(§2. The cohomology group of a Lie algebra (フブ王))

§2.3. Cohomology groups of low degree.

$\mathbb{K}$: field of char. 0
$\mathfrak{g}$: Lie algebra/$\mathbb{K}$
M : $\mathfrak{g}$-module

$H^0(\mathfrak{g};M) = M^{\mathfrak{g}}$

(∵) $m \in M$, ($dm = 0 \Leftrightarrow \forall X \in \mathfrak{g} \quad Xm = 0$) //

$H^1(\mathfrak{g};M) = ?$

semi-direct product

$M \rtimes \mathfrak{g}$, $m_i \in M$, $X_i \in \mathfrak{g}$

$[(m_0, X_0), (m_1, X_1)] := (X_0 m_1 - X_1 m_0, [X_0, X_1])$

Jacobi identity

(略)

(∵) $[[(m_0, X_0), (m_1, X_1)], (m_2, X_2)]$
$= [(X_0 m_1 - X_1 m_0, [X_0, X_1]), (m_2, X_2)]$
$= ([X_0, X_1] m_2 - X_2 X_0 m_1 + X_2 X_1 m_0, [[X_0, X_1], X_2])$
$[X_0, X_1] m_2 - X_2 X_0 m_1 + X_2 X_1 m_0$
$+ [X_1, X_2] m_0 - X_0 X_1 m_2 + X_0 X_2 m_1$
$+ [X_2, X_0] m_1 - X_1 X_2 m_0 + X_1 X_0 m_2 = 0$ //

"obvious automorphism" $a \in M$.

$\varphi_a : M \rtimes \mathfrak{g} \to M \rtimes \mathfrak{g}$
$(m, X) \mapsto (m + Xa, X)$

$\varphi_a \varphi_b = \varphi_{a+b} \Rightarrow$ isom

$M \to \mathrm{Aut}(M \rtimes \mathfrak{g})$
action

homom

(∵) $[(m_0 + X_0 a, X_0), (m_1 + X_1 a, X_1)]$
$= (X_0 m_1 + X_0 X_1 a - X_1 m_0 - X_1 X_0 a, [X_0, X_1])$
$= \varphi_a([(m_0, X_0), (m_1, X_1)])$ //

$\pi : M \rtimes \mathfrak{g} \to \mathfrak{g}$, $(m, X) \mapsto X$, Lie algebra homomorphism.

— ∘ —.

$f : \mathfrak{g} \to M$ linear map

$s_f : \mathfrak{g} \to M \rtimes \mathfrak{g}$, $X \mapsto (f(X), X)$.

$X, Y \in \mathfrak{g}$.

$[(f(X), X), (f(Y), Y)] = (Xf(Y) - Yf(X), [X, Y])$

s_f : Lie algebra homom.

$\Leftrightarrow \forall X, \forall Y \in \mathfrak{g} \quad Xf(Y) - Yf(X) - f([X, Y]) = 0$

$\Leftrightarrow df = 0$

<u>Lemma 2.6.</u> $H^1(\mathfrak{g}; M) = \{\text{sections of } \pi : M \rtimes \mathfrak{g} \to \mathfrak{g}\}/M$

<u>Abelianization</u>

$[\mathfrak{g}, \mathfrak{g}] = \mathbb{K}$-lin. subsp. gen. by $\{[X, Y] : X, Y \in \mathfrak{g}\} \subset \mathfrak{g}$

derived ideal

$\mathfrak{g}^{abel} \overset{\text{def}}{=} \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ the abelianization of $\mathfrak{g}$

<u>Lemma 2.7</u> $H^1(\mathfrak{g}; \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}^{abel}, \mathbb{K})$.

(∵) $c : \mathfrak{g} \to \mathbb{K}$ $\mathbb{K}$-lin. map $\in C^1(\mathfrak{g}; \mathbb{K})$

$dc = 0 \Leftrightarrow c|_{[\mathfrak{g}, \mathfrak{g}]} = 0$ //

example

<u>Lemma 2.8.</u> $\forall n \geq 1$.

