HARMONIC MAGNUS EXPANSION ON THE UNIVERSAL FAMILY OF RIEMANN SURFACES

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§1. Introduction

Let $g \geq 1$ be an integer. The purpose of my research is to construct "canonical" differential forms representing the Morita-Mumford classes (or the tautological classes) $e_i = (-1)^{i+1}\kappa_i$ on the moduli space $\mathbb{M}_g$ of compact Riemann surfaces of genus g, using a higher analogue of the period matrices of compact Riemann surfaces, **the harmonic Magnus expansion**.

There are two classical approaches to constructing differential forms representing the classes e_i's. Uniformization Theorem tells us the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$ of the universal family of compact Riemann surfaces has a canonical Hermitian metric, the hyperbolic metric. A notable work of S. Wolpert (Invent. math. **85**(1986), 119–145,) gives an explicit description of the differential forms on $\mathbb{M}_g$ representing the Morita-Mumford classes induced by the hyperbolic metric in terms of the resolvent of the hyperbolic Laplacian. On the other hand, from Grothendieck-Riemann-Roch Formula, the pullbacks of the Chern forms on the Siegel upper half space $\mathfrak{H}_g$ by the period matrix map represent all the <u>odd</u> Morita-Mumford classes.

In order to construct differential forms representing <u>all</u> the Morita-Mumford classes without making use of the hyperbolic metric, **we introduce a higher analogue of the period matrices of compact Riemann surfaces**. For simplicity, we consider the moduli space $\mathbb{M}_{g,1}$ of **triples** (C, P_0, v) of genus g instead of the space $\mathbb{M}_g$. Here C is a compact Riemann surface of genus g, $P_0 \in C$, and v a non-zero tangent vector of C at P_0. The space $\mathbb{M}_{g,1}$ is an aspherical $(3g-1)$-dimensional complex analytic manifold, and the fundamental group is equal to the mapping class group $\mathcal{M}_{g,1} := \pi_0 \operatorname{Diff}_+(\Sigma_g, p_0, v_0)$, where Σ_g is an oriented closed connected C^∞ 2-manifold of genus g, $p_0 \in \Sigma_g$, and $v_0 \in T_{p_0}\Sigma_g \setminus \{0\}$. The universal covering space is just the Teichmüller space $\mathcal{T}_{g,1}$ for the topological triple (Σ_g, p_0, v_0).

For any triple (C, P_0, v) one can define the fundamental group of the complement $C \setminus \{P_0\}$ with the tangential basepoint v, which we denote by $\pi_1(C, P_0, v)$. If we choose a symplectic generator of $\pi_1(C, P_0, v)$, we can identify it with a free group of rank $2g$, F_{2g}. This induces a homomorphism $\mathcal{M}_{g,1} \to \operatorname{Aut}(F_{2g})$, which is known to be an injection from a theorem of Nielsen.

Let $n \geq 2$ be an integer, F_n a free group of rank n with free basis $x_1, x_2, \dots, x_n$, $F_n = \langle x_1, x_2, \dots, x_n \rangle$. We denote by $H = H_{\mathbb{R}} := H_1(F_n; \mathbb{R})$ the first real homology group of the group F_n, by $[\gamma] \in H$ the homology class induced by $\gamma \in F_n$, and $X_i := [x_i] \in H$ for i, $1 \leq i \leq n$. The completed tensor algebra generated by H, $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}$, $p \geq 1$. It should be remarked that the subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We define a Magnus expansion of the free group F_n in our generalized sense.

Definition 1.1. *A map $\theta : F_n \to 1 + \widehat{T}_1$ is* **a** *(real-valued)* **Magnus expansion** *of the free group F_n, if*

(1) *$\theta : F_n \to 1 + \widehat{T}_1$ is a group homomorphism, and*
(2) *$\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$.*

We write $\theta(\gamma) = \sum_{m=0}^{\infty} \theta_m(\gamma)$, $\theta_m(\gamma) \in H^{\otimes m}$. The m-th component $\theta_m : F_n \to H^{\otimes m}$ is a map, but *not* a group homomorphism. We denote by $\Theta_n = \Theta_{n,\mathbb{R}}$ the set of all the real-valued Magnus expansions.

