(§2. The cohomology group of a Lie algebra (つづき))
(§2.3. Cohomology groups of low degree (つづき))

<u>Definition</u> (H. Weyl)
$\mathfrak{g}$: (fin. dim'l) semi-simple Lie algebra $/\mathbb{K}$
$\overset{\text{def}}{\Longleftrightarrow}$ (0) $\mathfrak{g}$: fin. $\dim_{\mathbb{K}}$'l Lie algebra $/\mathbb{K}$
(1) $\forall M$: fin. $\dim_{\mathbb{K}}$ $\mathfrak{g}$-module M : completely reducible

$\mathfrak{g} = \mathbb{K}$: abelian ; <u>not</u> semi-simple
(∵) $M_0 := \mathbb{K} \oplus \mathbb{K}$ fin. dim'l.
$\rho_0 : \mathfrak{g} \to \mathfrak{gl}(M_0)$, $a \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, Lie algebra homom.
$N \subset M_0$: 1-dim. $\mathfrak{g}$-submodule
$\Longrightarrow N = \mathbb{K} \oplus 0$
$\left(∵ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \ \det \begin{pmatrix} x & y \\ y & 0 \end{pmatrix} = 0 \Rightarrow y = 0 \right)$
M_0 : <u>not</u> complete reducible //

<u>Corollary 2.12</u> $\mathfrak{g}$: semi-simple $\Rightarrow \mathfrak{g}^{abel} = 0$ i.e., $H^1(\mathfrak{g}:\mathbb{K}) = 0$
<u>proof</u> $\mathfrak{g}^{abel} \neq 0$ とする
$\mathfrak{g}^{abel} \overset{\exists}{\twoheadrightarrow} \mathbb{K} \xrightarrow{\rho_0} \mathfrak{gl}(M_0)$ <u>not</u> complete reducible //

ex) $\mathfrak{gl}_n(\mathbb{K})$: <u>not</u> semi-simple

<u>Theorem 2.13</u> $\mathfrak{g}$: semi-simple $\Rightarrow H^2(\mathfrak{g}:\mathbb{K}) = 0$
(pf) $\mathbb{K} \xrightarrow{\iota} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}$ extension
$\tilde{\mathfrak{g}} \ni \tilde{Y}$, $\mathfrak{g} \ni X$

$\rho(X)(\tilde{Y}) := [\tilde{X}, \tilde{Y}]$, $\tilde{X} \in \pi^{-1}(X)$
<u>well-defined</u>
(∵) $\operatorname{Ker} \pi = \iota(\mathbb{K}) \subset \operatorname{Center} \tilde{\mathfrak{g}}$)

$\tilde{\mathfrak{g}}$: $\mathfrak{g}$-module via ρ

$(\because)$ $\tilde{X}_1 \in \pi^{-1}(X_1)$, $\tilde{X}_2 \in \pi^{-1}(X_2)$

$[\tilde{X}_1, [\tilde{X}_2, \tilde{Y}]] - [\tilde{X}_2, [\tilde{X}_1, \tilde{Y}]] = [[\tilde{X}_1, \tilde{X}_2], \tilde{Y}]$

$\pi([\tilde{X}_1, \tilde{X}_2]) = [X_1, X_2]$ //

$\tilde{\mathfrak{g}}$: fin. dim. $\mathfrak{g}$-module $\Longrightarrow$ completely reducible

$\exists M \subset \tilde{\mathfrak{g}}$: $\mathfrak{g}$-submodule $\tilde{\mathfrak{g}} = \iota(\mathbb{K}) \oplus M$

$\sigma := (\pi|_M)^{-1} : \mathfrak{g} \xrightarrow{\cong} M$

$$\underbrace{[\sigma(X), \sigma(Y)]}_{\substack{\| \\ \sigma([X,Y])}} = \rho(X)\sigma(Y) \in M \qquad \sigma : \text{Lie algebra homom.}$$

$\tilde{\mathfrak{g}} \cong \mathbb{K} \rtimes \mathfrak{g}$ isom as Lie algebras

$\iota(a) + \sigma(X) \longleftrightarrow (a, X)$

$e(\tilde{\mathfrak{g}}) = 0 \in H^2(\mathfrak{g} : \mathbb{K})$

ゆえに

$H^2(\mathfrak{g} : \mathbb{K}) = 0$ //

次の§では半単純Lie代数および簡約Lie代数のcohomologyの消滅をもう少しくわしくみる.

