

# 特異圏の傾理論

京都大学集中講義 2021/1/4 ~ 8

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単位が必要な学生は, 1/22 までに以下のレポート問題を提出すること

- p4 上
- p8 中
- p15 下
- p20 下
- p40 下
- p43 下

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- §3-§4: dg 代数, dg 圏の導来圏は, 1 つ目が初出. 2 つ目の講義録が丁寧で, 傾理論への応用も説明されている.  
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§12: 最初の解説記事の 3.2 章と, そこに挙げられた文献.  
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# §1 Frobenius categories and Cohen-Macaulay modules ①

$\mathcal{A}$  : abelian category,  $\mathcal{A} \supset \mathcal{E}$  : full subcategory

Assume  $\mathcal{E}$  is **extension-closed** i.e.

$$\begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z \rightarrow 0 \\ & & \cap & & \cap & & \\ & & \mathcal{E} & & \mathcal{E} & & \end{array} : \text{exact sequence in } \mathcal{A} \Rightarrow Y \in \mathcal{E}$$

$$\mathcal{S} := \left\{ 0 \rightarrow \underbrace{X \rightarrow Y \rightarrow Z}_{\mathcal{E}} \rightarrow 0 : \text{exact seq. in } \mathcal{A} \right\}$$

以上の  $(\mathcal{E}, \mathcal{S})$  の公理化

Def [Quillen 73, Keller 90]

$\mathcal{E} = (\mathcal{E}, \mathcal{S})$  が **exact category** (**完全圏**)  $\iff$

- $\mathcal{E}$  は additive category

- $\mathcal{S}$  は  $\mathcal{E}$  の列  $0 \rightarrow X \xrightarrow{i} Y \xrightarrow{d} Z \rightarrow 0$  で

$i = \ker d$ ,  $d = \operatorname{cok} i$  をみたすもの (の一部) からなる

( $\mathcal{S}$  の元を **conflation**,  $i$  を **inflation**,  $d$  を **deflation** とよぶ)

- 以下が満たされる.

(Ex0)  $\mathcal{S}$  は同型で閉じる.

$$\bullet \forall X, Y \in \mathcal{E}, \left[ 0 \rightarrow X \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} X \oplus Y \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} Y \rightarrow 0 \right] \in \mathcal{S}$$

(Ex1) deflation の合成は deflation

(Ex1)<sup>op</sup> inflation  $\longrightarrow$  inflation





$A$ : ring    $\text{Mod } A$ : the category of (right)  $A$ -modules ③

$\text{mod } A$ : \_\_\_\_\_ finitely generated \_\_\_\_\_

$\text{proj } A$ : \_\_\_\_\_ fin. gen. projective \_\_\_\_\_

Def    $A$ : Noetherian ring

$A$ : Iwanaga-Gorenstein (or Gorenstein)

$$\iff \text{inj. dim } A_A < \infty, \text{ inj. dim } A^e < \infty$$

$$\iff \text{inj. dim } A_A = \text{inj. dim } A^e < \infty$$

[Zaks]

Def    $A$ : Iwanaga-Gorenstein ring

$X \in \text{mod } A$ : Cohen-Macaulay (or Gorenstein-projective)

$$\iff \forall i \geq 1, \text{Ext}_A^i(X, A) = 0$$

$\text{CMA}$ : the category of Cohen-Macaulay  $A$ -modules

$$\text{mod } A \supset \text{CMA} \supset \text{proj } A$$

Ex ①  $k$ : field    $A$ : finite dimensional  $k$ -algebra

$$A: \text{self-injective} \iff A_A: \text{injective} \iff A^e: \text{injective}$$

このとき  $\text{mod } A$  は Frobenius category

②  $A$ : Noetherian ring

$$\text{gl. dim } A = \sup \{ \text{proj. dim } X_A \mid X \in \text{mod } A \}$$

$$= \sup \{ \text{proj. dim } A^e Y \mid Y \in \text{mod } A^{\text{op}} \}$$

④

$\text{gl-dim } A < \infty \iff A : \text{Iwanaga-Gorenstein}, \text{CMA} = \text{proj } A$

↑

レポート これを確かめよ

Thm  $A : \text{Iwanaga-Gorenstein ring}$

①  $\text{CMA} : \text{Frobenius category}$

②  $( )^* := \text{Hom}_A( -, A) : \text{CMA} \simeq \text{CMA}^{\text{op}}, ( )^{**} \simeq 1_{\text{CMA}}$

Rem  $\forall \text{ ring } A, ( )^* : \text{proj } A \simeq \text{proj } A^{\text{op}}, ( )^{**} \simeq 1_{\text{proj } A}$

proof •  $\text{CMA}$  は  $\text{mod } A$  の extension-closed subcat. なのて.  
exact category

•  $\text{proj } A$  の object は  $\text{CMA}$  で projective かつ injective  
( $\text{Ext}_A^1(\text{CMA}, A) = 0$ )

•  $\forall X \in \text{CMA}, \exists \text{ exact seq } 0 \rightarrow \Omega X \rightarrow \underset{\cap}{P} \rightarrow X \rightarrow 0$

$\forall i > 0, \text{Ext}_A^i(\Omega X, A) \simeq \text{Ext}_A^{i+1}(X, A) = 0 \quad \text{proj } A$

$\therefore \Omega X \in \text{CMA}$  より  $\text{CMA}$  has enough proj. obj.

•  $\forall X \in \text{CMA}, \text{proj. resol. } \dots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \rightarrow 0 \text{ ㉔㉔}$   
 $\text{Ext}_A^i(X, A) = 0$  より  $0 \rightarrow X^* \xrightarrow{f_0^*} P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \dots \text{ exact } (*)$   
 $\searrow Y_1 \subset \quad \searrow Y_2 \subset$

$\forall i > 0, \text{Ext}_{A^{\text{op}}}^i(X^*, A) \simeq \text{Ext}_{A^{\text{op}}}^{i+1}(Y_1, A) \simeq \text{Ext}_{A^{\text{op}}}^{i+2}(Y_2, A) \simeq \dots \simeq 0$

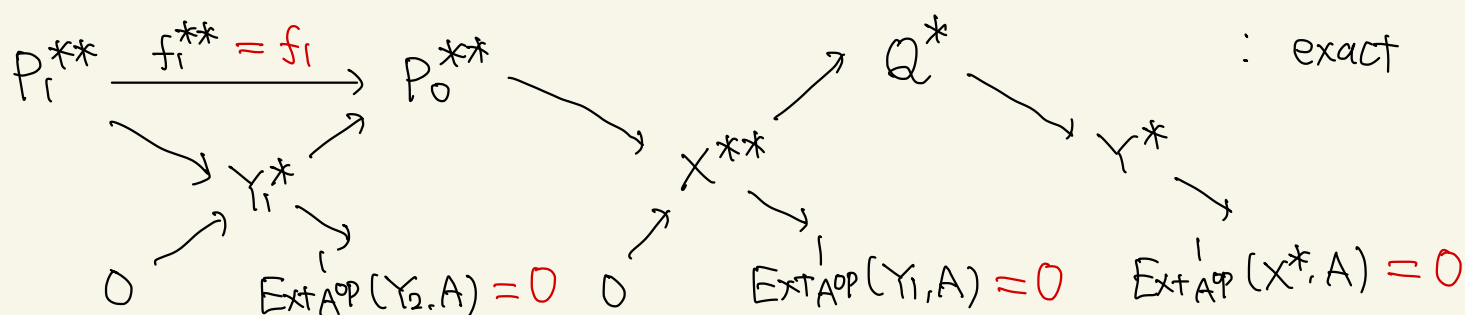
より  $X^*, Y_i \in \text{CMA}^{\text{op}} \quad (\forall i \geq 1)$

by  $\text{inj-dim } A < \infty$

exact seq.  $0 \rightarrow Y \rightarrow \underset{\cap}{Q} \rightarrow X^* \rightarrow 0 \text{ ㉔㉔. } (**)$

