(2) the image $= \{$ pseudo-periodic maps of negative twist $\}$

2) $S = \Sigma_g$: closed surface of genus $g \geq 2$.

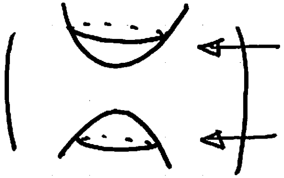
$$\mathrm{CpxStr}(S) := \left\{ \begin{array}{l} J : TS \to TS \text{ ; } C^\infty \text{ homom. of vect. b'dles}/S \ \ (\text{over } S) \\ J^2 = -1, \quad {}^{\forall} p \in S,\ 0 \neq {}^{\forall} v \in T_p S \\ \{v, Jv\} \subset T_p S \text{ positive basis} \end{array} \right\}$$

(<u>Remark</u> Since $\dim_{\mathbb{R}} S = 2$, $\forall J \in \mathrm{Cpx\,Str}(S)$ almost complex str is <u>integrable</u>)

$\mathrm{Cpx\,Str}(S) \simeq *$ contractible

$(\because)$ $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \ ; \ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$

$= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in GL_2(\mathbb{R}) : a^2+bc=0 \right\} =$



$\simeq * \amalg *$

$\mathrm{Diff}_0(S)\ (= \mathrm{Diff}_0(S, \partial S)) \curvearrowright \mathrm{Cpx\,Str}(S)$

free action $\left(\Leftarrow g \geqq 2 \right)$, Abel-$\overset{g \geqq 1}{\text{Jacobi}}$ map, Letschetz $\overset{2-2g \neq 0}{\text{fixed}}$ point theorem)

$\mathcal{T}(S) \overset{\text{def}}{=} \mathrm{Cpx\,Str}(S) / \mathrm{Diff}_0(S)$ Teichmüller space

$\approx \mathbb{R}^{6g-6}$ (Teichmüller) $\simeq *$

$\mathrm{Diff}_0(S) \to \mathrm{Cpx\,Str}(S) \to \mathcal{T}(S)$ fiber bundle (Earle-Eells)

$\qquad\qquad\qquad\qquad \mathrm{Cpx\,Str}(S) \simeq * , \quad \mathcal{T}(S) \simeq *$

(<u>Theorem</u> (Earle-Eells)

$\mathrm{Diff}_0(S) \simeq *$ contractible

<u>Rmk</u> $\forall$ compact surface S, $\mathrm{Diff}_0(S, \partial S) \simeq *$ (Earle-Schatz)

G : group (with discrete topology)
the Eilenberg-MacLane space of type $(G,1)$
$K(G,1)$ is characterized by
$$\pi_q(K(G,1)) = \begin{cases} G & \text{if } q=1 \\ 0 & \text{if } q \neq 1 \end{cases}$$
unique up to (weak) htpy equiv

$$\mathrm{Diff}_0(S) \simeq *$$
$$\Rightarrow \mathrm{Diff}^+(S) \simeq \mathcal{M}(S) \left(= \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)\right)$$
$$\Rightarrow \underset{\text{classifying space}}{B\mathrm{Diff}^+(S)} \simeq \underset{\text{the Eilenberg-MacLane space of type } (\mathcal{M}(S),1)}{K(\mathcal{M}(S),1)}$$

i.e., $\forall X$: (path-conn) paracompact space homotopic to a $\exists$ CW complex

$$\left\{ \begin{matrix} \pi : E \to X \\ \text{oriented } \Sigma_g\text{-bundles} \end{matrix} \right\} \Big/ \text{topological isom}/X \cong [X, K(\mathcal{M}(\Sigma_g),1)]$$
$$\xrightarrow[\cong]{\text{monodromy}} \mathrm{Hom}(\pi_1(X), \mathcal{M}(\Sigma_g))/\text{conjugate}$$

<u>The monodromy completely classifies oriented Σ_g-bundles.</u>

$$\left\{ \begin{matrix} \text{The Characteristic Classes} \\ \text{of oriented } \Sigma_g\text{-bundles} \end{matrix} \right\} = H^*(B\mathrm{Diff}^+(S)) \cong H^*(K(\mathcal{M}(\Sigma_g),1))$$

the group cohomology algebra of the mapping class group $\mathcal{M}(\Sigma_g)$

§1.5. Mumford-Morita-Miller classes

<u>the Mumford-Morita-Miller classes</u> (the MMM classes)

$\pi : E \to X$ $\quad C^\infty$ oriented Σ_g-bundle

$T_{E/X} := \mathrm{Ker}(d\pi : TE \to \pi^* TX)$ the relative tangent bundle of π

oriented $\mathbb{R}$-vector bundle of rank 2 over E ($\because$ $\pi : E \to X$ oriented)

$e(T_{E/X}) \in H^2(E ; \mathbb{Z})$ the Euler class of $T_{E/X}$

$\int_{\mathrm{fiber}} : H^*(E;\mathbb{Z}) \to H^{*-2}(X:\mathbb{Z})$ integration along the fibers = the Gysin map

$i \geq 1$.

$$e_i(E) \left(= (-1)^{i+1}\kappa_i(E)\right) \overset{\mathrm{def}}{=} \int_{\mathrm{fiber}} e(T_{E/X})^{i+1} \in H^{2i}(X:\mathbb{Z})$$

the i^{th} MMM class $= (-1)^{i+1}$ the i^{th} tautological class

Theorem (Madsen-Weiss) If $\underbrace{* < \frac{2}{3} g}_{\text{stable range}}$, then

$$H^*(K(\mathcal{M}(\Sigma_g),1), \mathbb{Q}) = H^*(B\mathrm{Diff}^+(\Sigma_g) : \mathbb{Q}) \cong \mathbb{Q}[e_i \,;\, i \geq 1]$$

In general, for a group G with discrete topology,

the cohomology group $H^*(K(G,1))$ can be described in an algebraic way

G : a group, M : left G-module

$p \geq 0$, $C^p(G:M) := \{ c : \underbrace{G \times \cdots \times G}_{p} \to M \text{ ; map, } c(\cdots, {}^{\exists}1, \cdots) = 0 \}$