$[\mathfrak{sl}_n(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K})] = [\mathfrak{gl}_n(\mathbb{K}), \mathfrak{gl}_n(\mathbb{K})] = \mathfrak{sl}_n(\mathbb{K})$

∵) 2つの($\subset$)は明らか. . .

$\mathfrak{sl}_n(\mathbb{K}) \subset [\mathfrak{sl}_n(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K})]$

を示せばよい。 $1 \leq i, j \leq n$ $\quad E_{ij} = \begin{array}{c} \\ i) \end{array} \begin{pmatrix} & \overset{j}{\vdots} & \\ \cdots & 1 & \bigcirc \end{pmatrix}$ $\quad (E_{ij})_{kl} = \delta_{ik}\delta_{jl}$

$\mathfrak{sl}_n(\mathbb{K}) = \langle E_{ij}, E_{ii} - E_{jj} ; i \neq j \rangle$

$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = -2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ //

$H^1(\mathfrak{sl}_n(\mathbb{K});\mathbb{K})=0$
$H^1(\mathfrak{gl}_n(\mathbb{K});\mathbb{K})\cong\mathbb{K}$.

$H^2(\mathfrak{g};M)=?$

M: abelian Lie algebra とみなす.

Definition $M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g}$ extension
$\overset{\text{def}}{\Longleftrightarrow}$ 0) E: Lie algebra $/\mathbb{K}$
ι,π: Lie algebra homom's
1) $0\to M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g}\to0$ (exact)
2) $\forall X\in E$, $\forall m\in M$
$[X,m]=\iota(\pi(X)m)$.

$\left.\begin{array}{l}M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g}\\ M\overset{\iota'}{\hookrightarrow}E'\overset{\pi'}{\twoheadrightarrow}\mathfrak{g}\end{array}\right\}$ extensions

Definition $(M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g})\underset{\text{isom}}{\cong}(M\overset{\iota'}{\hookrightarrow}E'\overset{\pi'}{\twoheadrightarrow}\mathfrak{g})$
$\overset{\text{def}}{\Longleftrightarrow}$
$$\begin{tikzcd} M \arrow[r, hook, "\iota"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, two heads, "\pi"] \arrow[d, "\exists", "\text{Lie alg isom}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{g} \arrow[d, equal] \\ M \arrow[r, hook, "\iota'"] & E' \arrow[r, two heads, "\pi'"] & \mathfrak{g} \end{tikzcd}$$

$\mathcal{E}(\mathfrak{g};M)\overset{\text{def}}{=}\{M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g}\text{ extensions}\}/\cong$

Euler class $e:\mathcal{E}(\mathfrak{g};M)\longrightarrow H^2(\mathfrak{g};M)$

どのくらい半直積でないか?

$M\overset{\iota}{\hookrightarrow}E\overset{\pi}{\twoheadrightarrow}\mathfrak{g}$
$\exists\sigma$: $\mathbb{K}$-lin. map. s.t $\pi\circ\sigma=1_{\mathfrak{g}}$

σはどれくらいLie代数準同型でないか?

$f_\sigma(X,Y)\overset{\text{def}}{=}\iota^{-1}([\sigma(X),\sigma(Y)]-\sigma([X,Y]))\in M$.

$df_\sigma=0\in C^3(\mathfrak{g};M)$ ----- ①
$[f_\sigma]\in H^2(\mathfrak{g};M)$ indep of the choice of σ --- ②

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$e(E) \overset{\text{def}}{=} [f_\sigma] \in H^2(\mathfrak{g}:M)$ the Euler class of the extension E

$(M \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} \mathfrak{g}) \cong (M \overset{\iota'}{\hookrightarrow} E' \overset{\pi'}{\twoheadrightarrow} \mathfrak{g}) \Rightarrow e(E) = e(E') \in H^2(\mathfrak{g}:M)$ ... ③

逆に $f \in C^2(\mathfrak{g}:M)$ について

$E_f := M \oplus \mathfrak{g}$ as $\mathbb{K}$-vec. sp.