$\cap$   
 $\text{CMA}^{\text{op}} \quad \text{proj } A^{\text{op}}$

$(*) \vee (**)$  に  $( )^*$  を apply :



$$0 \rightarrow X^{**} \xrightarrow{\cap} Q^* \xrightarrow{\cap} Y^* \rightarrow 0 \quad \text{よ) CMA has enough inj.}$$

$\begin{matrix} \cap & \cap & \cap \\ X & \text{proj } A & \text{CMA} \end{matrix}$

Def  $\underline{\Sigma}$  : exact cat. with enough proj. obj.

•  $\underline{\Sigma}$  : stable category (安定圏)

object は  $\Sigma$  と同じ

$$\underline{\Sigma}(X, Y) := \Sigma(X, Y) / P(X, Y)$$

$$P(X, Y) := \{ f \in \Sigma(X, Y) \mid f \text{ factors through a proj. obj.} \}$$

•  $\Omega : \underline{\Sigma} \rightarrow \underline{\Sigma}$  : syzygy functor

$$\begin{array}{ccccccc} \forall X \in \underline{\Sigma}, & \text{fix a conflation} & 0 \rightarrow \Omega X \rightarrow P_X \rightarrow X \rightarrow 0 \\ f \downarrow & & \downarrow \Omega f & \downarrow & \downarrow f \\ Y \in \underline{\Sigma} & & 0 \rightarrow \Omega Y \rightarrow P_Y \rightarrow Y \rightarrow 0 \end{array}$$

$\Omega$  gives a well-defined functor  $\underline{\Sigma} \rightarrow \underline{\Sigma}$

Def  $\underline{\Sigma}$  : exact cat. with enough inj. obj. に対し. 双対的に

$\overline{\Sigma}$  : costable category

$\overline{\Sigma} : \overline{\Sigma} \rightarrow \overline{\Sigma}$  : cosyzygy functor

} を定める

Prop  $\underline{\Sigma}$  : Frobenius category  $(\Rightarrow \underline{\Sigma} = \overline{\Sigma})$

$\Omega : \underline{\Sigma} \rightarrow \underline{\Sigma}$  は equivalence 也.  $\overline{\Sigma}$  が quasi-inverse

## §2 Triangulated category

⑥

Def  $\mathcal{T} = (\mathcal{T}, [1], \Delta)$  が **triangulated category** (三角圏)  $\iff$

- $\mathcal{T}$ : additive category
- $[1]: \mathcal{T} \rightarrow \mathcal{T}$  は equivalence
- $\Delta$  は  $\mathcal{T}$  の列  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$  のクラスで、以下を満たす  
( $\Delta$  の元を **triangle** とよぶ)

(TR1)  $\Delta$  は 同型で閉じる

$$\forall X \in \mathcal{T}, [X \xrightarrow{1_X} X \rightarrow 0 \rightarrow X[1]] \in \Delta$$

$$\forall f \in \mathcal{T}(X, Y) \exists [X \xrightarrow{f} Y \rightarrow Z \xrightarrow{\quad} X[1]] \in \Delta$$

||  
cone f

(TR2)  $[X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]] \in \Delta$

$$\Rightarrow [Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]] \in \Delta$$

(TR3)  $[X \rightarrow Y \rightarrow Z \rightarrow X[1]] \in \Delta$

$$a \downarrow \quad b \downarrow \quad \text{red } \downarrow \quad \text{red } \downarrow \quad a[1] \downarrow$$

$$[X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]] \in \Delta$$

(TR4)  $[X \rightarrow Y \rightarrow A \rightarrow X[1]] \in \Delta$

$$\parallel \quad \text{red } \downarrow \quad \text{red } \downarrow \quad \parallel$$

$$[X \rightarrow Z \rightarrow C \rightarrow X[1]] \in \Delta$$

$$\downarrow \quad \text{red } \downarrow \quad \text{red } \downarrow \quad \downarrow$$

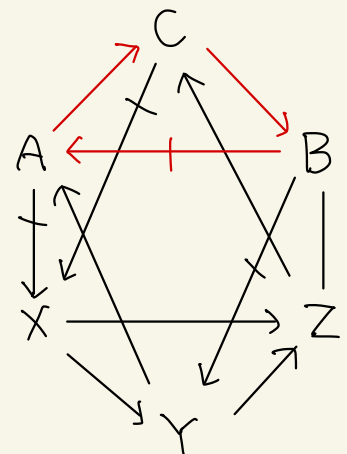
$$B \cong B \rightarrow Y[1]$$

$$\downarrow \quad \text{red } \downarrow \quad \downarrow$$

$$Y[1] \rightarrow A[1]$$

$$\underbrace{\quad} \quad \underbrace{\quad}$$

$$\triangle \quad \triangle$$



Thm [Happel 88]  $\mathcal{E} : \text{Frobenius cat.}$   $\Sigma : \mathcal{E} \rightarrow \mathcal{E} : \text{cosyzygy}$   
 $\Rightarrow (\mathcal{E}, \Sigma, \exists \Delta) : \text{triangulated cat.}$

$\Delta$  の定め方  $\forall f \in \mathcal{E}(X, Y)$

$$\begin{array}{ccccccc} \text{conflation} & 0 & \longrightarrow & X & \xrightarrow{\text{injective}} & I_X & \longrightarrow \Sigma X \longrightarrow 0 \\ & & & f \downarrow & & \text{PO} \downarrow & \parallel \\ & & & 0 & \longrightarrow & Y & \xrightarrow{g} Z \xrightarrow{h} \Sigma X \longrightarrow 0 \end{array} \quad \text{by } (\text{Ex 2})^{\text{op}}$$

$\mathcal{E}$  の列  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  と同型なもの全体を  $\Delta$  とする

Def このようにして得られる triang. cat. を **algebraic** とよぶ

Ex A : Iwanaga-Gorenstein ring  $\Rightarrow$  **CMA** : alg. triangulated cat.

Ex  $\mathcal{A} : \text{additive category}$

**$C[\mathcal{A}]$**  the category of chain complexes

obj :  $\mathcal{A}$  の列  $X = [\dots \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \dots]$  s.t.  $d^{i+1} d^i = 0$

morph :  $\mathcal{A}$  の commutative diagram

$$\begin{array}{ccccccc} X & \dots & \xrightarrow{d_X^{i-1}} & X^i & \xrightarrow{d_X^i} & X^{i+1} & \xrightarrow{d_X^{i+1}} \dots \\ \downarrow f & & & \downarrow f^i & & \downarrow f^{i+1} & \\ Y & \dots & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} & \xrightarrow{d_Y^{i+1}} \dots \end{array}$$

Prop  $\mathcal{A} : \text{additive cat.} \Rightarrow (C[\mathcal{A}], \exists \mathcal{S})$  は Frobenius category

$\mathcal{S} := \{ 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \mid \forall i. 0 \rightarrow X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \rightarrow 0 \text{ は split exact seq. } \}$

Def  $\mathcal{A}$  : additive cat.