§3. Cohomology of reductive Lie algebras

$\mathbb{K}$: field of char. 0
$\mathfrak{g}$: Lie algebra / $\mathbb{K}$
M : fin. dim$_{\mathbb{K}}$ $\mathfrak{g}$-module $\Rightarrow \rho : \mathfrak{g} \to \mathfrak{gl}(M)$ Lie alg. homom.

$\beta = \beta_M : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$
$(X_1, X_2) \mapsto \beta(X_1, X_2) \overset{\mathrm{def}}{=} \mathrm{trace}(\rho(X_1)\rho(X_2))$
$\mathfrak{g}$-invariant, symmetric

$\because$) $\forall Y \in \mathfrak{g}$ $\beta([X_1, Y], X_2) = \mathrm{tr}(\rho(X_1)\rho(Y)\rho(X_2)) - \mathrm{tr}(\rho(Y)\rho(X_1)\rho(X_2))$
$= \mathrm{tr}(\rho(X_1)\rho(Y)\rho(X_2)) - \mathrm{tr}(\rho(X_1)\rho(X_2)\rho(Y))$
$= \beta(X_1, [Y, X_2])$ //

$\hat{\beta} : \mathfrak{g} \to \mathfrak{g}^* = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathbb{K}),\ X \mapsto \beta(X, \cdot)$
$\mathfrak{g}$-homomorphism.

$\beta([Y, X], Z) = -\beta(X, [Y, Z])$ $(\forall X, \forall Y, \forall Z \in \mathfrak{g})$
$\hat{\beta}([Y, X]) = Y\hat{\beta}(X)$

$\mathfrak{g}$: fin. dim. Lie alg / $\mathbb{K}$ のとき $\beta_{\mathfrak{g}}$: Killing form という

§3.1. Vanishing lemma

Lemma 3.1. $\mathfrak{g}$: Lie algebra / $\mathbb{K}$
M : fin. dim. simple $\mathfrak{g}$-module $\left(\Rightarrow \rho : \mathfrak{g} \to \mathfrak{gl}(M) \text{ Lie alg. homom.} \atop \Rightarrow \beta = \beta_M : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \right)$
仮定 $\exists \mathfrak{h} < \mathfrak{g}$ ideal s.t.

① $\mathfrak{g} = \mathrm{Ker}\,\rho \oplus \mathfrak{h}$ $(\Rightarrow \mathfrak{h} < \mathfrak{gl}(M)$ finite dimensional$)$
② $\mathfrak{h} \neq 0$ $(\Rightarrow \mathfrak{g} \cdot M \neq 0)$
③ $\beta_M|_{\mathfrak{h}}$: non-degenerate

$\Longrightarrow H^*(\mathfrak{g} ; M) = 0$

No.

Date

proof $r = \dim_{\mathbb{K}} \mathfrak{h} \neq 0$ (∵ ②)

$\{Y_1, \cdots, Y_r\} \subset \mathfrak{h}$ $\mathbb{K}$-basis

$\exists \{Y_1', \cdots, Y_r'\} \subset \mathfrak{h}$ $\mathbb{K}$-basis

s.t. $\beta(Y_i, Y_j') = \delta_{ij}$ $(1 \leq i, j \leq r)$ (∵ ③)

$$t_M := \sum_{j=1}^{r} Y_j \otimes Y_j' \in \mathfrak{h} \otimes \mathfrak{h} \quad \text{Casimir element}$$

indep of the choice of $\{Y_i\}$

(∵) $\hat{\beta} : \mathfrak{h} \xrightarrow{\simeq} \mathfrak{h}^*$, $X \mapsto \beta(X, \circ)$

$\{\hat{\beta}(Y_j')\}_{j=1}^{r} \subset \mathfrak{h}^*$: dual basis of $\{Y_j\}_{j=1}^{r}$

$$1_{\mathfrak{h}} = \sum_{j=1}^{r} Y_j \otimes \hat{\beta}(Y_j') \in \mathfrak{h} \otimes \mathfrak{h}^* = \mathrm{End}(\mathfrak{h})$$

$$(1 \otimes \hat{\beta})^{-1}(1_{\mathfrak{h}}) = \sum_{j=1}^{r} Y_j \otimes Y_j' = t_M \in \mathfrak{h} \otimes \mathfrak{h} \ //$$