$[(m_0, X_0), (m_1, X_1)] := (X_0 m_1 - X_1 m_0 + f(X_0, X_1), [X_0, X_1])$

$m_0, m_1 \in M$, $X_0, X_1 \in \mathfrak{g}$

E_f : Lie algebra $/\mathbb{K}$ $\Leftrightarrow$ $df = 0$ ... ①′

Proposition 2.9

$e : \mathcal{E}(\mathfrak{g}:M) \xrightarrow{\cong} H^2(\mathfrak{g}:M)$ isomorphism $[M \rtimes \mathfrak{g}] \mapsto 0$

逆は $[f] \mapsto [E_f]$ で与えられる

レポート問題4 ①②③①′および Prop. 2.9 を示せ

Example (Virasoro cocycle)

$\mathfrak{dl} = \mathfrak{dl}_{\mathbb{K}} = \mathbb{K}[x, x^{-1}]\frac{d}{dx}$, the Lie algebra of Laurent polynomial vector fields

$(\overset{\text{dense}}{\subset} \mathrm{Vect}(S^1))$

Theorem 2.10 (Gel'fand - Fuks)

$H^*_{\text{conti}}(\mathrm{Vect}(S^1):\mathbb{R}) = \Lambda^*_{\mathbb{R}}(\mathrm{vir}, \theta)$

$\deg \mathrm{vir} = 2$, $\deg \theta = 3$

$\Leftarrow$ distribution (original proof)

Bott-Segal + Haefliger

$H^*_{\text{cont}}(\mathrm{Vect}(S^1); \mathbb{R}) \cong H^*(\mathrm{Map}(S^1, S^3); \mathbb{R})$

$C^*_{(0)}(\mathfrak{dl}_{\mathbb{R}} : \mathbb{R}) \cong C^*_{\text{conti}}(\mathrm{Vect}(S^1), \mathbb{R})^{SO(2)} \simeq C^*_{\text{conti}}(\mathrm{Vect}(S^1) : \mathbb{R})$

$\Rightarrow H^*(\mathfrak{dl}_{\mathbb{R}} : \mathbb{R}) \cong H^*_{\text{conti}}(\mathrm{Vect}(S^1); \mathbb{R})$

レポート問題5 Serre spectral sequence を使って次を示せ

$$H^*(\mathrm{Map}(S^1, S^3); \mathbb{R}) = \Lambda^*_{\mathbb{R}}(e, \theta)$$

$$(\deg e = 2,\ \deg \theta = 3)$$

以下 $H^1(\mathfrak{d}; \mathbb{K}) = 0$, $H^2(\mathfrak{d}; \mathbb{K}) \cong \mathbb{K}$ を直接計算で示す

$e_k := x^{k+1}\frac{d}{dx} \in \mathfrak{d}$ $(k \in \mathbb{Z})$ basis

$[e_k, e_\ell] = [x^{k+1}\frac{d}{dx}, x^{\ell+1}\frac{d}{dx}] = (\ell - k)\, x^{k+\ell+1}\frac{d}{dx} = (\ell - k)\, e_{k+\ell}$

$[e_0, e_k] = k e_k$

$\delta_k \in C^1(\mathfrak{d}; \mathbb{K})$, $\delta_k(e_\ell) := \delta_{k\ell}$, $(k, \ell \in \mathbb{Z})$

$\mathcal{L}_{e_0}\delta_k = -k\,\delta_k$

$C^*(\mathfrak{d}; \mathbb{K}) = \prod_{\lambda \in \mathbb{Z}} C^*_{(\lambda)}$, $C^*_{(\lambda)} = C^*_{(\lambda)}(\mathfrak{d}, \mathbb{K}) = \mathrm{Ker}(\mathcal{L}_{e_0} - \lambda)$

$H^*(C^*_{(\lambda)}) = 0$ $(\because\ 1_{C^*_{(\lambda)}} = d\,i(\frac{1}{\lambda}e_0) + i(\frac{1}{\lambda}e_0)\,d$ //$)$

$H^*(\mathfrak{d}; \mathbb{K}) = H^*(C^*_{(0)})$.