$K[\mathcal{A}] := \underline{C[\mathcal{A}]}$  : homotopy category

Rem ① 上の  $X$ ,  $f$  に対し

$$X[1] := \left[ \dots \xrightarrow{-d_x^i} X^{i+1} \xrightarrow{-d_x^{i+1}} X^{i+2} \xrightarrow{-d_x^{i+2}} \dots \right]$$

$$\text{cone } f := \left[ \dots \longrightarrow Y^i \oplus X^{i+1} \xrightarrow{\begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix}} Y^{i+1} \oplus X^{i+2} \longrightarrow \dots \right]$$

$$\text{よおしく} \quad [0 \rightarrow Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \text{cone } f \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X[1] \rightarrow 0] \in \mathcal{S}$$

$$[X \xrightarrow{f} Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \text{cone } f \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X[1]] \in \Delta$$

②  $\forall X \in C[\mathcal{A}]$ ,  $\text{cone } 1_X \in C[\mathcal{A}]$  は projective  $\Leftrightarrow$  injective

③  $f \in C[\mathcal{A}](X, Y)$  が  $\underline{C[\mathcal{A}]}$  で  $0 \iff$  null-homotopic

レポート問題 Rem をたしかめよ

Def  $(\mathcal{T}, [1], \Delta)$ ,  $(\mathcal{T}', [1]', \Delta')$  : triang. cat.

• A **triangle functor** is a pair  $(F, \alpha)$  of an additive functor

$F: \mathcal{T} \rightarrow \mathcal{T}'$  and an isomorphism  $\alpha: F_0[1] \xrightarrow{\sim} [1]' \circ F$  of functors

s.t.

$$[X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]] \in \Delta \Rightarrow [FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\alpha_Z Fh} (FX)[1]'] \in \Delta'$$

•  $\pm \mathcal{S}$  に  $F$  が equivalence のとき,  $(F, \alpha)$  は **triangle equivalence** といふ

Ex  $(\mathcal{E}, \mathcal{S})$  : Frobenius cat.  $\mathcal{E} \supset \mathcal{P}$  : the cat. of projective obj.

$$C[\mathcal{P}] \supset \text{Ca}[\mathcal{P}] := \left\{ \dots \xrightarrow{\quad} P^{i-1} \xrightarrow{\quad} P^i \xrightarrow{\quad} P^{i+1} \xrightarrow{\quad} \dots \right\}$$

$\begin{array}{ccccc} & & \swarrow & \searrow & \\ & & X^i & \rightarrow & X^{i+1} \\ & \swarrow & & \searrow & \\ X^{i-1} & & & & X^{i+2} \end{array}$

$\in \mathcal{S}$

Prop  $\text{Ca}[\mathcal{P}]$  is Frobenius category であり.

$$\exists \text{ triangle equivalence } \begin{array}{ccc} \text{Ca}[\mathcal{P}] & \xrightarrow{\sim} & \underline{\mathcal{E}} \\ (P^i, d^i) & \longmapsto & X^0 \end{array}$$

Rem  $\text{Ca}[\mathcal{P}]$  は dg category の構造を持ち.  $H^0(\text{Ca}[\mathcal{P}]_{\text{dg}}) = \text{Ca}[\mathcal{P}]$

特に. triang. cat. が algebraic  $\iff$  dg enhancement を持つ

### §3 Derived categories and Singularity categories

$R$ : commutative ring

Def ①  $(A, d)$  is a differential graded (dg)  $R$ -algebra  $\iff$

•  $A = \bigoplus_{i \in \mathbb{Z}} A^i$  is a  $\mathbb{Z}$ -graded  $R$ -algebra ( $A^i \cdot A^j \subset A^{i+j}$ )

•  $\left[ \cdots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots \right] \in C[\text{Mod } R]$

• (Leibniz rule)  $d(ab) = d(a)b + (-1)^i a d(b)$  ( $a \in A^i, b \in A$ )

② このとき  $(X, d)$  is a dg  $A$ -module  $\iff$

•  $X = \bigoplus_{i \in \mathbb{Z}} X^i$  is a  $\mathbb{Z}$ -graded  $A$ -module ( $X^i \cdot A^j \subset X^{i+j}$ )

•  $\left[ \cdots \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots \right] \in C[\text{Mod } R]$

• (Leibniz rule)  $d(ab) = d(a)b + (-1)^i a d(b)$  ( $a \in X^i, b \in A$ )

Ex  $\mathcal{A}$ : add. cat.  $\forall X \in C[\mathcal{A}]$  gives a dg end. alg.  $\text{End}_{\mathcal{P}}(X)$

③ A morphism  $f: X \rightarrow Y$  of dg  $A$ -modules is a morphism of  $\mathbb{Z}$ -graded  $A$ -modules s.t.  $f d_X = d_Y f$

$C(A)$ : the category of dg  $A$ -modules

Ex  $A = A^0 \Rightarrow C(A) = C[\text{Mod } A]$

⑩

Prop  $A : \text{dg algebra} \Rightarrow (C(A), \exists \mathcal{S})$  is a Frobenius cat.

$\mathcal{S} := \{ 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \mid \mathbb{Z}\text{-graded } A\text{-module の } \square \text{ で split exact} \}$

$K(A) := \underline{C(A)} : \text{homotopy category of } A$

Def  $\mathcal{T} : \text{triang. cat}$   $\mathcal{T} \supset \mathcal{U} : \text{full subcat.}$

$\mathcal{U} : \text{thick subcategory} \iff$

- $\mathcal{U}[\pm 1] \subset \mathcal{U}$
- $\underbrace{[X \rightarrow Y \rightarrow Z \rightarrow X[1]]}_{\mathcal{U}} \in \Delta \Rightarrow \underbrace{Z}_{\mathcal{U}}$
- direct summand で閉じる

Ex  $A : \text{dg algebra}$

$K_a(A) := \{ X \in K(A) \mid X : \text{acyclic} \}$  は thick subcat. of  $K(A)$   
 $(\forall i \in \mathbb{Z}, H^i(X) = 0)$

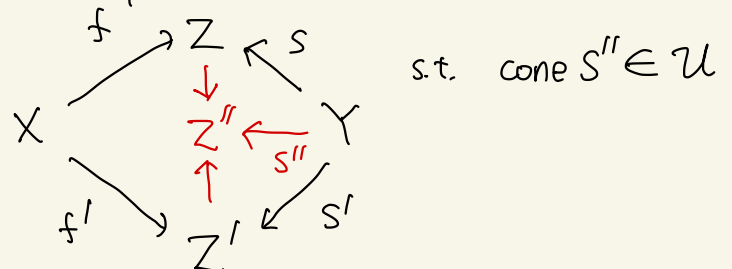
Def  $\mathcal{T} \supset \mathcal{U} : \text{as above}$

Verdier quotient  $\mathcal{T}/\mathcal{U}$

• obj. は  $\mathcal{T}$  と同じ

•  $(\mathcal{T}/\mathcal{U})(X, Y) := \{ [X \xrightarrow{f} Z \xleftarrow{s} Y] \mid Z \in \mathcal{T}, \text{ cone } S \in \mathcal{U} \} / \sim$

$(f, s) \sim (f', s') \iff \exists \text{ commu. diag}$



Prop  $\mathcal{T}/\mathcal{U}$  は canonical な triangulated cat. の構造を持つ

Def  $A : \text{dg algebra}$

$D(A) := K(A) / K_a(A) : \text{derived category of } A$



Rem  $f \in C(A)(X, Y)$  が **quasi-isom.**  $\iff H^i(f): H^i(X) \simeq H^i(Y) \quad \forall i \in \mathbb{Z}$  ①  
 $\iff \text{cone } f \in K_a(A)$

$D(A)$  では quasi-isomorphism が iso. になる.