$d : C^p(G:M) \to C^{p+1}(G:M)$, $c \mapsto dc$. $(\gamma_i \in G)$

$$(dc)(\gamma_0, \gamma_1, \cdots, \gamma_p) := \gamma_0(c(\gamma_1, \cdots, \gamma_p)) + \sum_{i=0}^{p-1} (-1)^{i+1} c(\gamma_0, \cdots, \gamma_i\gamma_{i+1}, \cdots \gamma_p) + (-1)^{p+1} c(\gamma_0, \cdots, \gamma_{p-1})$$

$dd = 0$, $C^*(G:M) := \{ C^p(G:M), d \}_{p \geq 0}$ the normalized standard cochain complex of G with values in M

$H^p(G:M) \overset{\text{def}}{=} H^p(C^*(G:M))$ the p^{th} cohomology group of G with values in M

cup product M', M'': left G-modules

$c' \in C^p(G:M')$, $c'' \in C^q(G:M'')$

$c' \cup c'' \in C^{p+q}(G : M' \otimes M'')$ Alexander-Whitney cup product

$$(c' \cup c'')(\gamma_1, \cdots, \gamma_{p+q}) := c'(\gamma_1, \cdots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p \, c''(\gamma_{p+1}, \cdots, \gamma_{p+q}) \in M' \otimes M'' \quad (\gamma_i \in G)$$

$$d(c' \cup c'') = (dc') \cup c'' + (-1)^p c' \cup (dc'')$$

$\cup : H^p(G:M') \times H^q(G:M'') \longrightarrow H^{p+q}(G : M' \otimes M'')$ cup product

$H^*(G : \mathbb{Z}) = H^*(K(G,1) : \mathbb{Z})$ the cohomology algebra of the group G

$\mathbb{Q}$ $\mathbb{Q}$

$\vdots$ $\vdots$

$\underline{p=0}$ $C^0(G:M) = M.$

$d: M \to C^1(G:M), \quad m \mapsto (dm : \gamma \mapsto \gamma m - m)$

$\underline{p=1}$ $c \in C^1(G:M)$

$(dc)(\gamma_0, \gamma_1) = \gamma_0 c(\gamma_1) - c(\gamma_0\gamma_1) + c(\gamma_0)$

$Z^1(G:M) := \mathrm{Ker}(d : C^1(G:M) \to C^2(G:M))$

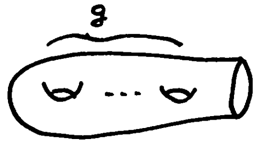
$= \{ z : G \to M \text{ map} ; \ \forall\gamma_0, \forall\gamma_1 \in G. \ z(\gamma_0\gamma_1) = z(\gamma_0) + \gamma_0 z(\gamma_1) \}$

1-cocycle = crossed homomorphism

$c' \in C^1(G:M'), \quad c'' \in C^1(G:M'')$

$(c' \cup c'')(\gamma_1, \gamma_2) = c'(\gamma_1) \otimes \gamma_1 c''(\gamma_2) \qquad (\gamma_i \in G)$

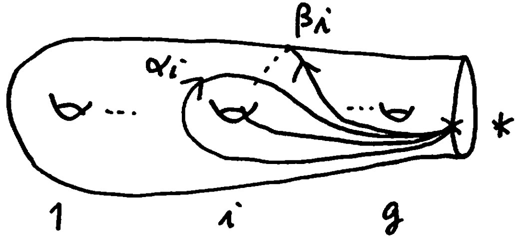
go back to the mapping class group $G = \mathcal{M}(\Sigma_g)$

For simplicity, we consider $G = \mathcal{M}(\Sigma_{g,1})$, $\Sigma_{g,1} =$ 

<u>group</u> <u>cocycles</u> <u>for</u> <u>the</u> i^{th} <u>MMM</u> <u>class</u> e_i <u>on</u> $\mathcal{M}(\Sigma_{g,1})$

$\Uparrow$

<u>the Johnson homomorphisms</u>



$* \in \partial\Sigma_{g,1}$

$\pi := \pi_1(\Sigma_{g,1}, *) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \rangle$

<u>free</u> group of rank $2g$

$\{\Gamma_k \pi\}_{k=1}^{\infty}$ lower central series of the group π

$\Gamma_1 \pi := \pi, \quad \Gamma_{k+1}\pi := [\Gamma_k \pi, \pi], \; k \geq 1$

$\Gamma_1\pi / \Gamma_2\pi = \pi/[\pi,\pi] = \pi^{abel} = H_1(\Sigma_g ; \mathbb{Z}) =: H_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$

$\Gamma_2\pi/\Gamma_3\pi \cong \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}}, \quad [\gamma_1, \gamma_2] \bmod \Gamma_3\pi \longmapsto [\gamma_1] \wedge [\gamma_2]$

$$\begin{array}{ccccccccc} 0 \to & \Gamma_2\pi/\Gamma_3\pi & \to & \Gamma_1\pi/\Gamma_3\pi & \to & \Gamma_1\pi/\Gamma_2\pi & \to 0 & (\text{exact}) \\ & \wr\| & & & & \| & & \\ & \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}} & & & & H_{\mathbb{Z}} & & \end{array}$$

$\mathcal{I}(\Sigma_{g,1}) := \mathrm{Ker}(\mathcal{M}(\Sigma_{g,1}) \to \mathrm{Aut}(H_{\mathbb{Z}}))$ Torelli group

$\varphi \in \mathcal{I}(\Sigma_{g,1}), \; \gamma \in \pi, \quad \varphi(\gamma) \equiv \gamma \bmod \Gamma_2\pi \quad (\because \varphi \in \mathcal{I}(\Sigma_{g,1}))$

$\tau_1(\varphi)(\gamma) := \gamma^{-1}\varphi(\gamma) \bmod \Gamma_3\pi \in \Gamma_2\pi/\Gamma_3\pi \cong \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}}$

depends only on $[\gamma] \in H_{\mathbb{Z}}$

$$\begin{array}{rl} \tau_1(\varphi) \in & \mathrm{Hom}(H_{\mathbb{Z}}, \Lambda^2_{\mathbb{Z}} H_{\mathbb{Z}}) \\ & \cup \\ \in & \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \end{array}$$