$C^0_{(0)} = \mathbb{K}$

$C^1_{(0)} = \mathbb{K}\,\delta_0$

$C^2_{(0)} = \{ f = \sum_{n=1}^{\infty} a_n\,\delta_n \vee \delta_{-n} ;\ a_n \in \mathbb{K}$, 無限和 $\}$

$a_n = f(e_n, e_{-n})$, $a_{-n} := -a_n = f(e_{-n}, e_n)$ $(n \geq 1)$

$n \in \mathbb{Z}$ $(d\delta_0)(e_n, e_{-n}) = -\delta_0([e_n, e_{-n}]) = 2n\,\delta_0(e_0) = 2n$

$a_n = 2n$ に対応。とくに $d\delta_0 \neq 0$

$\Longrightarrow H^1(\mathfrak{d}; \mathbb{K}) = 0$

$f = \sum_{n=1}^{\infty} a_n\,\delta_n \vee \delta_{-n} \in C^2_{(0)}$

$n, m \in \mathbb{Z}$

$(df)(e_n, e_m, e_{-n-m})$

$= -f([e_n, e_m], e_{-n-m}) - f([e_m, e_{-n-m}], e_n) - f([e_{-n-m}, e_n], e_m)$

$= -(m-n) f(e_{n+m}, e_{-n-m}) - (-2m-n) f(e_{-n}, e_n) - (2n+m) f(e_{-m}, e_m)$

$= -(m-n)\, a_{n+m} - (2m+n)\, a_n + (m+2n)\, a_m$

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$$df = 0$$
$$\Leftrightarrow \left(\begin{array}{l} \forall n, \forall m \in \mathbb{Z} \\ (m-n)a_{m+n} + (2m+n)a_n - (m+2n)a_m = 0 \end{array} \right) \cdots (\#)$$

$d\delta_0 = \sum_{n=1}^{\infty} 2n\,\delta_n \cup \delta_{-n}$ より $d(\frac{1}{2}a_1\delta_0)$ を ひいて $a_1 = 0$ としてよい

$(\#)|_{n=1}$; $(m-1)a_{m+1} = (m+2)a_m$

$$a_m = \frac{m+1}{m-2}a_{m-1} = \cdots = (m+1)m(m-1)a_2$$

$a_m = m^3 - m$ は $(\#)$ をみたす

(略) [∵) $(m-n)((m+n)^3-(m+n)) + (2m+n)(n^3-n) - (m+2n)(m^3-m)$
$= (m-n)(m+n)((m+n)^2-1) + (m+n)(n^3-n-m^3+m) + mn^3 - nm^3$
$= (m-n)(m+n)[(m+n)^2 - 1 + 1 - m^2 - mn - n^2 - mn] = 0$ //

$vir(e_n, e_m) \overset{def}{=} \frac{n^3-n}{12}\delta_{n+m,0}$ Virasoro cocycle

$$H^2(dl;\mathbb{K}) = \mathbb{K}[vir] \cong \mathbb{K}$$

$Vir := E_{vir}$, Virasoro algebra (dl よりも豊富な表現をもつ)

(fini. dim'l) semi-simple Lie algebra

$\mathfrak{g}$: Lie algebra/$\mathbb{K}$

M : $\mathfrak{g}$-module $\neq 0$

(Definition M : simple $\mathfrak{g}$-module
$\overset{def}{\Leftrightarrow}$ $0 \subsetneq \not\exists N \subsetneq M$ $\mathfrak{g}$-submodule