- $D(A)$  の morphism の計算や.  $D(A)$  間の functor の構成のためには, projective resolution, injective resolution が必要

Def  $K_p(A) := \{ X \in K(A) \mid K(A)(X, K_a(A)) = 0 \}$  K-projective  
 $K_i(A) := \{ \text{---} (K_a(A), X) \text{---} \}$  K-injective

Thm [Keller]  $K_p(A) \simeq D(A)$  triangle equiv.  
 $K_i(A) \simeq D(A)$

Derived functors  $A, B, C : \text{dg } \mathcal{R}\text{-algebras} \quad \otimes := \otimes_{\mathcal{R}}$

$A \otimes B^{\text{op}}$  は自然に dg alg の構造を持つ

$X \in C(A \otimes B^{\text{op}}), Y \in C(A \otimes C^{\text{op}}), Z \in C(B \otimes C^{\text{op}})$

自然に  $\text{Hom}_A(X, Y) \in C(B \otimes C^{\text{op}})$ .  $X \otimes_B Z \in C(A \otimes C^{\text{op}})$  となる

$C = \mathcal{R}$  とし

$\text{Hom}_A(X, -) : C(A) \rightarrow C(B)$  は  $K(A) \rightarrow K(B)$  を induce

Caution しかし  $K_a(A) \rightarrow K_a(B)$  を induce しない

$R\text{Hom}_A(X, -) := \left[ D(A) \simeq K_i(A) \xrightarrow{\text{Hom}_A(X, -)} K(B) \rightarrow D(B) \right]$

同様にし

$R\text{Hom}_A(-, X) : D(A) \rightarrow D(B^{\text{op}})$   
 $-\otimes_B^L X : D(B) \rightarrow D(A)$  } を定める.

Ex  $f: A \rightarrow B$  : dg alg. の quasi-isom.

( $\cdot := \mathbb{Z}$ -graded alg の morph. で.  $f$  が quasi-isom.)

$$\Rightarrow -\bigoplus_A^L B : D(A) \simeq D(B) : -\bigoplus_B^L B : \text{triang. equiv.}$$

$$\text{per } A \simeq \text{per } B$$

$A$  : Noetherian ring

Def  $D(A) \supset D^b(A) := \{ X \in D(A) \mid H(X) \in \text{mod } A \}$

$\supset \text{per } A := \text{thick}_{D(A)} A (\simeq K^b[\text{proj } A])$

( $A$  を含む  $D(A)$  の最小の thick subcat.)

$D_{\text{sg}}(A) := D^b(A) / \text{per } A$  : singularity category

Ex  $\text{gl. dim } A < \infty \Rightarrow D^b(A) = \text{per } A \iff D_{\text{sg}}(A) = 0$

Thm [Buchweitz 87]  $A$  : Iwanaga-Gorenstein ring

$\exists$  triangle equivalence  $\underline{\text{CM}} A \simeq D_{\text{sg}}(A)$

Recall  $\text{gl. dim } A < \infty \iff \text{CM } A = \text{proj } A \iff \underline{\text{CM}} A = 0$

## §4 Tilting theory

(13)

Def  $\mathcal{T}$ : triangulated category  $\mathcal{T} \ni U$ : **tilting object**

$$\Leftrightarrow \textcircled{1} \mathcal{T}(U, U[i]) = 0 \quad \forall i \in \mathbb{Z} \setminus \{0\}$$

$$\textcircled{2} \mathcal{T} = \text{thick}_{\mathcal{T}} U \quad (:= U \text{ を含む } \mathcal{T} \text{ の最小の thick sub.})$$

Ex  $\textcircled{1} A: \text{ring} \Rightarrow A \in \text{per} A$  は tilting object

$\textcircled{2} A$ : finite dimensional Iwanaga-Gorenstein  $k$ -algebra

$\Rightarrow DA \in \text{per} A$  は tilting object

Rem  $\mathcal{T}$  が  $D(A)$  などの 大きな (= 無限余積をもつ) triang. cat. の場合は,  $U$  に **compact** 性 (=  $\mathcal{T}(U, -)$  と無限余積が可換) を課し,  $\textcircled{2}$  を  $\mathcal{T} = \text{Loc}_{\mathcal{T}} U$  ( $:= U$  を含み無限余積で閉じた最小の thick sub.) におきかえる.

Thm [Keller]  $\mathcal{T}$ : triang. cat. に対し 以下は同値

$$\textcircled{1} \exists A: \text{ring} \quad \exists \text{ triang. equiv. } \mathcal{T} \simeq \text{per} A$$

$$\textcircled{2} \mathcal{T} \text{ は algebraic, idempotent complete で tilting obj. を持つ}$$

$$\mathcal{T}: \text{ idempotent complete} \Leftrightarrow \forall X \in \mathcal{T}, \forall e = e^2 \in \text{End}_{\mathcal{T}}(X)$$

$$\exists Y \xrightleftharpoons[p]{i} X \quad \text{s.t.} \quad pi = 1_Y, \quad ip = e$$

Sketch of  $\textcircled{2} \Rightarrow \textcircled{1}$

$\textcircled{I}$  一般に  $\text{thick } U = \mathcal{T}$  なる  $U \in \mathcal{T}$  に対し.

$$\exists \text{ dg } k\text{-alg. } B \quad \text{s.t.} \quad \mathcal{T} \simeq \text{per } B$$

①  $\mathcal{T} = \underline{\Sigma}$  とする  $\S 2$  Prop より  $\mathcal{T} = \text{Ka}[\mathcal{P}]$  としてよい. (4)

$C[\mathcal{P}] \ni U$  の dg end. alg.  $B := \text{End}_{\mathcal{P}}(U)$  に対し.

functor  $\text{Hom}_{\mathcal{P}}(U, -) : C[\mathcal{P}] \longrightarrow C(B)$  が

triangle equiv.  $\text{Ka}[\mathcal{P}] \longrightarrow \text{per } B$  と  $\exists$  する.  
 $U \longmapsto B$

$$\text{II) } H^i(B) = \text{Hom}_{\text{per } B}(B, B[i]) = \mathcal{T}(U, U[i]) = \begin{cases} 0 & i \neq 0 \\ \text{End}_{\mathcal{T}}(U) & i = 0 \end{cases}$$

$$\begin{array}{ccccccc} B & \cdots & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \xrightarrow{d^0} B^1 \longrightarrow \cdots \\ U & & & \parallel & & U & \uparrow \\ C & \cdots & \longrightarrow & B^{-1} & \longrightarrow & Z^0(B) & \longrightarrow 0 \longrightarrow \cdots \\ \downarrow & & & \downarrow & & \downarrow & \downarrow \\ H^0(B) & \cdots & \longrightarrow & 0 & \longrightarrow & H^0(B) & \longrightarrow 0 \longrightarrow \cdots \end{array}$$

dg algebras の quasi-isomorphisms  $C \hookrightarrow B, C \twoheadrightarrow \text{End}_{\mathcal{T}}(U)$  と  
 する.  $\therefore$   $\text{per } B \simeq \text{per } C \simeq \text{per } \text{End}_{\mathcal{T}}(U)$   $\square$