(D. Johnson) the 1^{st} Johnson homomorphism

<u>Theorem</u> (D. Johnson)

$\tau_1 : \mathcal{I}(\Sigma_{g,1})^{abel} \longrightarrow \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}$

(1) surjective

(2) Kernel = fin. gen. 2-torsion

(__Remark__ : $\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}$: non-trivial $\mathcal{M}(\Sigma_{g,1})$-module)

$H_{\mathbb{Z}} \subset H := H_1(\Sigma_{g,1} : \mathbb{Q}) \;(\cong \mathbb{Q}^{2g})$

__Theorem__ (Morita)

$\exists !\, \tilde{k} \in H^1(\mathcal{M}(\Sigma_{g,1}) : \Lambda^3 H)$ s.t. $\tilde{k}|_{\mathcal{I}(\Sigma_{g,1})} = \tau_1 : \mathcal{I}(\Sigma_{g,1}) \to \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \subset \Lambda^3 H$

$f \in \mathrm{Hom}(\Lambda^p(\Lambda^3 H), \mathbb{Q})^{Sp(H)}$, $p \geq 0$.

$$\begin{array}{ccc} f_* : H^p(\mathcal{M}(\Sigma_{g,1}) : \Lambda^p(\Lambda^3 H)) & \longrightarrow & H^p(\mathcal{M}(\Sigma_{g,1}) : \mathbb{Q}) \\ \Downarrow & & \Downarrow \\ \tilde{k}^p & \longmapsto & f_*(\tilde{k}^p) =: \tilde{k}_*(f) \end{array}$$

the p^{th} power of $\tilde{k}$

__Theorem__ (Morita) The MMM classes e_i's are in the image of

$\tilde{k}_* : \mathrm{Hom}(\Lambda^*(\Lambda^3 H), \mathbb{Q})^{Sp(H)} \longrightarrow H^*(\mathcal{M}(\Sigma_{g,1}) : \mathbb{Q})$

__Theorem__ (Morita-K.) $\forall * \geq 0$ (not only in the stable range)

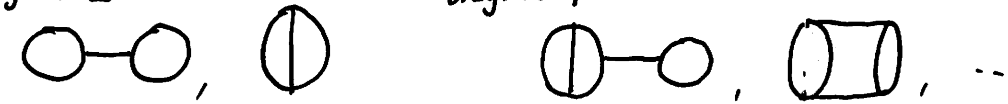
$\mathrm{Im}\, \tilde{k}_*$ is generated by the MMM classes e_i's

__Morita's description__

$\mathbb{Q}[\text{trivalent graphs}] \twoheadrightarrow \mathrm{Hom}(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})^{Sp(H)}$

vertex of deg 3 $\leftrightarrow \Lambda^3 H$
edge $\leftrightarrow H \otimes H \to \mathbb{Q}$ intersection number

ex) degree 2 degree 4



Outline of Part II : (higher) Johnson homomorphisms

- cohomology
  - construction of $\tilde{k}$
  - computation of $\mathrm{Im}\, \tilde{k}_*$
- geometric interpretation of the Johnson homomorphisms by the Goldman Lie algebra
  - Dehn twist $t_C = \exp\left(\frac{1}{2} (\log C)^2\right)$

II. Johnson homomorphisms

§2.1. Higher Johnson homomorphisms

--- <u>algebraic</u> <u>approach</u>

π : finitely generated free group (ex). $\pi_1(\Sigma_{g,r})$ if $r \geq 1$)

$H_{\mathbb{Z}} := \pi^{abel} = \pi/_{[\pi,\pi]} = H_1(\pi:\mathbb{Z})$ abelianization of π

$\{\Gamma_k \pi\}_{k=1}^{\infty}$: lower central series of π — commutator

$\Gamma_1 \pi := \pi, \quad \Gamma_{k+1}\pi := [\Gamma_k \pi, \pi], \ k \geq 1, \quad \bigcap_{k=1}^{\infty} \Gamma_k \pi = \{1\}$

$\mathcal{L}(H_{\mathbb{Z}}) = \bigoplus_{k=1}^{\infty} \mathcal{L}_k(H_{\mathbb{Z}})$, the free Lie algebra over $H_{\mathbb{Z}}$

$\mathcal{L}_k(H_{\mathbb{Z}})$: the degree k component

$\mathcal{L}_k(H_{\mathbb{Z}}) \overset{\cong}{\Rightarrow} \Gamma_k\pi/_{\Gamma_{k+1}\pi}$, $\left(\begin{array}{c}\text{a Lie word} \\ \text{of } [\gamma_1], \cdots, [\gamma_k]\end{array}\right)_{\gamma_i \in \pi} \longmapsto \left(\begin{array}{c}\text{corresponding} \\ \text{commutator} \\ \text{of } \gamma_1, \cdots, \gamma_k\end{array}\right)$ (Magnus - Witt)

$\mathcal{L}(H_{\mathbb{Z}}) \cong \bigoplus_{k=1}^{\infty} \Gamma_k\pi/_{\Gamma_{k+1}\pi}$ Lie algebra isomorphism

$k \geq 0$. $A(k) \overset{def}{=} \mathrm{Ker}(\mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(\pi/_{\Gamma_k\pi})) \triangleleft \mathrm{Aut}(\pi)$ — normal subgroup

$\mathrm{Aut}(\pi) = A(0) \supset A(1) \supset \cdots \supset A(k) \supset A(k+1) \supset \cdots \quad \bigcap_{k=0}^{\infty} A(k) = \{1\}$

<u>Andreadakis</u> <u>filtration</u>

$\Gamma_2 \pi = [\pi, \pi]$

$\pi/_{\Gamma_2 \pi} = \Gamma_1 \pi /_{\Gamma_2 \pi} = \pi/_{[\pi,\pi]} = H_{\mathbb{Z}} = \mathcal{L}_1(H_{\mathbb{Z}})$

$A(0)/A(1) \cong GL(H_{\mathbb{Z}})$ $\left(\because \begin{pmatrix} x_i \longmapsto x_i x_j \\ x_k \longmapsto x_k \text{ if } k \neq i \end{pmatrix} , \overset{\text{where}}{\pi = \langle x_1, \cdots, x_n \rangle} \right)$

<u>Observation</u> $\varphi \in \mathrm{Aut}(\pi)$, $k \geq 1$.