In this report we will explain some properties of the space Θ_n on which some Maurer-Cartan differential forms η_p, $p \geq 1$, are defined, its close relation to the (twisted) Morita-Mumford classes, and how we construct a canonical map we call the harmonic Magnus expansion $\theta : \mathcal{T}_{g,1} \to \Theta_{2g}$. The pullbacks $\theta^* \eta_p$, $p \geq 1$, give us the canonical differential forms representing the Morita-Mumford classes and their higher relations.

§2. Magnus Expansions and Johnson Maps

We start with an observation the space Θ_n has two kinds of group actions. The first one is given by the automorphism group $\mathrm{Aut}(F_n)$ of the group F_n. It acts on Θ_n by

$$\varphi \cdot \theta := |\varphi| \circ \theta \circ \varphi^{-1}, \quad (\varphi \in \mathrm{Aut}(F_n),\ \theta \in \Theta_n). \tag{2.1}$$

Here $|\varphi|$ is the induced map on $H = H_1(F_n; \mathbb{R})$, which acts on the tensor algebra $\widehat{T}$ in a natural way. The second is given by the (projective limit of) Lie group(s) $\mathrm{IA}(\widehat{T})$ of all the $\mathbb{R}$-algebra automorphisms $U : \widehat{T} \to \widehat{T}$ which satisfies $U(\widehat{T}_p) = \widehat{T}_p$, for any $p \geq 1$ and $U = 1_H$ on $\widehat{T}_1/\widehat{T}_2 = H$. It acts on Θ_n by $U \cdot \theta := U \circ \theta$, $(U \in \mathrm{IA}(\widehat{T}),\ \theta \in \Theta_n)$. The second action is **free and transitive**. It is easy to show we have a natural bijection of sets

$$\mathrm{IA}(\widehat{T}) \cong \prod_{m=1}^{\infty} \mathrm{Hom}(H, H^{\otimes m+1}) \cong \prod_{m=1}^{\infty} H^* \otimes H^{\otimes m+1}, \quad U \mapsto U|_H. \tag{2.2}$$

The difference of these two kinds of group actions induces **the Johnson map** associated to a fixed $\theta \in \Theta_n$

$$\tau^\theta : \mathrm{Aut}(F_n) \to \mathrm{IA}(\widehat{T}) = \prod_{m=1}^{\infty} H^* \otimes H^{\otimes m+1}, \quad \varphi \mapsto \tau^\theta(\varphi) = (\tau^\theta_m(\varphi))$$

by

$$\tau^\theta(\varphi) \circ \theta = \varphi \cdot \theta \ (= |\varphi| \circ \theta \circ \varphi^{-1}), \quad (\varphi \in \mathrm{Aut}(F_n)). \tag{2.3}$$

The restriction of **the p-th Johnson map** $\tau_p^\theta : \mathrm{Aut}(F_n) \to H^* \otimes H^{\otimes p+1}$ to some subgroup $\mathcal{M}(p) \subset \mathcal{M}_{g,1} \subset \mathrm{Aut}(F_{2g})$ coincides with the classical p-th Johnson homomorphism of the mapping class groups.

Since the bijection (2.2) is **not** a homomorphism, the Johnson maps τ_p^θ's are **not** homomorphisms. They satisfy an infinite sequence of cochain relations including

$$\tau_1^\theta(\varphi\psi) = \tau_1^\theta(\varphi) + |\varphi|\tau_1^\theta(\psi) \tag{2.4}$$

$$\tau_2^\theta(\varphi\psi) = \tau_2^\theta(\varphi) + (\tau_1^\theta(\varphi)\otimes 1 + 1\otimes\tau_1^\theta(\varphi))|\varphi|\tau_1^\theta(\psi) + |\varphi|\tau_2^\theta(\psi) \tag{2.5}$$

for any φ and $\psi \in \mathrm{Aut}(F_n)$.