### Application

Cor [Rickard 89]  $A, B$  : ring

①  $\exists$  triangle equiv.  $D(A) \simeq D(B)$

②  $\exists$  \_\_\_\_\_  $\text{per } A \simeq \text{per } B$

③  $\exists$  **tilting complex** (= tilting obj in  $\text{per } A$ )  $U$   
 s.t.  $\text{End}_{\text{per } A}(U) \simeq B$

Rem  $U \in \text{Mod } A$  なら  $\bigoplus_B^L U \hookrightarrow \text{RHom}_A(U, -)$  が equiv.  
 と  $\exists$  する

- $\mathcal{A}$  : additive category

$$X \in \mathcal{A} : \text{indecomposable} \iff (X \simeq Y \oplus Z \implies Y=0 \text{ or } Z=0)$$

- ring  $A$  が **local**  $\iff A$  has a unique maximal right ideal  
 $\iff A \xrightarrow{\hspace{2cm}} \text{left} \xrightarrow{\hspace{2cm}}$   
 $\iff$  非可逆元の全体が ideal をなす

Fact  $X \in \mathcal{A}$ ,  $\text{End}_{\mathcal{A}}(X) : \text{local} \implies X : \text{indecomposable}$

$\Leftarrow$  は正しくないが, それが成立する状況を考える

Def  $\mathcal{A}$  : additive category が **Krull-Schmidt (KS)**  $\iff \forall X \in \mathcal{A}$

$$\exists X_1 \sim \dots \sim X_n \in \mathcal{A} \text{ s.t. } X \simeq X_1 \oplus \dots \oplus X_n \text{ and } \text{End}_{\mathcal{A}}(X_i) \text{ is local}$$

Thm [Krull-Schmidt]  $\mathcal{A}$  : KS category

$$X_1 \oplus \dots \oplus X_n \simeq Y_1 \oplus \dots \oplus Y_m \quad (X_i, Y_j : \text{indecomp.}) \implies$$

$$n=m \text{ かつ 番号をうまくつけかえて } X_i \simeq Y_i \quad (\forall i)$$

Sketch iso. を  $m \times n$  行列  $[f_{ij}]$  ( $f_{ij} \in \mathcal{A}(X_j, Y_i)$ )

と表し, この行列に掃き出し法を行う  $\square$

レポート 証明を完成せよ

Def  $\mathcal{A}$  : KS category  $f \in \mathcal{A}(X, Y)$

- $f$  : **right almost split**  $\iff f$  is not split epi. and

$$\forall g : Z \rightarrow Y : \text{not split epi.} \quad \exists h : Z \rightarrow X \text{ s.t. } g = fh$$

- $f$  : **right minimal**  $\iff \left[ \begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & \searrow & \uparrow \\ X & \xrightarrow{f} & Y \end{array} \right] \implies g : \text{iso.}$

- $f$ : **sink morphism** (or **minimal right almost split**) of  $Y$  (16)

$\iff$  right almost split  $\hookrightarrow$  right minimal

双対的に **left almost split**, **left minimal**, **source morph.** を定める

Fact  $f_i \in \mathcal{A}(X_i, Y)$ : sink morph of  $Y$

$\Rightarrow \exists g: X_1 \rightarrow X_2$ : iso. s.t.  $f_1 = f_2 g$

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y \\ g \downarrow & \circlearrowleft & \uparrow \\ X_2 & \xrightarrow{f_2} & Y \end{array}$$

Def-Prop  $\mathcal{T}$ : KS triangulated cat.

$X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ : triangle

$f$ : source morph.  $\iff g$ : sink morph.

このとき この triangle を **Auslander-Reiten triangle** とよぶ

Rem 同様のことは KS exact cat. の conflation でも成立

proof  $\Leftarrow$  を示す  $\forall a \in \mathcal{A}(X, W)$ : not split mono.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \rightarrow & X[1] \\ a \downarrow & & b \downarrow & & \parallel & & \downarrow a[1] \\ W & \xrightarrow{f'} & V & \xrightarrow{g'} & Z & \rightarrow & W[1] \\ & \xleftarrow{c} & & & & & \end{array} \quad (\text{TR4})$$

- $f'$  が split mono なら,  $\exists c$  s.t.  $cf' = 1_W$  すると  $a = cbf$
- さもないと,  $g'$  は split epi. ではない  $g$ : right almost spl. より

$$\begin{array}{ccccccc} W & \xrightarrow{f'} & V & \xrightarrow{g'} & Z & \rightarrow & W[1] \\ a' \downarrow & & b' \downarrow & & \parallel & & \downarrow a'[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \rightarrow & X[1] \end{array}$$

$g = gb'b$  かつ  $g$  は right min. より  $b'b$  は iso.

5 Lemma より  $a'a$  も iso. これは  $a$  のとり方に反する  $\square$

Def  $k$ : field  $\mathcal{T}$ : KS triang. cat.

$\mathcal{T}$  が  $k$ -linear Hom-finite ( i.e.  $\forall X, Y \in \mathcal{T}. \mathcal{T}(X, Y)$  は fin. dim.  $k$ -vector space で. 合成は  $k$ -bilinear )

$\mathcal{T}$  の Serre functor とは.  $k$ -linear equivalence  $S: \mathcal{T} \rightarrow \mathcal{T}$  で  $\forall X, Y \in \mathcal{T}, \exists$  bifunctorial iso.  $\mathcal{T}(X, Y) \simeq D \mathcal{T}(Y, SX)$   
 $\uparrow$   $k$ -dual

Thm [Happel, Reiten-Van den Bergh]  $k$ : field

$\mathcal{T}$ :  $k$ -linear Hom-finite KS triangulated cat.

$\mathcal{T}$  has a Serre functor  $\iff \mathcal{T}$  has AR triangles

i.e.  $\forall X \in \mathcal{T}$ : indecomp.  $\exists$  AR triangles  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$   
 $Z' \rightarrow Y' \rightarrow X \rightarrow Z'[1]$

このときさらに  $Z \simeq S^{-1}X[1], Z' \simeq SX[-1]$  が成立

Rem 同様のことは KS exact cat. でも成立

[I-Nakaoka-Palu, arXiv 1805.03766] for extriangulated cat.

proof  $\Rightarrow$  のみ示す.  $\forall X \in \mathcal{T}$ : indecomp.

$$\begin{array}{ccc} D \mathcal{T}(X, X) & \xrightarrow{\sim} & \mathcal{T}(X, SX) \\ \cup & & \cup \\ D(\text{End}_{\mathcal{T}}(X)/\mathcal{J}) & \ni \varphi \xrightarrow{\#} \hbar & \end{array} \quad \mathcal{J}: \text{非同型の全体}$$

$\exists$  triangle  $SX[-1] \rightarrow Y \xrightarrow{f} X \xrightarrow{\hbar} SX$