($\varphi \in A(k) \Leftrightarrow \forall \gamma \in \pi$, $(\tau_k(\varphi)(\gamma) :=) \gamma^{-1}\varphi(\gamma) \in \Gamma_{k+1}\pi$

$\tau_k(\varphi)(\gamma_1\gamma_2) = (\gamma_1\gamma_2)^{-1}\varphi(\gamma_1\gamma_2) = \gamma_2^{-1}\gamma_1^{-1}\varphi(\gamma_1)\,\gamma_2\,\gamma_2^{-1}\varphi(\gamma_2)$

$= \gamma_2^{-1}\tau_k(\varphi)(\gamma_1)\gamma_2 \cdot \tau_k(\varphi)(\gamma_2)$

$\tau_k(\varphi) \in \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}H_{\mathbb{Z}})$ if $k \in A(k)$ $\left(\mathcal{L}_{k+1}H_{\mathbb{Z}} \overset{\text{identity}}{=} \Gamma_{k+1}\pi/_{\Gamma_{k+2}\pi} \right)$

$\tau_k : A(k)/_{A(k+1)} \hookrightarrow \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}H_{\mathbb{Z}})$ <u>the</u> k^{th} <u>Johnson</u> <u>homomorphism</u>

<u>injective</u> group homomorphism

$$\begin{tikzcd} \tau : \bigoplus_{k=1}^{\infty} A(k)/A(k+1) \arrow[r, hook] \arrow[dr] & \bigoplus_{k=1}^{\infty} \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}H_{\mathbb{Z}}) \arrow[d] \\ & \mathrm{Der}(\mathcal{L}(H_{\mathbb{Z}})) \end{tikzcd}$$

Lie algebra

$\left(\begin{matrix} \because \text{(Andreadakis)} \\ [A(k), A(l)] \subset A(k+l) \end{matrix} \right)$

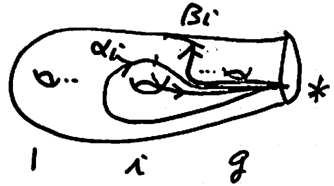
derivation Lie algebra

<u>injective</u> <u>Lie</u> <u>algebra</u> <u>homomorphism</u>

<u>mapping class group</u> $g \geq 1$, $\Sigma_{g,1} =$  $* \in \partial\Sigma_{g,1}$

$\pi = \pi_1(\Sigma_{g,1}, *) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \rangle$

free group of rank $2g$



$\mathcal{M}(\Sigma_{g,1}) \hookrightarrow \mathrm{Aut}(\pi)$, $\varphi \mapsto \varphi_*$,

<u>injective</u> group homomorphism (Dehn-Nielsen)

$k \geq 0$ $\quad \mathcal{M}(k) := \mathcal{M}(\Sigma_{g,1}) \cap A(k)$ $\quad$ <u>Johnson filtration</u>

$\mathcal{M}(1) = \mathcal{I}(\Sigma_{g,1})$ Torelli group

$\tau : \bigoplus_{k=1}^{\infty} \mathcal{M}(k)/\mathcal{M}(k+1) \hookrightarrow \mathrm{Der}(\mathcal{L}(H_{\mathbb{Z}}))$ <u>injective</u> Lie algebra homomorphism

$\omega := \sum_{i=1}^{g} [\,[\alpha_i], [\beta_i]\,] \in \mathcal{L}_2(H_{\mathbb{Z}})$

bracket

homology class

<u>Morita</u>:

(1) $\mathrm{Im}\,\tau < \mathrm{Der}_\omega(\mathcal{L}(H_{\mathbb{Z}})) := \{ D \in \mathrm{Der}(\mathcal{L}(H_{\mathbb{Z}})) \,;\, D\omega = 0 \}$

(2) $0 \neq {}^{\exists}\mathrm{Tr} : \mathrm{Der}_\omega(\mathcal{L}(H_{\mathbb{Z}})) \to \mathrm{Sym}(H_{\mathbb{Z}})$ (Morita trace)

symmetric tensor algebra over $H_{\mathbb{Z}}$

s.t. $\mathrm{Tr} \circ \tau = 0$ on $\bigoplus_{k=1}^{\infty} \mathcal{M}(k)/\mathcal{M}(k+1)$

In particular, $\mathrm{Im}\,\tau \subsetneqq \mathrm{Der}_\omega(\mathcal{L}(H_{\mathbb{Z}}))$

(ex) $\mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_2(H_{\mathbb{Z}})) \cap \mathrm{Der}_\omega(\mathcal{L}(H_{\mathbb{Z}})) = \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}$)

$H := H_{\mathbb{Z}} \otimes \mathbb{Q} = H_1(\Sigma_{g,1}; \mathbb{Q})$

$\tau_{\mathbb{Q}} : \bigoplus_{k=1}^{\infty} (\mathcal{M}(k)/\mathcal{M}(k+1)) \otimes \mathbb{Q} \to \mathrm{Der}_\omega(\mathcal{L}(H))$

(<u>Hain.</u> $\mathrm{Im}\,\tau_{\mathbb{Q}}$ is generated by the degree 1 component $\tau_{\mathbb{Q}}((\mathcal{M}(1)/\mathcal{M}(2)) \otimes \mathbb{Q})$.