$\varphi \in \mathfrak{gl}(M)$, $c \in C^p(\mathfrak{g}; M)$ について

$$\varphi c := \varphi \circ c$$

とおく。とくに $X \in \mathfrak{g}$ について $\rho(X)c := \varphi(X) \circ c$ と表す。

このとき $\forall X_i \in \mathfrak{g}$ について

$$d\varphi c - \varphi d c$$

$$= \sum_{i=1}^{p+1} (-1)^{i+1} (\rho(X_i)\varphi - \varphi \rho(X_i)) c(X_1, \overset{\hat{i}}{\cdots}, X_{p+1}) \quad \cdots\cdots (1)$$

がなりたつ。

$$\Phi : C^p(\mathfrak{g}; M) \to C^{p-1}(\mathfrak{g}; M)$$

$$\Phi := \sum_{j=1}^{r} \rho(Y_j) i(Y_j')$$

$$d\Phi + \Phi d = \sum_{j=1}^{r} d\rho(Y_j) i(Y_j') - \rho(Y_j) d\, i(Y_j') + \rho(Y_j) \mathcal{L}_{Y_j'}$$

$c \in C^p(\mathfrak{g}; M)$

$$(d\Phi + \Phi d) c(X_1, \cdots, X_p)$$

$$= \sum_{j=1}^{r} \sum_{i=1}^{p} (-1)^{i+1} (\rho(X_i)\rho(Y_j) - \rho(Y_j)\rho(X_i)) (i(Y_j') c)(X_1, \overset{\hat{i}}{\cdots}, X_p)$$

$$+ \sum_{j=1}^{r} \rho(Y_j) \varphi(Y_j') c(X_1, \cdots, X_p) - \sum_{j=1}^{r} \sum_{i=1}^{p} \rho(Y_j) c(X_1, \cdots, \overset{\overset{i}{\smile}}{[Y_j', X_i]}, \cdots, X_p)$$

$$= \sum_{j=1}^{r}\sum_{i=1}^{p}(-1)^{i+1}\rho([X_i, Y_j])\, c(Y_j', X_1, \overset{\hat{i}}{\cdots}, X_p)$$

$$+ \sum_{j=1}^{r}\rho(Y_j)\rho(Y_j')c(X_1,\cdots,X_p) - \sum_{j=1}^{r}\sum_{i=1}^{p}(-1)^{i+1}\rho(Y_j)c([Y_j', X_i], X_1, \overset{\hat{i}}{\cdots}, X_p)$$

ここで $\mathfrak{h} < \mathfrak{g}$ ideal より $[X_i, Y_j], [Y_j', X_i] \in \mathfrak{h}$ である

$$[X_i, Y_j] = \sum_{k=1}^{r}\beta([X_i, Y_j], Y_k')Y_k$$

$$[Y_j', X_i] = \sum_{k=1}^{r}\beta(Y_k, [Y_j', X_i])Y_k'$$

そこで $\mathfrak{h}\otimes\mathfrak{h}$ において

$$\sum_{j=1}^{r}[X_i, Y_j]\otimes Y_j' = \sum_{j=1}^{r}\sum_{k=1}^{r}\beta([X_i, Y_j], Y_k')Y_k\otimes Y_j' \qquad \cdots (2)$$

$$= \sum_{j=1}^{r}\sum_{k=1}^{r}\beta(Y_j, [Y_k', X_i])Y_k\otimes Y_j' = \sum_{k=1}^{r}Y_k\otimes[Y_k', X_i]$$

となるから、 $\rho(t_M) := \sum_{j=1}^{r}\rho(Y_j)\circ\rho(Y_j') \in \mathfrak{gl}(M)$ とおくと (2) より

$$(d\Phi + \Phi d)c(X_1,\cdots,X_p) = \sum_{j=1}^{r}\rho(Y_j)\rho(Y_j')c(X_1,\cdots,X_p)$$

$$= \rho(t_M)c(X_1,\cdots,X_p) \qquad \cdots (3)$$

となる。

他方、再び (2) より $\forall X = X_i \in \mathfrak{g}$ について

$$0 = \sum_{j=1}^{r}\rho(X)\rho(Y_j)\rho(Y_j') - \rho(Y_j)\cancel{\rho(X)\rho(Y_j')} - \rho(Y_j)\rho(Y_j')\rho(X) + \rho(Y_j)\cancel{\rho(X)}\rho(Y_j')$$

$$= \rho(X)\rho(t_M) - \rho(t_M)\rho(X) \qquad \cdots (4)$$

となるから

$\rho(t_M) : M \longrightarrow M$: $\mathfrak{g}$-homom

M : simple より Schur's Lemma から

$\rho(t_M) = 0$ または $\rho(t_M) : M \xrightarrow{\cong} M$ isom

ここで

$$\mathrm{trace}\,\rho(t_M) = \sum_{j=1}^{r}\mathrm{tr}(\rho(Y_j)\rho(Y_j')) = r \neq 0$$

ゆえに

$\rho(t_M) : M \xrightarrow{\cong} M$

No.