$f$  が right almost split を示すには,  $\forall$  split epi でない  $a: Z \rightarrow X$ ,  $\hbar a = 0$  を示せば良い.

$$\begin{array}{ccccc} \mathcal{T}(X, Z) & & 0 \in D \mathcal{T}(X, Z) & \xrightarrow{\sim} & \mathcal{T}(Z, SX) \ni 0 \\ \cdot a \downarrow & \searrow & \uparrow D(\cdot a) \uparrow & & \uparrow a \cdot \uparrow \\ \mathcal{T}(X, X) \supset \mathcal{J} & & \varphi \in D \mathcal{T}(X, X) & \xrightarrow{\sim} & \mathcal{T}(X, SX) \ni \hbar \quad \square \end{array}$$

Thm [Happel]  $k$ : field  $A$ : finite dimensional  $k$ -algebra

Assume  $A$  is Iwanaga-Gorenstein

①  $\nu := -\overset{L}{\otimes}_A DA : \text{per } A \xrightarrow{\sim} \text{per } A$  gives a Serre functor

Nakayama functor

②  $\text{per } A$  has AR triangles

proof 
$$\begin{aligned} D\text{RHom}_A(X, Y \overset{L}{\otimes}_A DA) &\simeq D(Y \overset{L}{\otimes}_A \text{RHom}_A(X, DA)) \\ &\simeq D(Y \overset{L}{\otimes}_A DX) \simeq \text{RHom}_A(Y, \underbrace{DDX}_X) \end{aligned}$$

Taking  $H^0$ , we obtain  $D\text{Hom}_{\text{per } A}(X, \nu Y) \simeq \text{Hom}_{\text{per } A}(Y, X) \quad \square$

Cor  $k$ : field  $A$ : finite dimensional  $k$ -algebra

$\text{gl-dim } A < \infty \Rightarrow D^b(A)$  has AR triangles



## §6 Quiver representations

$\mathbb{k}$ : field     $A$ :  $\mathbb{k}$ -algebra

$A^e := A \otimes_{\mathbb{k}} A^{\text{op}}$ : enveloping algebra

$A^e$ -module  $\equiv$   $(A, A)$ -module s.t.  $\mathbb{k}$  acts centrally

Prop  $\text{gl. dim } A \leq \text{proj. dim } A_{A^e}$

proof  $l := \text{RHS}$  Take a projective resolution of  $A \in \text{Mod } A^e$

$$0 \rightarrow P_l \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0$$

$$\forall X \in \text{Mod } A, \quad 0 \rightarrow X \otimes_A P_l \rightarrow \dots \rightarrow X \otimes_A P_0 \rightarrow X \rightarrow 0$$

gives a projective resolution of  $X \in \text{Mod } A$   $\square$

Def  $A$ :  $\mathbb{k}$ -algebra     $M$ :  $A^e$ -module

$$T_A(M) := \bigoplus_{i \geq 0} M^{\otimes i}, \quad M^{\otimes i} := \underbrace{M \otimes_A \dots \otimes_A M}_i \quad (M^0 = A, M^1 = M)$$

tensor algebra

$$\left\{ \begin{array}{l} x = x_1 \otimes \dots \otimes x_i \in M^{\otimes i}, \quad y = y_1 \otimes \dots \otimes y_j \in M^{\otimes j} \\ xy := x_1 \otimes \dots \otimes x_i \otimes y_1 \otimes \dots \otimes y_j \in M^{\otimes i+j} \end{array} \right\}$$

Def  $Q = (Q_0, Q_1, s, t)$ : quiver  $\iff$

$Q_0, Q_1$ : (finite) sets

$s, t: Q_1 \rightarrow Q_0$ : maps

点 矢

始点 終点

- $\mathbb{k}Q_0$ :  $\mathbb{k}$ -vector spaces with basis  $\{e_i \mid i \in Q_0\}$
- $\mathbb{k}Q_1$ :  $\text{---} \text{---} \text{---} Q_1$
- $\mathbb{k}Q_0$  is a  $\mathbb{k}$ -algebra:  $e_i e_j = \delta_{ij} e_i$

- $kQ_1$  is a  $(kQ_0)^e$ -module :  $e_i a = \delta_{i, s(a)} a$   
 $a e_i = \delta_{i, t(a)} a$

$kQ := T_{kQ_0}(kQ_1) :$  path algebra of  $Q$

Rem •  $kQ$  の  $k$ -basis は  $Q$  上の path  $a_1 a_2 \cdots a_\ell$

$$\begin{aligned} & (a_i \in Q_1, \quad t(a_i) = s(a_{i+1})) \\ & (a_1 \cdots a_\ell)(b_1 \cdots b_m) = \begin{cases} a_1 \cdots a_\ell b_1 \cdots b_m & t(a_\ell) = s(b_1) \\ 0 & \text{else} \end{cases} \end{aligned}$$

- $\dim_k(kQ) < \infty \iff kQ : \text{acyclic}$  (サイクル (始点と終点の同じ path) を持たない)

Thm  $\text{gl. dim } kQ = \text{proj. dim } (kQ)_{(kQ)^e} = \begin{cases} 1 & Q_1 \neq \emptyset \\ 0 & \text{else} \end{cases}$   
i.e.  $kQ$  is hereditary

proof 一般の tensor algebra  $T = T_A(M)$  に対して

$$\begin{aligned} 0 \longrightarrow T \underset{A}{\otimes} M \underset{A}{\otimes} T &\longrightarrow T \underset{A}{\otimes} T \longrightarrow T \longrightarrow 0 \\ x \otimes y &\longmapsto xy \\ x \otimes y \otimes z &\longmapsto xy \otimes z - x \otimes yz \end{aligned}$$

は  $T^e$ -module の exact seq. である ①

$T = kQ$  のとき  $A = \prod_{Q_0} k$  であることより

$$T \underset{A}{\otimes} M \underset{A}{\otimes} T, \quad T \underset{A}{\otimes} T \in \text{proj } (kQ)^e \quad \text{②} \quad \square$$

レポート ①, ② をたしかめよ.

with. AR triangles

$k$  : algebraically closed field (簡単のため)



$\mathcal{T}$  :  $k$ -lin. Hom-fin. KS triangulated cat. with Serre functor  $S$

$$\begin{aligned} J_{\mathcal{T}}(X, Y) &:= \left\{ f \in \mathcal{T}(X, Y) \mid f = (f_{\bar{j}i})_{\bar{j}, i} \quad f_{\bar{j}i} : X_{\bar{j}} \rightarrow Y_i \right\} \\ &\quad \parallel \parallel \quad \text{\textcolor{red}{\mathcal{T} の Jacobson radical}} \quad \text{isom. ではない} \\ &\quad \bigoplus_{\bar{j}} X_{\bar{j}} \quad \bigoplus_i Y_i \quad (X_{\bar{j}}, Y_i : \text{indec.}) \end{aligned}$$

Def AR quiver of  $J$  (infinite quiver)

- vertices : isom. classes of indec. obj.
  - arrows :  $d_{XY} := \dim_k \left( \frac{J_T(X,Y)}{J_T^2(X,Y)} \right)$  本の矢を  $X$  から  $Y$  へ 描く
- $X, Y : \text{indec. obj}$