The free and transitive action of $\mathrm{IA}(\widehat{T})$ on Θ_n induces the Maurer-Cartan form $\eta = (\eta_p) \in \Omega^1(\Theta_n)\widehat{\otimes}\mathrm{LieIA}(\widehat{T}) = \prod_{p=1}^\infty \Omega^1(\Theta_n)\otimes H^*\otimes H^{\otimes(p+1)}$. Using Chen's iterated integrals we have an integral presentation of the total Johnson map

$$\tau^\theta(\varphi)^{-1} = 1 + \sum\nolimits_{m=1}^\infty \int_\theta^{|\varphi|\circ\theta\circ\varphi^{-1}} \overbrace{\eta\eta\cdots\eta}^{m} \tag{2.6}$$

for any $\varphi \in \mathrm{Aut}(F_n)$. Especially the closed 1-form η_1 represents the cohomology class $[\tau_1^\theta] \in H^1(\mathrm{Aut}(F_n); H^*\otimes H^{\otimes 2})$. From the Maurer-Cartan formula $d\eta = \eta\wedge\eta$ we obtain

$$d\eta_p = \sum_{s=1}^{p-1} (\underbrace{\eta_s\otimes 1\otimes\cdots\otimes 1 + \cdots + 1\otimes\cdots\otimes 1\otimes\eta_s}_{p-s+1}) \circ \eta_{p-s}, \tag{2.7}$$

which suggests us a close relation between the Johnson maps and the Stasheff associahedrons K_p (J.D. Stasheff, Trans. Amer. Math. Soc., **108**(1963) 275–292).

Consider the double cochain complex

$$C^{*,*} := C^*(K_{p+1}; \Omega^*(\Theta_n; H^*\otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)}),$$

that is, the cellular cochain complex of K_{p+1} with values in the de Rham complex of Θ_n with twisted coefficients in $\mathrm{Aut}(F_n)$-module $H^*\otimes H^{\otimes(p+1)}$. The formula (2.7) means the Maurer-Cartan forms η_p's induce a p-cocycle $Y_p \in Z^p(C^{*,*})$, whose cohomology class

$$[Y_p] \in H^p(C^{*,*}) \cong H^p(\Omega^*(\Theta_n; H^*\otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)})$$

induces the $(0, p+2)$-twisted Morita-Mumford class on the moduli space $\mathbb{M}_{g,1}$.

§3. Applications to Cohomology of $\mathrm{Aut}(F_n)$.

The formula (2.4) means τ_1^θ is a 1-cocycle of $\mathrm{Aut}(F_n)$ with values in $H^*\otimes H^{\otimes 2}$. If we restrict it to the mapping class group $\mathcal{M}_{g,1} \subset \mathrm{Aut}(F_{2g})$ we have

Theorem 3.1. $\tau_1^\theta|_{\mathcal{M}_{g,1}} = \frac{1}{6}m_{0,3} \in H^1(\mathcal{M}_{g,1}; H^{\otimes 3})$.

Here it should be remarked H and its dual H^* are $\mathcal{M}_{g,1}$-isomorphic to each other by the intersection form for the closed surface. The cohomology class $m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1}; \Lambda^j H)$ is the (i,j)-twisted Morita-Mumford class (K-., Invent. math. **131** (1998), 137–149.) As was shown in S. Morita and K-., Math. Research Lett. **3** (1996), 629–641, all the algebraic combinations of the twisted Morita-Mumford classes using by the intersection form are just the polynomials of all the Morita-Mumford classes. Consequently the 1-cocycle τ_1^θ, or equivalently, the closed 1-form η_1 yields all the Morita-Mumford classes.