<u>Enomoto-Satoh</u>

$0 \neq {}^{\exists}\hat{T}r : \mathrm{Der}_\omega(\mathcal{L}(H)) \to T(H)/[T(H), T(H)]$ (Enomoto-Satoh trace)

s.t. $\hat{T}r \circ \tau_{\mathbb{Q}} = 0$ on $\bigoplus_{k=0}^{\infty} (\mathcal{M}(k)/\mathcal{M}(k+1)) \otimes \mathbb{Q}$

where $T(H) := \bigoplus_{k=0}^{\infty} H^{\otimes k}$. tensor algebra over H

$[T(H), T(H)]$: linear span of $\{ab - ba \,;\, a, b \in T(H)\}$

($\Leftarrow$ Satoh trace for $\bigoplus_{k=1}^{\infty} \Gamma_k A(1)/\Gamma_{k+1} A(1)$)

(<u>Remark</u> When $\pi = \pi_1(X)$, X : path-conn. space

$\mathbb{Q}\pi/[\mathbb{Q}\pi, \mathbb{Q}\pi] = \mathbb{Q}(\pi/\mathrm{conj}) = \mathbb{Q}[S^1, X]$ ← free loops on X)

§2.2. Extension of the Johnson homomorphisms

- <u>Morita</u> $\exists$ extension of τ_1 and τ_2 to the whole $\mathcal{M}(\Sigma_{g,1})$
- $\forall k \geq 1$, τ_k : <u>Hain</u>, <u>K</u>, Day, <u>Massuyeau</u>. $\rightsquigarrow$ computation of $\operatorname{Im} \widetilde{k}_*$

We consider $/\mathbb{Q}$ for simplicity.

π : finitely generated free group

$\mathbb{Q}\pi = \{ \sum_{\gamma \in \pi} a_\gamma \gamma \ : \ a_\gamma \in \mathbb{Q}, \ a_\gamma = 0 \text{ for all but finite } \gamma \}$ group ring of π

$\varepsilon : \mathbb{Q}\pi \to \mathbb{Q}$, $\sum a_\gamma \gamma \mapsto \sum a_\gamma$, augmentation

$I\pi := \operatorname{Ker} \varepsilon = \{ \sum_{\gamma \in \pi} a_\gamma \gamma \in \mathbb{Q}\pi \ : \ \sum_{\gamma \in \pi} a_\gamma = 0 \}$ augmentation ideal

<u>Observation</u> $k \geq 1$, $\gamma \in \pi$.

$$\gamma \in \Gamma_k \pi \iff \gamma - 1 \in (I\pi)^k = \overbrace{(I\pi) \cdots (I\pi)}^{k}$$

$\widehat{\mathbb{Q}\pi} := \varprojlim_{k \to \infty} \mathbb{Q}\pi / (I\pi)^k$ completed group ring

$\widehat{(I\pi)^k} := \operatorname{Ker} (\widehat{\mathbb{Q}\pi} \to \mathbb{Q}\pi / (I\pi)^k)$ $k \geq 1$ $\rightsquigarrow$ topology on $\widehat{\mathbb{Q}\pi}$

$\widehat{\mathbb{Q}\pi}$: complete Hopf algebra. completed tensor product

$\Delta : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\pi}$ coproduct

$\gamma \in \pi \longmapsto \gamma \hat{\otimes} \gamma$

$$H = \pi^{abel} \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\pi;\mathbb{Q})$$

$$\hat{T} := \prod_{m=0}^{\infty} H^{\otimes m} \quad \text{completed tensor algebra over } H \text{ --- complete Hopf algebra}$$

$$\hat{T}_{\geq k} := \prod_{m \geq k} H^{\otimes m} \longrightarrow \text{topology on } \hat{T}$$

$$\Delta : \hat{T} \to \hat{T} \hat{\otimes} \hat{T}, \quad X \in H \mapsto X \hat{\otimes} 1 + 1 \hat{\otimes} X, \quad \text{coproduct}$$

$$\hat{\mathcal{L}} := \{ u \in \hat{T} : \Delta u = u \hat{\otimes} 1 + 1 \hat{\otimes} u \} \quad \text{Lie-like elements}$$

$$\hat{\mathcal{L}} \cap H^{\otimes k} = \mathcal{L}_k(H) = \mathcal{L}_k(H_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \qquad (\forall k \geq 1)$$

$$\mathrm{Aut}(\hat{T}) := \{ U : \hat{T} \to \hat{T} \text{ algebra automorphism} \ ; \ \forall k \geq 1, U(\hat{T}_{\geq k}) = \hat{T}_{\geq k} \}$$

$$|\cdot| : \mathrm{Aut}(\hat{T}) \rightleftarrows GL(H), \quad U \mapsto |U| := (U \text{ on } \hat{T}_{\geq 1}/\hat{T}_{\geq 2} = H)$$

$$A_* := \prod_{k=0}^{\infty} A^{\otimes k} \longleftarrow\!\shortmid A \qquad \text{section}$$

$$IA(\hat{T}) := \mathrm{Ker}(|\cdot|) \triangleleft \mathrm{Aut}(\hat{T})$$

$$\mathrm{Aut}(\hat{T}) = IA(\hat{T}) \rtimes GL(H) \quad \text{semi-direct product}$$

$$IA(\hat{T}) \cong \mathrm{Hom}(H, \hat{T}_{\geq 2}) = \prod_{p=1}^{\infty} \mathrm{Hom}(H, H^{\otimes(p+1)}) \xrightarrow{pr_k} \mathrm{Hom}(H, H^{\otimes(k+1)})$$

$$U \longmapsto U|_H - 1_H \longmapsto (u_p)_{p=1}^{\infty} \xmapsto{\text{the } k^{th} \text{ projection}} u_k$$

$$\left.\begin{array}{ccc} \mathrm{Aut}(\hat{T}) & = & IA(\hat{T}) \rtimes GL(H) \\ U & \leftarrow\!\shortmid & ((u_p)_{p=1}^{\infty}, A) \\ V & \leftarrow\!\shortmid & ((v_p)_{p=1}^{\infty}, B) \\ UV & \leftarrow\!\shortmid & ((w_p)_{p=1}^{\infty}, AB) \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} w_1 = u_1 + Av_1 & \text{--- } (*.1) \\ w_2 = u_2 + (u_1 \otimes 1 + 1 \otimes u_1) \circ Av_1 + Av_2 & \text{--- } (*.2) \end{array}\right.$$

$$\text{where} \quad Av_p := A^{\otimes(p+1)} \circ v_p \circ A^{-1} \in \mathrm{Hom}(H, H^{\otimes(p+1)})$$

many choices

Fact $\exists\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ isomorphism of complete Hopf algebras