Prop 次のとき  $X \rightarrow E \rightarrow S^1 X[1] \rightarrow X[1]$  AR tri

$$SY[-1] \rightarrow E' \rightarrow Y \rightarrow SY$$

$$\Rightarrow d_{XY} = (\text{E の直和分解に現れる Y の個数}) \\ = (E' \text{ の } \text{---} X \text{---})$$

Thm [Gabriel]  $Q$ : quiver

①  $kQ$ : representation-finite (i.e.  $\# \{ \text{indec. } kQ\text{-mod} \} / \text{iso} < \infty$ )

②  $Q$ : Dynkin quiver

以下のグラフの辺を矢に置き換えてえられる quiver

$A_n$        $\bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \text{---} \bullet$  ( $n \geq 1$ )

$D_n$        $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$  ( $n \geq 4$ )

$E_n$        $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$  ( $n = 6, 7, 8$ )

$\pm 1$  に 対応  $\{ \text{indec. } kQ\text{-mod.} \} / \text{iso} \xleftrightarrow{1-1} \{ \text{positive roots} \}$

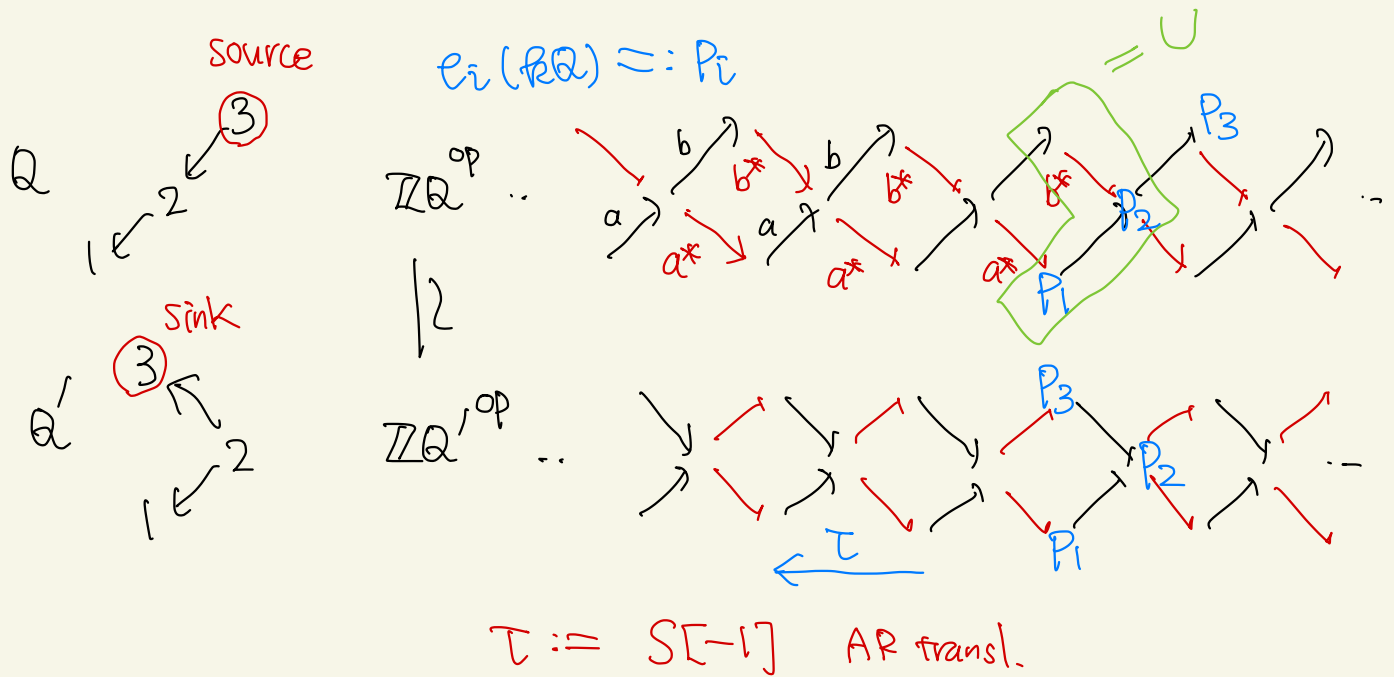
Thm [Happel] ①  $Q$ : quiver

$$\{ D^b(\mathbb{R}Q) = \text{per } \mathbb{R}Q \text{ の indec. obj. } \} / \text{iso} \ni X[\tilde{z}]$$

$$\updownarrow 1-1 \quad (\Leftarrow \text{gl.dim } \mathbb{R}Q \leq 1)$$

$$(\{ \text{indec. } \mathbb{R}Q\text{-module} \} / \text{iso}) \times \mathbb{Z} \ni (X, \tilde{z})$$

$$\textcircled{2} \quad Q: \text{Dynkin} \Rightarrow (D^b(\mathbb{R}Q) \text{ の AR quiver}) = \mathbb{Z}Q^{\text{op}}$$

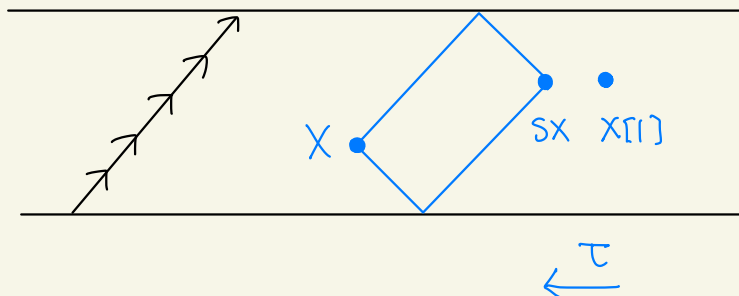


Prop quiver  $Q$  &  $Q'$  as reflection  $\Rightarrow D(\mathbb{R}Q) \simeq D(\mathbb{R}Q')$

$\Downarrow$   $\Updownarrow$

$\exists$  tilt.  $\mathbb{R}Q$ -module  $U$  s.t.  $\text{End}_{\mathbb{R}Q}(U) \simeq \mathbb{R}Q'$

Rem  $A_n$  のとき  $S$  &  $[-1]$  の AR quiver の作用



以下,  $R$  は commutative Noetherian ring of  $d = \dim R$ .

Def ①  $(R, \mathfrak{m})$ : local ring のとき  $\mathfrak{k} = R/\mathfrak{m}$   
 $\mathfrak{m}$  max. ideal.

•  $R$ : **regular**  $\iff \dim_{(R/\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) = d$

$$\iff \text{gl-dim } R < \infty \iff \text{gl-dim } R = d$$

[Auslander-Buchsbaum-Serre]

•  $R$ : **Gorenstein**  $\iff \text{inj-dim } R < \infty \iff \text{inj-dim } R = d$

$$\iff \text{Ext}_R^d(\mathfrak{k}, R) \simeq \mathfrak{k}, \text{Ext}_R^i(\mathfrak{k}, R) = 0 \ (i < d)$$

Rem  $\text{CM } R = \{ X \in \text{mod } R \mid \text{depth } X = d \text{ or } X = 0 \}$   
 $\uparrow$  §1で定義  $\uparrow$  ( $X$ -regular seq の長土の max.)

