HARMONIC MAGNUS EXPANSION ON THE
UNIVERSAL FAMILY OF RIEZMANN SURFACES

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§1. Introduction

Let \( g \geq 1 \) be an integer. The purpose of my research is to construct “canonical” differential forms representing the Morita-Mumford classes (or the tautological classes) \( e_i = (-1)^{i+1} \kappa_i \) on the moduli space \( \mathcal{M}_g \) of compact Riemann surfaces of genus \( g \), using a higher analogue of the period matrices of compact Riemann surfaces, the harmonic Magnus expansion.

There are two classical approaches to constructing differential forms representing the classes \( e_i \)’s. Uniformization Theorem tells us the relative tangent bundle \( T_{C_g/M_g} \) of the universal family of compact Riemann surfaces has a canonical Hermitian metric, the hyperbolic metric. A notable work of S. Wolpert (Invent. math. 85 (1986), 119–145,) gives an explicit description of the differential forms on \( \mathcal{M}_g \) representing the Morita-Mumford classes induced by the hyperbolic metric in terms of the resolvent of the hyperbolic Laplacian. On the other hand, from Grothendieck-Riemann-Roch Formula, the pullbacks of the Chern forms on the Siegel upper half space \( \mathcal{H}_g \) by the period matrix map represent all the odd Morita-Mumford classes.

In order to construct differential forms representing all the Morita-Mumford classes without making use of the hyperbolic metric, we introduce a higher analogue of the period matrices of compact Riemann surfaces. For simplicity, we consider the moduli space \( \mathcal{M}_{g,1} \) of triples \( (C, P_0, v) \) of genus \( g \) instead of the space \( \mathcal{M}_g \). Here \( C \) is a compact Riemann surface of genus \( g \), \( P_0 \in C \), and \( v \) a non-zero tangent vector of \( C \) at \( P_0 \). The space \( \mathcal{M}_{g,1} \) is an aspherical \((3g-1)\)-dimensional complex analytic manifold, and the fundamental group is equal to the mapping class group \( \mathcal{M}_{g,1} := \pi_0 \text{Diff}_+(\Sigma_g, p_0, v_0) \), where \( \Sigma_g \) is an oriented closed connected \( C^\infty \) 2-manifold of genus \( g \), \( p_0 \in \Sigma_g \), and \( v_0 \in T_{p_0} \Sigma_g \setminus \{0\} \). The universal covering space is just the Teichmüller space \( \mathcal{T}_{g,1} \) for the topological triple \( (\Sigma_g, p_0, v_0) \).

For any triple \( (C, P_0, v) \) one can define the fundamental group of the complement \( C \setminus \{P_0\} \) with the tangential basepoint \( v \), which we denote by \( \pi_1(C, P_0, v) \). If we choose a symplectic generator of \( \pi_1(C, P_0, v) \), we can identify it with a free group of rank \( 2g, F_{2g} \). This induces a homomorphism \( \mathcal{M}_{g,1} \to \text{Aut}(F_{2g}) \), which is known to be an injection from a theorem of Nielsen.
Let \( n \geq 2 \) be an integer, \( F_n \) a free group of rank \( n \) with free basis \( x_1, x_2, \ldots, x_n, \) \( F_n = \langle x_1, x_2, \ldots, x_n \rangle. \) We denote by \( H = H_\mathbb{R} := H_1(F_n; \mathbb{R}) \) the first real homology group of the group \( F_n, \) by \( [\gamma] \in H \) the homology class induced by \( \gamma \in F_n, \) and \( X_i := [x_i] \in H \) for \( i, 1 \leq i \leq n. \) The completed tensor algebra generated by \( H, \) \( \hat{T} = \hat{T}(H) := \prod_{m=0}^\infty H^{\otimes m} \), has a decreasing filtration of two-sided ideals \( \hat{T}_p := \prod_{m \geq p} H^{\otimes m}, p \geq 1. \) It should be remarked that the subset \( 1 + \hat{T}_1 \) is a subgroup of the multiplicative group of the algebra \( \hat{T}. \) We define a Magnus expansion of the free group \( F_n \) in our generalized sense.