s.t. $\forall k \geq 1$

1) $\theta\left(\widehat{(I\pi)^k}\right) = \widehat{T}_{\geq k}$

2)
$$\begin{tikzcd}
\Gamma_k\pi/\Gamma_{k+1}\pi \arrow[r, hook] \arrow[d, equal] & \widehat{(I\pi)^k}/\widehat{(I\pi)^{k+1}} \arrow[r, "\theta", "\cong"'] \arrow[d, phantom, "\circlearrowright"] & \widehat{T}_{\geq k}/\widehat{T}_{\geq k+1} \arrow[d, equal] \\
\mathcal{L}_k(H_{\mathbb{Z}}) \arrow[r, hook] & \widehat{\mathcal{L}}(H)\cap H^{\otimes k} \arrow[r, hook] & H^{\otimes k}
\end{tikzcd}$$

$$\mathrm{Aut}(\pi) \curvearrowright \widehat{\mathbb{Q}\pi} \underset{\theta}{\xrightarrow{\cong}} \widehat{T}$$

$\Rightarrow$ $T^\theta : \mathrm{Aut}(\pi) \to \mathrm{Aut}(\widehat{T})$

$$A(k) = \mathrm{Ker}\left(\mathrm{Aut}(\pi) \xrightarrow{T^\theta} \mathrm{Aut}(\widehat{T}) \to \mathrm{Aut}(\widehat{T}/\widehat{T}_{\geq k+1})\right) \quad (\forall k \geq 0)$$

$k \geq 1$, $\tau_k^\theta : \mathrm{Aut}(\pi) \xrightarrow{T^\theta} \mathrm{Aut}(\widehat{T}) = IA(\widehat{T}) \rtimes GL(H) \xrightarrow{pr_k} \mathrm{Hom}(H, H^{\otimes(k+1)})$

the k^{th} Johnson map (not homom)

Fox differential

general θ

Proposition (Kitano, K.)

$$\tau_k^\theta|_{A(k)} = \tau_k : A(k) \xrightarrow{\tau_k} \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_k(H_{\mathbb{Z}})) \hookrightarrow \mathrm{Hom}(H, H^{\otimes(k+1)})$$

Corollary (K.) $\forall k \geq 1$, $\exists$ extension of τ_k to the whole $\mathrm{Aut}(\pi)$

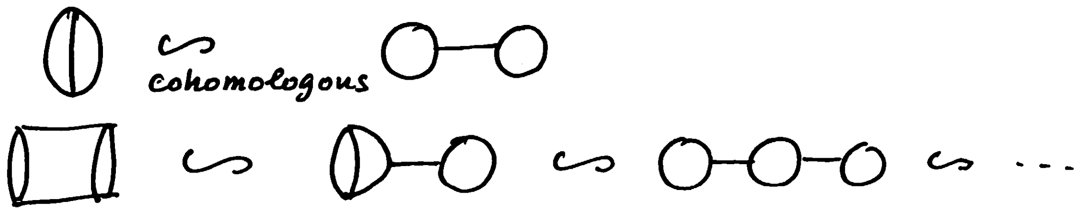
$$\begin{cases} -(*.1) \quad w_1 = u_1 + A v_1 \\ -(*.2) \quad w_2 = u_2 + (u_1 \otimes 1 + 1 \otimes u_1) \circ A v_1 + A v_2 \\ - \forall \varphi, \forall \psi \in \mathrm{Aut}(\pi) \quad T^\theta(\varphi\psi) = T^\theta(\varphi)\, T^\theta(\psi) \end{cases}$$

Alexander-Whitney cup product $\circ : \mathrm{Hom}(H, H^{\otimes 2}) \times \mathrm{Hom}(H^{\otimes 2}, H^{\otimes 3}) \to \mathrm{Hom}(H, H^{\otimes 3})$

$$\Longrightarrow \begin{cases} \tau_1^\theta(\varphi\psi) = \tau_1^\theta(\varphi) + |\varphi|\, \tau_1^\theta(\psi) \\ \tau_2^\theta(\varphi\psi) = \tau_2^\theta(\varphi) + \big((\tau_1^\theta \otimes 1 + 1 \otimes \tau_1^\theta) \cup \tau_1^\theta\big)(\varphi, \psi) + |\varphi|\, \tau_2^\theta(\psi) \end{cases}$$

$$\Longleftrightarrow \begin{cases} \tau_1^\theta \in Z^1(\mathrm{Aut}(\pi), H^* \otimes H^{\otimes 2}) \\ -d\tau_2^\theta = (\tau_1^\theta \otimes 1 + 1 \otimes \tau_1^\theta) \cup \tau_1^\theta \in C^2(\mathrm{Aut}(\pi), H^* \otimes H^{\otimes 3}) \end{cases}$$

$\Longrightarrow$ moves of trivalent graphs



$\rightsquigarrow$ computation of $\mathrm{Im}\, \tilde{k}_*$ (Morita-K.)

$\tau^\theta := (\tau_p^\theta) : \mathrm{Aut}(\pi) \to \prod_{p=1}^{\infty} \mathrm{Hom}(H, H^{\otimes(p+1)})$, $\varphi \mapsto (\tau_p^\theta(\varphi))$, total Johnson map (K.)

fits to the group cohomology, but does <u>not</u> respect the coproduct Δ

$\swarrow$

<u>an improvement of</u> τ^θ (Massuyeau)

does <u>not</u> fit to the group cohomology, but respects the coproduct

$$T^\theta : \mathcal{A}(1) \to IA_\Delta(\hat{T}) := \{ U \in IA(\hat{T}) \;;\; (U \hat{\otimes} U) \circ \Delta = \Delta \circ U \}$$

$$\log : IA_\Delta(\hat{T}) \underset{\log}{\xrightarrow{\cong}} \mathrm{Der}^+(\hat{\mathcal{L}}(H)) \;,\quad U \mapsto \log U := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(U-1)^k \text{ converges}$$

$+$ ← positive degree part

$$\log \circ T^\theta : \mathcal{A}(1) \to \mathrm{Der}^+(\hat{\mathcal{L}}(H)) \quad \text{Massuyeau's total Johnson map}$$

group homomorphism (group law on $\mathrm{Der}^+(\hat{\mathcal{L}}(H)) \Leftarrow$ Baker-Campbell-Hausdorff series)

$$\hat{\mathcal{L}}(H) \overset{\theta}{\cong} \text{Lie-like element of } \widehat{\mathbb{Q}\pi}$$

geometric re-construction of $\log T^\theta : \mathcal{M}(1) \to \mathrm{Der}^+(\widehat{\mathbb{Q}\pi})$ (Kuno-K.)