Date

ゆえに

$$p(t_M) : C^*(\mathfrak{g} : M) \xrightarrow{\cong} C^*(\mathfrak{g} : M)$$

さらにこれは cochain map (∵ (1)(4))

$$p(t_M)_* : H^*(\mathfrak{g} : M) \xrightarrow{\cong} H^*(\mathfrak{g} : M)$$

$\| \Leftarrow (3)$

0

ゆえに $H^*(\mathfrak{g} : M) = 0$ // Lemma

以下これを使って cohomology を計算する。
そのために半単純 Lie 代数と簡約 Lie 代数の
よく知られた事実を
[B] N. Bourbaki "Groupes et algèbres de Lie" Chapitre 1, Hermann (和訳あり)
から引用する

§3.2. Semi-simple Lie algebras

以下

$\mathfrak{g}$: finite dimensional Lie algebra $/\mathbb{K}$

とする。

Definition (再) (Weyl)

$\mathfrak{g}$: semi-simple
$\overset{\text{def}}{\Longleftrightarrow}$ $\forall M$: fin.dim$_{\mathbb{K}}$ $\mathfrak{g}$-module ; completely reducible

$\Rightarrow$ Cor 2.12 $H^1(\mathfrak{g} : \mathbb{K}) = 0$
Thm 2.13 $H^2(\mathfrak{g} : \mathbb{K}) = 0$

Theorem 3.2. (J.H.C. Whitehead)

$\mathfrak{g}$: semi-simple, M : fin.dim. $\mathfrak{g}$-module

$$\Longrightarrow H^*(\mathfrak{g} : M) = H^*(\mathfrak{g} : M^{\mathfrak{g}}) \; (= H^*(\mathfrak{g} : \mathbb{K}) \otimes M^{\mathfrak{g}})$$

proof M : completely reducible ゆえ $\Rightarrow \rho : \mathfrak{g} \to \mathfrak{gl}(M)$
M : non-trivial simple $\mathfrak{g}$-module について
$H^*(\mathfrak{g} : M) = 0$ を示せばよい

$M \Rightarrow \rho : \mathfrak{g} \to \mathfrak{gl}(M)$

$\mathrm{Ker}\,\rho \subset \mathfrak{g}$ ideal ($\Leftrightarrow$ $\mathfrak{g}$-submodule)

$\exists \mathfrak{h} < \mathfrak{g}$: $\mathfrak{g}$-submodule ($\Leftrightarrow$ ideal)

$\mathfrak{g} = \mathfrak{h} \oplus \mathrm{Ker}\,\rho$

$\mathfrak{h} \neq 0$ ($\Leftarrow$ M : non-trivial)

$\mathfrak{h}$: semi-simple ($\because$ $\mathfrak{g} \overset{\rho}{\twoheadrightarrow} \mathfrak{h}$ より $\mathfrak{h}$-module $\Rightarrow$ $\mathfrak{g}$-module
$\cup$ $\mathfrak{h}$-submodule $\Leftrightarrow$ $\mathfrak{g}$-submodule)

Theorem 3.3 (E. Cartan) ([B] §6. no. 1, Prop. 1)

$\mathfrak{g}$: semi-simple Lie algebra

M : fin. $\dim_{\mathbb{K}}$ $\mathfrak{g}$-module ($\Rightarrow \rho : \mathfrak{g} \to \mathfrak{gl}(M)$)

ρ : injective

$\Longrightarrow \beta_M : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, $(X, Y) \mapsto \mathrm{tr}(\rho(X)\rho(Y))$, non-degenerate

いまの場合

$\beta_M|_{\mathfrak{h} \times \mathfrak{h}}$: non-degenerate

ゆえに Lemma 3.1 より $H^*(\mathfrak{g} : M) = 0$.

これが示すべきことであった //

Corollary 3.4 (J. H. C. Whitehead)

$\mathfrak{g}$: semi-simple, M : fin. $\dim_{\mathbb{K}}$ $\mathfrak{g}$-module

$\Longrightarrow H^1(\mathfrak{g} : M) = H^2(\mathfrak{g} : M) = 0$

($\because$) Thm 3.2, Cor 2.12, Thm 2.13 //

次回

$H^3(\mathfrak{g} : \mathbb{K}) = ?$ $\Leftarrow$ $\mathfrak{g}$ の単純因子の個数

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