**Definition 1.1.** A map \( \theta : F_n \to 1 + \hat{T}_1 \) is a (real-valued) Magnus expansion of the free group \( F_n, \) if

1. \( \theta : F_n \to 1 + \hat{T}_1 \) is a group homomorphism, and
2. \( \theta(\gamma) \equiv 1 + [\gamma] \) (mod \( \hat{T}_2) \) for any \( \gamma \in F_n. \)

We write \( \theta(\gamma) = \sum_{m=0}^\infty \theta_m(\gamma), \theta_m(\gamma) \in H^{\otimes m}. \) The \( m \)-th component \( \theta_m : F_n \to H^{\otimes m} \) is a map, but not a group homomorphism. We denote by \( \Theta_n = \Theta_{n, \mathbb{R}} \) the set of all the real-valued Magnus expansions.

In this report we will explain some properties of the space \( \Theta_n \) on which some Maurer-Cartan differential forms \( \eta_p, p \geq 1, \) are defined, its close relation to the (twisted) Morita-Mumford classes, and how we construct a canonical map we call the harmonic Magnus expansion \( \theta : T_{g, 1} \to \Theta_{2g}. \) The pullbacks \( \theta^* \eta_p, p \geq 1, \) give us the canonical differential forms representing the Morita-Mumford classes and their higher relations.

§2. Magnus Expansions and Johnson Maps

We start with an observation the space \( \Theta_n \) has two kinds of group actions. The first one is given by the automorphism group \( \text{Aut}(F_n) \) of the group \( F_n. \) It acts on \( \Theta_n \) by

\[
\varphi \cdot \theta := |\varphi| \circ \theta \circ \varphi^{-1}, \quad (\varphi \in \text{Aut}(F_n), \ \theta \in \Theta_n).
\] (2.1)

Here \( |\varphi| \) is the induced map on \( H = H_1(F_n; \mathbb{R}), \) which acts on the tensor algebra \( \hat{T} \) in a natural way. The second is given by the (projective limit of) Lie group(s) \( \text{IA}(\hat{T}) \) of all the \( \mathbb{R} \)-algebra automorphisms \( U : \hat{T} \to \hat{T} \) which satisfies \( U(\hat{T}_p) = \hat{T}_p, \) for any \( p \geq 1 \) and \( U = 1_H \) on \( \hat{T}_1/\hat{T}_2 = H. \) It acts on \( \Theta_n \) by \( U \cdot \theta := U \circ \theta, \) \( (U \in \text{IA}(\hat{T}), \ \theta \in \Theta_n). \) The second action is free and transitive. It is easy to show we have a natural bijection of sets

\[
\text{IA}(\hat{T}) \cong \prod_{m=1}^\infty \text{Hom}(H, H^{\otimes m+1}) \cong \prod_{m=1}^\infty H^* \otimes H^{\otimes m+1}, \quad U \mapsto U|_H. \] (2.2)

The difference of these two kinds of group actions induces the Johnson map associated to a fixed \( \theta \in \Theta_n \)

\[
\tau^\theta : \text{Aut}(F_n) \to \text{IA}(\hat{T}) = \prod_{m=1}^\infty H^* \otimes H^{\otimes m+1}, \quad \varphi \mapsto \tau^\theta(\varphi) = (\tau^\theta_m(\varphi))
\]

by

\[
\tau^\theta(\varphi) \circ \theta = \varphi \cdot \theta \ (|\varphi| \circ \theta \circ \varphi^{-1}), \quad (\varphi \in \text{Aut}(F_n)). \] (2.3)
The restriction of the \( p \)-th Johnson map \( \tau_p^\theta : \text{Aut}(F_n) \to H^* \otimes H^{\otimes p+1} \) to some subgroup \( \mathcal{M}(p) \subset \mathcal{M}_{g,1} \subset \text{Aut}(F_{2g}) \) coincides with the classical \( p \)-th Johnson homomorphism of the mapping class groups.