D. Johnson, Topology **24** (1985), 127–144, proved the first Johnson homomorphism induces an isomorphism of the abelianization or the Torelli group onto the space $\Lambda^3 H_\mathbb{Z}$ (or $\Lambda^3 H_\mathbb{Z}/H_\mathbb{Z}$) up to 2-torsions. A similar result holds for the group $\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}(F_n) \to \mathrm{GL}(H))$.

Theorem 3.2. *The first Johnson map τ_1^θ induces an isomorphism $\tau_1 : {IA_n}^{\mathrm{abel}} \xrightarrow{\cong} {H_\mathbb{Z}}^* \otimes \Lambda^2 H_\mathbb{Z}$.*

Using the cohomology class $[\tau_1^\theta] \in H^1(\mathrm{Aut}(F_n); H_\mathbb{Z}^* \otimes H_\mathbb{Z}^{\otimes 2})$ induced by a $\mathbb{Z}$-valued Magnus expansion θ, we obtain

Theorem 3.3. *Suppose $1-n$ is invertible in a commutative ring R. Then we have a natural decomposition of the cohomology group*

$$H^*(\mathrm{Aut}(F_n); M) = H^*(\mathrm{Out}(F_n); M) \oplus H^{*-1}(\mathrm{Out}(F_n); {H_R}^* \otimes M)$$

for any $R[\mathrm{Out}(F_n)]$-module M. Especially, $\pi^* : H^*(\mathrm{Out}(F_n); M) \to H^*(\mathrm{Aut}(F_n); M)$ is an injection.

For details, see K.-, preprint, arXiv:math.GT/0505497.

§4. Harmonic Magnus Expansions

The map $H^* = H^1(C; \mathbb{R}) \to \Omega^1(C)$ assigning each cohomology class the harmonic 1-form representing it can be regarded as a H-valued 1-form $\omega_{(1)} \in \Omega^1(C) \otimes H$. We denote by φ' and φ'' the $(1,0)$- and the $(0,1)$-parts of $\varphi \in \Omega^1(C) \otimes \mathbb{C}$, respectively. Then ${\omega_{(1)}}'$ is holomorphic, and ${\omega_{(1)}}''$ anti-holomorphic. We have $\int_C \omega_{(1)} \wedge \omega_{(1)} = I \in H^{\otimes 2}$, the intersection form. We denote by $\delta_{P_0} : C^\infty(C) \to \mathbb{R}$, $f \mapsto f(P_0)$, the delta 2-current on C at P_0. Then we have a $\widehat{T}$-valued 1-current $\omega = \sum_{p \geq 1} \omega_{(p)}$, $\omega_{(p)} \in \Omega^1(C) \otimes H^{\otimes p}$, satisfying the (modified) integrability condition

$$d\omega = \omega \wedge \omega - I \cdot \delta_0, \tag{4.1}$$

$\omega_{(p)} = \omega_{(1)}$ for $p = 1$, and the normalization condition $\int_C \omega_{(p)} \wedge *\varphi = 0$ for any closed 1-form φ and each $p \geq 2$. Here $*$ is the Hodge $*$-operator on $\Omega^1(C)$, which is conformal invariant of the Riemann surface C. Moreover, using Chen's iterated integrals, we can define a Magnus expansion

$$\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1 + \widehat{T}_1, \quad [\ell] \mapsto 1 + \sum_{m=1}^\infty \int_\ell \overbrace{\omega\omega\cdots\omega}^{m}.$$

The Magnus expansions $\theta^{(C,P_0,v)}$ for all the triples (C,P_0,v) define a canonical real analytic map $\theta : \mathcal{T}_{g,1} \to \Theta_{2g}$, which we call **the harmonic Magnus expansion on the universal family of Riemann surfaces**. The pullbacks of the Maurer-Cartan forms η_p's give the canonical differential forms representing the Morita-Mumford classes and their higher relations. Our main result in this report is