$\Uparrow$

Goldman Lie algebra

§2.3. Goldman Lie algebra

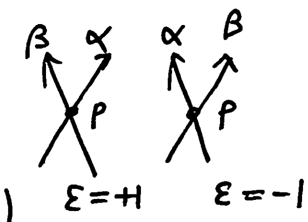
S : (connected oriented) surface.

$\hat{\pi} = \hat{\pi}(S) \overset{\text{def}}{=} \pi_1(S)/_{\text{conjugate}} = [S^1, S]$ free loops on S

$p \in S$. $|\cdot| : \pi_1(S,p) \to \hat{\pi}(S)$ forgetful map of the basepoint p

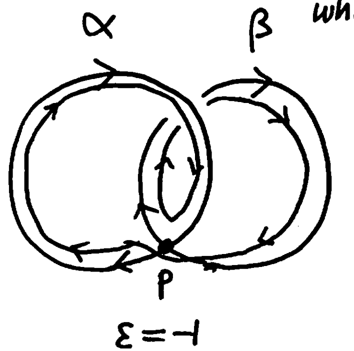
$\alpha, \beta \in \hat{\pi}$ (Choose their representatives in general position) $\rightsquigarrow$ $\alpha \cap \beta$ <u>finite</u> and <u>transverse</u>

$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$ <u>Goldman</u> <u>bracket</u>



where $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ local intersection number

α_p (resp. β_p) $\in \pi_1(S, p)$ based loop along α (resp. β)



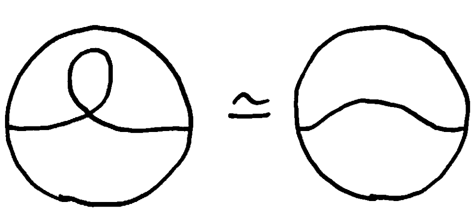
<u>Theorem</u> (Goldman)

(1) $[\ ,\ ]$: well-defined i.e., independent of the choice of representatives

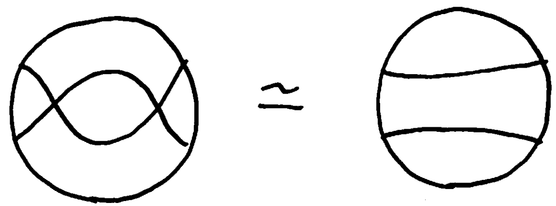
(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra $\rightsquigarrow$ <u>Goldman</u> <u>Lie</u> <u>algebra</u> <u>of</u> S

(1) $\Leftarrow$ 3 moves

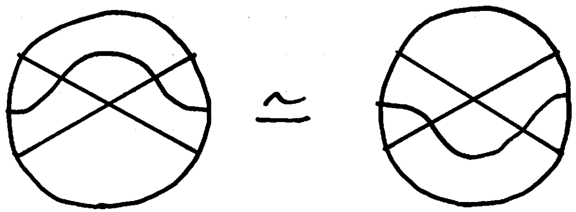
(ω1) monogon



(ω2) bigon



(ω3) jumping over a double point



<u>another proof of (1)</u> $[\ ,\ ]$ is the intersection form on the twisted homology
$$H_1(\pi_1(S), (\mathbb{Z}\pi_1(S))^{\mathrm{conj}})$$

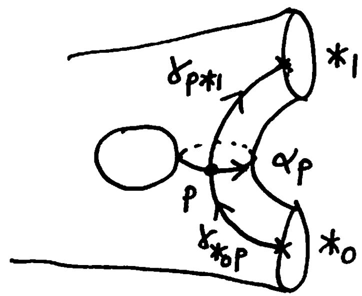
<u>Remarks</u> (1) Wolpert-Goldman formula for the Poisson bracket on the moduli space of flat bundles over S

(2) Higher dimensional generalization: Chas-Sullivan's string topology

<u>Assume</u> $\partial S \neq \phi$,

$*_0, *_1 \in \partial S$, $\quad \Pi S(*_0, *_1) = \pi_1(S, *_0, *_1) \overset{\mathrm{def}}{=} [([0,1], 0, 1), (S, *_0, *_1)]$ paths from $*_0$ to $*_1$

fundamental groupoid



$\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_0, *_1)$ in general position

$$\sigma(\alpha)(\gamma) \overset{\mathrm{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma)\, \gamma_{*_0 p}\, \alpha_p\, \gamma_{p *_1} \in \mathbb{Z}\Pi S(*_0, *_1)$$

$\gamma_{*_0p} \in \Pi S(*_0, p)$, $\gamma_{p*_1} \in \Pi S(p, *_1)$ segments of γ

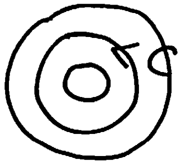
<u>Theorem</u> (Kuno-K.)

(1) $\sigma(\cdot)(\cdot)$ is well-defined

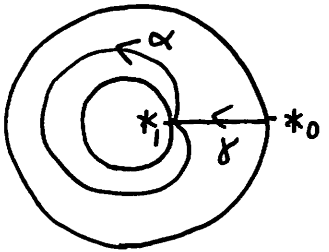
(2) $\sigma : \mathbb{Z}\hat{\pi} \to \mathrm{Der}(\mathbb{Z}\Pi S|_{\partial S})$ Lie algebra homomorphism

(restriction of the $\mathbb{Z}$-linear small category $\mathbb{Z}\Pi S$ to the boundary ∂S)

<u>Remark</u> (Massuyeau–Turaev) The homomorphism $\sigma: \mathbb{Z}\hat{\pi} \to \mathrm{Der}(\mathbb{Z}\pi_1(S, *_0))$ and the Goldman bracket can be interpreted under the framework of van den Bergh's theory of double brackets (non-commutative Poisson geometry)