Since the bijection (2.2) is not a homomorphism, the Johnson maps \( \tau_p^\theta \)'s are not homomorphisms. They satisfy an infinite sequence of cochain relations including

\[
\begin{align*}
\tau_1^\theta(\varphi \psi) &= \tau_1^\theta(\varphi) + |\varphi|\tau_1^\theta(\psi) \\
\tau_2^\theta(\varphi \psi) &= \tau_2^\theta(\varphi) + (\tau_1^\theta(\varphi) \otimes 1 + 1 \otimes \tau_1^\theta(\varphi))|\varphi|\tau_1^\theta(\psi) + |\varphi|\tau_2^\theta(\psi)
\end{align*}
\]

for any \( \varphi \) and \( \psi \in \text{Aut}(F_n) \).

The free and transitive action of \( \text{IA}(\hat{T}) \) on \( \Theta_n \) induces the Maurer-Cartan form \( \eta = (\eta_p) \in \Omega^1(\Theta_n) \otimes \text{Lie}(\hat{T}) = \prod_{p=1}^\infty \Omega^1(\Theta_n) \otimes H^* \otimes H^{\otimes (p+1)} \). Using Chen’s iterated integrals we have an integral presentation of the total Johnson map

\[
\tau^\theta(\varphi)^{-1} = 1 + \sum_{m=1}^\infty \int_{\theta}^{\varphi} \cdots \eta \eta \eta \quad (2.6)
\]

for any \( \varphi \in \text{Aut}(F_n) \). Especially the closed 1-form \( \eta_1 \) represents the cohomology class \( [\tau_1^\theta] \in H^1(\text{Aut}(F_n); H^* \otimes H^{\otimes 2}) \). From the Maurer-Cartan formula \( d\eta = \eta \wedge \eta \) we obtain

\[
d\eta_p = \sum_{s=1}^{p-1} (\eta_s \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \eta_s) \circ \eta_{p-s}, \quad (2.7)
\]


Consider the double cochain complex

\[
C^{*,*} := C^*(K_{p+1}; \Omega^*(\Theta_n; H^* \otimes H^{\otimes (p+1)})^{\text{Aut}(F_n)}),
\]

that is, the cellular cochain complex of \( K_{p+1} \) with values in the de Rham complex of \( \Theta_n \) with twisted coefficients in \( \text{Aut}(F_n) \)-module \( H^* \otimes H^{\otimes (p+1)} \). The formula (2.7) means the Maurer-Cartan forms \( \eta_p \)'s induce a \( p \)-cocycle \( Y_p \in Z^p(C^{*,*}) \), whose cohomology class

\[
[Y_p] \in H^p(C^{*,*}) \cong H^p(\Omega^*(\Theta_n; H^* \otimes H^{\otimes (p+1)})^{\text{Aut}(F_n)})
\]

induces the \((0, p+2)\)-twisted Morita-Mumford class on the moduli space \( \mathcal{M}_{g,1} \).

§3. Applications to Cohomology of \( \text{Aut}(F_n) \).

The formula (2.4) means \( \tau_1^\theta \) is a 1-cocycle of \( \text{Aut}(F_n) \) with values in \( H^* \otimes H^{\otimes 2} \). If we restrict it to the mapping class group \( \mathcal{M}_{g,1} \subset \text{Aut}(F_{2g}) \) we have
Theorem 3.1. \( \tau^0_1|_{\mathcal{M}_{g,1}} = \frac{1}{6} m_{0,3} \in H^1(\mathcal{M}_{g,1}; H^{\otimes 3}). \)

Here it should be remarked \( H \) and its dual \( H^* \) are \( \mathcal{M}_{g,1} \)-isomorphic to each other by the intersection form for the closed surface. The cohomology class \( m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1}; \Lambda^j H) \) is the \( (i,j) \)-twisted Morita-Mumford class ( K., Invent. math. 131 (1998), 137–149.) As was shown in S. Morita and K., Math. Research Lett. 3 (1996), 629–641, all the algebraic combinations of the twisted Morita-Mumford classes using by the intersection form are just the polynomials of all the Morita-Mumford classes. Consequently the 1-cocycle \( \tau^0_1 \), or equivalently, the closed 1-form \( \eta_1 \) yields all the Morita-Mumford classes.