Theorem 4.1. *For any $[C,P_0,v] \in \mathbb{M}_{g,1}$ we have*

$$(\theta^*\eta)_{[C,P_0,v]} = 2\Re(N(\omega'\omega') - 2{\omega_{(1)}}'{\omega_{(1)}}') \in T^*_{[C,P_0,v]}\mathbb{M}_{g,1} \otimes \widehat{T}_3.$$

Here $N : \widehat{T}_1 \to \widehat{T}_1$ is defined by $N|_{H^{\otimes m}} := \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}^k$, and the meromorphic quadratic differential $N(\omega'\omega')$ is regarded as a $(1,0)$-cotangent vector at $[C,P_0,v] \in \mathbb{M}_{g,1}$ in a natural way.

The second homogeneous term of $N(\omega'\omega')$ coincides with $2{\omega_{(1)}}'{\omega_{(1)}}'$, which is just the first variation of the period matrices (H.E. Rauch, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 42–49.)

The space $\mathbb{M}_{g,1}$ can be regarded as the complement of the zero section in the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$ of the universal family $\mathbb{C}_g$ over the moduli space $\mathbb{M}_g$. Let $\varpi : \mathbb{M}_{g,1} \to \mathbb{C}_g$ be the projection. Then there exists a unique 1-form $Q_\tau \in \Omega^1(\mathbb{C}_g; \Lambda^3 H)$ such that $\varpi^* Q_\tau = \theta^*\eta_1$. The 1-form Q_τ represents the extended first Johnson homomorphism on the mapping class group $\mathcal{M}_{g,*} := \pi_0 \operatorname{Diff}_+(\Sigma_g, p_0)$. In view of a theorem of Morita (in 'Topology and Teichmüller Spaces,' World Scientific, 1996, 159–186,) there exist unique α_0 and $\alpha_1 \in \operatorname{Hom}((\Lambda^3 H)^{\otimes 2}, \mathbb{R})^{Sp_{2g}(\mathbb{R})}$ such that $e^J := \alpha_{0*}({Q_\tau}^{\otimes 2})$ and ${e_1}^J := \alpha_{1*}({Q_\tau}^{\otimes 2}) \in \Omega^2(\mathbb{C}_g)$ represent the first Chern class of $T_{\mathbb{C}_g/\mathbb{M}_g}$ and the first Morita-Mumford class, respectively. The form e_1^J can be regarded as a 2-form on $\mathbb{M}_g$. The 1-form Q_τ is known as the first variation of the pointed harmonic volume (B. Harris, Acta Math., **150** (1983), 91–123.) In the context of Hodge theory R. Hain and D. Reed (J. Diff. Geom., **67** (2004), 195–228,) introduced the 1-form Q_τ and studied the difference $e_1^J - 12c_1(\lambda, L^2)$ in detail, where λ is the Hodge line bundle over $\mathfrak{H}_g$.

The Chern form e^J seems to be related to Arakelov's admissible metric. Let $B := \frac{\sqrt{-1}}{2g}\sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$ be the volume form on a compact Riemann surface C induced by the orthonormal basis $\{\psi_i\}_{i=1}^g$ of the holomorphic 1-forms, $\frac{\sqrt{-1}}{2}\int_C \psi_i \wedge \overline{\psi_j} = \delta_{i,j}$, $(1 \le i,j \le g.)$ Let h be the function on $\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ satisfying the conditions $\frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}h\big|_{C\times\{P_0\}} = B - \delta_{P_0}$ and $\int_C \left(h\big|_{C\times\{P_0\}}\right) B = 0$. Then we obtain

Theorem 4.2.

$$\left(\frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}h\right)\bigg|_{\text{diagonal}} = e^J + \frac{1}{(2-2g)^2}({e_1}^J - {e_1}^F) \ \in \ \Omega^2(\mathbb{C}_g).$$

Here ${e_1}^F := \int_{\text{fiber}} (e^J)^2 \in \Omega^2(\mathbb{M}_g)$.

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