Schur's Lemma M, N : simple $\mathfrak{g}$-modules

$\cdot\ \varphi : M \to N$ $\mathfrak{g}$-homom, $\varphi \neq 0$

$\Rightarrow \varphi : M \to N$ isom

(∵) $\mathrm{Ker}\,\varphi = 0 \Rightarrow \varphi$: inj
$\mathrm{Im}\,\varphi = M \Rightarrow \varphi$: surj //

M : $\mathfrak{g}$-module

<u>Lemma 2.11</u> M : $\mathfrak{g}$-module について次の (a)(b) は互いに同値である

(a) $\exists \{M_\lambda\}_{\lambda\in\Lambda}$: simple $\mathfrak{g}$-submodules of M s.t. $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$

(b) $\forall N \subset M$ $\mathfrak{g}$-submodule $\exists L \subset M$ $\mathfrak{g}$-submodule $M = N \oplus L$

このとき $\forall N \subset M$ $\mathfrak{g}$-submodule も (a)(b) をみたす

<u>レポート問題6</u> これを示せ ($\Leftarrow$ Zorn's Lemma)

<u>Definition</u> M : completely reducible 完全可約

$\overset{\text{def}}{\Longleftrightarrow}$ Lem 2.11 の (a)(b) をみたす

<u>Definition</u> (H. Weyl)

$\mathfrak{g}$: (fin. dim'l) semi-simple Lie algebra $/\mathbb{K}$

$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\mathfrak{g}$: fin. $\dim_{\mathbb{K}}$'l Lie algebra $/\mathbb{K}$

1) $\forall M$: fin. $\dim_{\mathbb{K}}$ $\mathfrak{g}$-module M : completely reducible

<u>Theorem 2.12</u> $\mathfrak{g}$: (fin. dim'l) semi-simple Lie algebra

$\Longrightarrow$ $H^2(\mathfrak{g} : \mathbb{K}) = 0$

(pf) $\mathbb{K} \xrightarrow{\iota} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}$ extension

$\tilde{\mathfrak{g}} \ni \tilde{Y}$, $\mathfrak{g} \ni X$

$\rho(X)(\tilde{Y}) := [\tilde{X}, \tilde{Y}]$, $\tilde{X} \in \pi^{-1}(X)$

<u>well-defined</u> ($\because$ $\operatorname{Ker}\pi = \iota(\mathbb{K}) <$ Center $\tilde{\mathfrak{g}}$)

$\tilde{\mathfrak{g}}$: $\mathfrak{g}$-module via ρ

($\because$) $\tilde{X}_1 \in \pi^{-1}(X_1)$, $\tilde{X}_2 \in \pi^{-1}(X_2)$

$[\tilde{X}_1, [\tilde{X}_2, \tilde{Y}]] - [\tilde{X}_2, [\tilde{X}_1, \tilde{Y}]] = [[\tilde{X}_1, \tilde{X}_2], \tilde{Y}]$

$\pi([\tilde{X}_1, \tilde{X}_2]) = [X_1, X_2]$ //

$\tilde{\mathfrak{g}}$: fin. dim. $\mathfrak{g}$-module $\Longrightarrow$ completely reducible

$\exists M < \tilde{\mathfrak{g}}$: $\mathfrak{g}$-submodule, $\tilde{\mathfrak{g}} = \iota(\mathbb{K}) \oplus M$

$\sigma := (\pi|_M)^{-1} : \mathfrak{g} \to M$

$[\sigma(X), \sigma(Y)] = \rho(X)\sigma(Y) \in M$

$\| $

$\sigma([X, Y])$ σ : Lie algebra homom

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$\tilde{\mathfrak{g}} \cong \mathbb{K} \oplus \mathfrak{g}$ isom. as Lie algebras

$e(\tilde{\mathfrak{g}}) = 0 \in H^2(\mathfrak{g}; \mathbb{K})$

ゆえに

$H^2(\mathfrak{g}; \mathbb{K}) = 0$ //

次の§では半単純Lie代数および完約Lie代数の cohomology の消滅をもう少しくわしくみる