D. Johnson, Topology 24 (1985), 127–144, proved the first Johnson homomorphism induces an isomorphism of the abelianization or the Torelli group onto the space \( \Lambda^3 H \) (or \( \Lambda^3 H_*/H_* \)) up to 2-torsions. A similar result holds for the group \( IA_n := \text{Ker}(\text{Aut}(F_n) \to \text{GL}(H)) \).

Theorem 3.2. The first Johnson map \( \tau^0_1 \) induces an isomorphism \( \tau_1 : IA_n^{\text{abel}} \cong H_*^{\text{rel}} \otimes \Lambda^2 H_* \).

Using the cohomology class \([\tau^0_1] \in H^1(\text{Aut}(F_n); H_*^{\text{rel}} \otimes H_*^{\text{rel}}) \) induced by a \( \mathbb{Z} \)-valued Magnus expansion \( \theta \), we obtain

Theorem 3.3. Suppose \( 1-n \) is invertible in a commutative ring \( R \). Then we have a natural decomposition of the cohomology group

\[
H^*(\text{Aut}(F_n); M) = H^*(\text{Out}(F_n); M) \oplus H^{*-1}(\text{Out}(F_n); H_*^R \otimes M)
\]

for any \( R[\text{Out}(F_n)] \)-module \( M \). Especially, \( \pi^* : H^*(\text{Out}(F_n); M) \to H^*(\text{Aut}(F_n); M) \) is an injection.


§4. Harmonic Magnus Expansions

The map \( H^* = H^1(C; \mathbb{R}) \to \Omega^1(C) \) assigning each cohomology class the harmonic 1-form representing it can be regarded as a \( H \)-valued 1-form \( \omega^{(1)} \in \Omega^1(C) \otimes H \). We denote by \( \varphi' \) and \( \varphi'' \) the \((1,0)\) and the \((0,1)\)-parts of \( \varphi \in \Omega^1(C) \otimes \mathbb{C} \), respectively. Then \( \omega^{(1)'} \) is holomorphic, and \( \omega^{(1)''} \) anti-holomorphic. We have \( \int_C \omega^{(1)} \wedge \omega^{(1)} = I \in H^{\otimes 2} \), the intersection form. We denote by \( \delta_{P_0} : C^\infty(C) \to \mathbb{R}, f \mapsto f(P_0) \), the delta 2-current on \( C \) at \( P_0 \). Then we have a \( \widehat{T} \)-valued 1-current \( \omega = \sum_{p \geq 1} \omega_{(p)}, \omega_{(p)} \in \Omega^1(C) \otimes H^{\otimes p} \), satisfying the (modified) integrability condition

\[
d\omega = \omega \wedge \omega - I \cdot \delta_0, \quad (4.1)
\]

\( \omega_{(p)} = \omega_{(1)} \) for \( p = 1 \), and the normalization condition \( \int_C \omega_{(p)} \wedge * \varphi = 0 \) for any closed 1-form \( \varphi \) and each \( p \geq 2 \). Here * is the Hodge operator on \( \Omega^1(C) \), which is conformal invariant of the Riemann surface \( C \). Moreover, using Chen’s iterated integrals, we can define a Magnus expansion

\[
\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1 + \widehat{T}, \quad [\ell] \mapsto 1 + \sum_{m=1}^{\infty} \int_\ell \omega \wedge \cdots \wedge \omega.
\]
The Magnus expansions \( \theta^{(C,P_0,v)} \) for all the triples \( (C, P_0, v) \) define a canonical real analytic map \( \theta : T_{g,1} \to \Theta_{2g} \), which we call the harmonic Magnus expansion on the universal family of Riemann surfaces. The pullbacks of the Maurer-Cartan forms \( \eta_p \)'s give the canonical differential forms representing the Morita-Mumford classes and their higher relations. Our main result in this report is

**Theorem 4.1.** For any \( [C, P_0, v] \in \mathcal{M}_{g,1} \) we have

\[
(\theta^* \eta)_{[C, P_0, v]} = 2\Re(N(\omega') - 2\omega(1)') \in T^*_{[C, P_0, v]} \mathcal{M}_{g,1} \otimes \hat{T}_3.
\]

Here \( N : \hat{T}_1 \to \hat{T}_1 \) is defined by \( N|_{H^{g,m}} := \sum_{k=0}^{m-1} \binom{1}{2} \binom{2}{3} \cdots \binom{m-1}{m} \binom{m}{1} \), and the meromorphic quadratic differential \( N(\omega') \) is regarded as a \((1,0)\)-cotangent vector at \([C, P_0, v] \in \mathcal{M}_{g,1} \) in a natural way.