<u>example</u> $S =$ 

annulus, $t_C \in \mathcal{M}(S)$ right-handed Dehn twist



$C = |\alpha|$

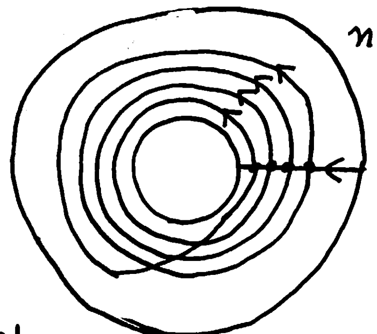
$$\begin{cases} t_C(\alpha) = \alpha \\ t_C(\gamma) = \gamma\alpha \end{cases}$$

completion with respect to $I\pi_1(S, *_0)$

$$\log t_C \overset{\mathrm{def}}{=} \sum_{k=1}^{\infty} \frac{(-1)^k}{k}(t_C - 1)^k \in \mathrm{End}(\widehat{\mathbb{Q}\pi S|_{\partial S}})$$

$$(\#1) \begin{cases} (\log t_C)(\alpha) = 0 \\ (\log t_C)(\gamma) = \gamma \log \alpha \end{cases}$$

$n \in \mathbb{Z}_{>0}$
$$\begin{cases} \sigma(C^n)(\alpha) = 0 \quad (\because\ C \cap \alpha = \phi) \\ \sigma(C^n)(\gamma) = n\gamma\alpha^n \quad (\because) \end{cases}$$



$\forall f(t) \in \mathbb{Q}[[t-1]]$

$$(\#2) \begin{cases} \sigma(f(C))(\alpha) = 0 \\ \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \end{cases}$$

Compare (#1) and (#2):

Find $f(t)$ satisfying $t f'(t) = \log t$

<u>Answer</u> $f(t) = \int_1 \frac{1}{t} \log t \, dt = \frac{1}{2}(\log t)^2$

Hence we have

$$\sigma\left(\tfrac{1}{2}(\log C)^2\right) = \log t_C \in \mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_{\partial S}) \quad \text{for } S = \text{annulus}$$

<u>General case</u>:

S : compact (connected oriented) surface with $\partial S \neq \phi$

Choose a finite subset $E \subset \partial S$ s.t. $E = \pi_0(E) \xrightarrow[\cong]{\mathrm{incl}_*} \pi_0(\partial S)$

$*, *' \in E$, $\exists \gamma_0 \in \Pi S(*, *')$ ($\because$ S : connected)

$\mathbb{Q}\Pi S(*, *') = \mathbb{Q}\pi_1(S, *)\gamma_0$

$\widehat{\mathbb{Q}\Pi S}(*, *') \overset{\mathrm{def}}{=} \varprojlim_{k\to\infty} \left(\mathbb{Q}\pi_1(S,*)\gamma_0 / (I\pi_1(S,*))^k\gamma_0 \right)$ independent of the choice of γ_0

$\rightsquigarrow \widehat{\mathbb{Q}\Pi S}|_E$: $\mathbb{Q}$-linear small category whose object set is E.

$\widehat{\mathbb{Q}\hat{\pi}}(S) \overset{\mathrm{def}}{=} \varprojlim_{k\to\infty} \mathbb{Q}\hat{\pi}(S) / |\mathbb{Q}1 + (I\pi_1(S,*))^k|$ completed Goldman Lie algebra

$\qquad \Psi$

(ex) $\tfrac{1}{2}(\log C)^2$)

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E)$ Lie algebra homomorphism

<u>Theorem</u> ($\underline{\Sigma_{g,1}}$ Kuno-K., $\underline{\text{general}}$ Kuno-K., Massuyeau-Turaev)

$\forall C \subset S \setminus \partial S$ simple closed curve

$$t_C = \exp\left(\sigma\left(\tfrac{1}{2}(\log C)^2\right)\right) \in \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E)$$

Johnson homomorphism ?

(Observation $\forall \alpha \in \hat{\pi}(S)$ $\sigma(\alpha)(\forall \text{boundary loop}) = 0$ $(\because \alpha \subset S \setminus \partial S)$

$\mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) := \{ D : \text{continuous derivation of } \widehat{\mathbb{Q}\Pi S}|_E \ ;\ D(\forall \text{boundary loop}) = 0 \}$

(Theorem (Kuno-K.)

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$ isomorphism of Lie algebras

$\mathcal{M}^0(S) := \{ \varphi \in \mathcal{M}(S) ;\ \log \varphi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\varphi - 1)^k \text{ converges in } \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) \}$

subset of $\mathcal{M}(S)$ (ex) $t_C \in \mathcal{M}^0(S)$, $\mathcal{I}(\Sigma_{g,1}) \subset \mathcal{M}(\Sigma_{g,1})$)

$\tau : \mathcal{M}^0(S) \xrightarrow{\log} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) \underset{\sigma^{-1}}{\xrightarrow{\cong}} \widehat{\mathbb{Q}\hat{\pi}}(S) \left(\underset{\theta}{\cong} \prod_{k=1}^{\infty} (H_1(S;\mathbb{Q})^{\otimes k})^{\text{cyclic}} \right)$

"geometric" Johnson homomorphism (Kuno-K.)

- τ is equivalent to Massuyeau's $\log T^\theta$ when $S = \Sigma_{g,1}$
- $\tau(t_C) = \frac{1}{2}(\log C)^2$
- $\delta : \widehat{\mathbb{Q}\hat{\pi}}(S) \to (\widehat{\mathbb{Q}\hat{\pi}}(S))^{\hat{\otimes} 2}$ Turaev cobracket

  $\delta \circ \tau = 0 : \mathcal{M}^0(S) \to (\widehat{\mathbb{Q}\hat{\pi}}(S))^{\hat{\otimes} 2}$ (Kuno-K.) $\rightsquigarrow$ Morita trace
- "framed" version of this construction (K.)

  $\Longrightarrow$ Enomoto-Satoh trace ($S = \Sigma_{g,1}$)

  $\Longrightarrow$ divergence cocycle in the Kashiwara-Vergne problem ($S = \Sigma_{0,n+1}$)