The second homogeneous term of \( N(\omega') \) coincides with \( 2\omega(1)')', \) which is just the first variation of the period matrices (H.E. Rauch, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 42–49.)

The space \( \mathcal{M}_{g,1} \) can be regarded as the complement of the zero section in the relative tangent bundle \( T_{\mathcal{C}_g/\mathcal{M}_g} \) of the universal family \( \mathcal{C}_g \) over the moduli space \( \mathcal{M}_g \). Let \( \varpi : \mathcal{M}_{g,1} \to \mathcal{C}_g \) be the projection. Then there exists a unique 1-form \( Q_\tau \in \Omega^1(\mathcal{C}_g; \Lambda^3 \mathcal{H}) \) such that \( \varpi^* Q_\tau = \theta^* \eta_1 \). The 1-form \( Q_\tau \) represents the extended first Johnson homomorphism on the mapping class group \( \mathcal{M}_{g,*} := \pi_0 \text{Diff}_+ (\Sigma_g, p_0) \). In view of a theorem of Morita (‘Topology and Teichmüller Spaces,’ World Scientific, 1996, 159–186) there exist unique \( \alpha_0 \) and \( \alpha_1 \in \text{Hom}(\Lambda^3 \mathcal{H}^{\otimes 2}, \mathbb{R})^{Sp_{2g}(\mathbb{R})} \) such that \( e^J := \alpha_0(Q_\tau^\otimes 2) \) and \( e^J_1 := \alpha_1(Q_\tau^\otimes 2) \in \Omega^2(\mathcal{C}_g) \) represent the first Chern class of \( T_{\mathcal{C}_g/\mathcal{M}_g} \) and the first Morita-Mumford class, respectively. The form \( e^J_1 \) can be regarded as a 2-form on \( \mathcal{M}_g \). The 1-form \( Q_\tau \) is known as the first variation of the pointed harmonic volume (B. Harris, Acta Math., **150** (1983), 91–123.) In the context of Hodge theory R. Hain and D. Reed (J. Diff. Geom., **67** (2004), 195–228,) introduced the 1-form \( Q_\tau \) and studied the difference \( e^J_1 - 12c_1(\lambda, L^2) \) in detail, where \( \lambda \) is the Hodge line bundle over \( \mathcal{H}_g \).

The Chern form \( e^J \) seems to be related to Arakelov’s admissible metric. Let \( B := \sqrt{\frac{2}{g}} \sum_{i=1}^{g} \psi_i \wedge \overline{\psi}_i \) be the volume form on a compact Riemann surface \( C \) induced by the orthonormal basis \( \{ \psi_i \}_{i=1}^{g} \) of the holomorphic 1-forms, \( \sqrt{\frac{2}{g}} \int_C \psi_i \wedge \overline{\psi}_j = \delta_{i,j}, \) \( (1 \leq i, j \leq g) \). Let \( h \) be the function on \( \mathcal{C}_g \times \mathcal{M}_g \) satisfying the conditions \( \frac{1}{2\pi \sqrt{-1}} \partial \overline{\partial} h \big|_{C \times \{ P_0 \}} = B - \delta_{P_0} \) and \( \int_C (h \big|_{C \times \{ P_0 \}})^2 = 0 \). Then we obtain

**Theorem 4.2.**

\[
\left( \frac{1}{2\pi \sqrt{-1}} \partial \overline{\partial} h \right)_{\text{diagonal}} = e^J + \frac{1}{(2 - 2g)^2} (e^J_1 - e^J) \in \Omega^2(\mathcal{C}_g).
\]

Here \( e^J_1 := \int_{\text{fiber}} (e^J)^2 \in \Omega^2(\mathcal{M}_g) \).

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