② 一般のとき  $R$ : regular  $\iff \forall \mathfrak{p} \in \text{Spec } R, R_{\mathfrak{p}}: \text{reg. Gor.}$

Rem  $\text{CM } R = \{ X \in \text{mod } R \mid \forall \mathfrak{p} \in \text{Spec } R, X_{\mathfrak{p}} \in \text{CM } R_{\mathfrak{p}} \}$   
 $\uparrow$  §1で定義

Def  $R$ : Gor. ring.  $\dim d$

$\text{CM}_0 R := \{ X \in \text{CM } R \mid \forall \mathfrak{p} \in \text{Spec } R \text{ with } \dim R_{\mathfrak{p}} < d, X_{\mathfrak{p}} \in \text{proj } R_{\mathfrak{p}} \}$   
 $\uparrow$

( $R$ : local のとき **locally free on punctured spec**)

Rem  $\text{CM}_0 R$  も Frobenius cat. をなす.  $\text{CM}_0 R$  は tri. cat. となる

$\text{CM}_0 R$  は Hom-finite (Hom set が  $R$ -mod として長土有限)

Def  $n \in \mathbb{Z}$ . triang. cat.  $\mathcal{T}$  is  $n$ -Calabi-Yau ( $n$ -CY) (24)

$\Leftrightarrow [n]: \mathcal{T} \rightarrow \mathcal{T}$  is Serre functor

Thm [Auslander 78]  $R$ : Gorenstein ring  $\dim R = d$

$\Rightarrow \underline{\text{CM}}_0 R$  is  $(d-1)$ -CY

Ex  $\underline{\text{CM}}_0 R = \underline{\text{CM}} R \Leftrightarrow$

$\forall \mathfrak{p} \in \text{Spec} R$  with  $\dim R_{\mathfrak{p}} < d$ .  $R_{\mathfrak{p}}$ : regular.

( $R$ : local のとき  $R$  は高  $d$  isolated sing.)

このとき  $\underline{\text{CM}} R$  is  $(d-1)$ -CY

Rem より一般に non-commutative  $R$ -alg  $A$  is symmetric  $R$ -order

とよばれるものにに対し Thm が成立 ( $R$ : Gor) (Gorenstein)

Ex  $R = \mathbb{k}$ : field  $A$ : fin. dim. self-injective  $\mathbb{k}$ -alg.

$(DA \simeq A \text{ as } A^e\text{-modules} \Leftrightarrow) \uparrow \text{symmetric}$

$\Rightarrow \underline{\text{mod}} A$  has a Serre functor  $\forall 0 [-1]$   $\left( \begin{array}{c} \text{per } A \\ \uparrow \\ V = -\bigoplus_A^L DA \end{array} \right)$

$\Downarrow$   
 $\text{Dsg}(A)$   $(-1)$ -CY

$R$ : complete local Gorenstein  $\mathbb{k}$ -alg. of  $d = \dim R$

$\mathbb{k} \simeq R/\mathfrak{m}$  is algebraically closed of char. 0.

Thm [ $\Leftarrow$  Knörrer,  $\Rightarrow$  Buchweitz-Greuel-Schreyer 87]

$R$ : CM-finite (i.e.  $\# \{ \text{indecomp. obj. in } \underline{\text{CM}} R \} / \simeq < \infty$ )

$\Leftrightarrow$  simple singularity i.e.  $R \simeq \mathbb{k}[[x, y, z_2, \dots, z_d]] / (f)$

$$f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \dots + z_d^2 & A_n \\ x^{n-1} + xy^2 & D_n \\ x^4 + y^3 & E_6 \\ x^3y + y^3 & E_7 \\ x^5 + y^3 & E_8 \end{cases}$$

Rem • CMR は  $d$  の parity のみによる (Knörrer periodicity)

- $d : \text{even} \Rightarrow \underline{\text{CMR}}$  の AR quiver は

$\mathbb{Q}$  の double  $\overline{\mathbb{Q}} \rightleftarrows \overline{\mathbb{Q}} \xleftarrow{\exists} \mathbb{Z}\mathbb{Q}$

$$\underline{\text{CMR}} \simeq \mathcal{C}_1(\overline{\mathbb{Q}}) \quad (:= D^b(\overline{\mathbb{Q}}) / \nu_0[-1])$$

1-cluster cat [Amiot]

$$\left. \begin{array}{l} A : \text{I-Gorenstein ring} \rightsquigarrow \underline{\text{CM}} A = D_{\text{sg}}(A) \\ B : \text{ring} \rightsquigarrow \text{per } B \end{array} \right\} \begin{array}{l} \text{2種類の} \\ \text{三角圏} \end{array}$$

適切な仮定のもと、ともに Serre functor を持つ

Q  $\underline{\text{CM}} A \simeq \text{per } B$  となることはあるか?

$\Leftrightarrow \underline{\text{CM}} A$  が tilting object を持つことはあるか?

A 自明な場合 ( $\underline{\text{CM}} A = 0$ ) を除いて “無い”

Fact  $A : \text{I-Gor. ring.}$   $\text{gl-dim } A = \infty \quad (\Leftrightarrow \underline{\text{CM}} A \neq 0)$

$\text{proj } A : \text{KS}$

$\Rightarrow \underline{\text{CM}} A$  は tilting object をもたない

$(\Leftrightarrow \nexists B \quad \underline{\text{CM}} A \simeq \text{per } B)$

Sketch 次を用いる

(26)

Exercise  $\mathcal{T}$ : tri. cat. with tilting object  $\Rightarrow$

$$\forall X, Y \in \mathcal{T}, \quad \mathcal{T}(X, Y[i]) = 0 \text{ for almost all } i \in \mathbb{Z}$$

一方  $A$  の Jacobson radical  $J_A$  に対し  $S := A/J_A$

$d = \inf. \dim A$  に対し.

$$0 \rightarrow \underbrace{\Omega^d(S)}_{\cap CMA} \rightarrow \underbrace{P_{d-1} \rightarrow \cdots \rightarrow P_0}_{\text{proj } A} \rightarrow S \rightarrow 0 : \text{exact seq.}$$

$$\forall X \in CMA, \quad \text{Ext}_A^i(X, S) = \text{Ext}_A^{i+d}(X, \Omega^d(S))$$

$$= \mathcal{T}(X, \Omega^d(S)[i+d]) = 0 \text{ for } i \gg 0 \text{ by Exercise}$$

$$\text{proj } A : \text{KS なので } \text{proj-dim } X < \infty \text{ より } \text{gl-dim } A < \infty \quad \square$$

以下. graded ring だと  $\underline{CM}^{\mathbb{Z}} A \simeq \text{per } B$  となる例が  
たくさんある ことを見る

$G$ : abelian group  $A = \bigoplus_{i \in G} A_i$ :  $G$ -graded ring

$\text{Mod}^G A$ : the cat. of  $G$ -graded  $A$ -module

$$\text{Hom}_A^G(X, Y) = \{ f \in \text{Hom}_A(X, Y) \mid f(X_i) \subset Y_i \ \forall i \in G \}$$

$\text{mod}^G A, \text{proj}^G A$ : 同様に定める

$$A: \text{I-Gor.} \Rightarrow \underline{CM}^G A := \left\{ X \in \text{mod}^G A \mid \begin{array}{l} X \in CMA \\ \text{as ungraded mod} \end{array} \right\}$$



Thm  $A$ :  $G$ -graded I-Gor. ring

(27)

$CM^G A$  is Frobenius cat.,  $\underline{CM}^G A$  is triang. cat.

$$\underline{CM}^G A \simeq D_{\text{sg}}^G(A) := D^b(\text{mod}^G A) / \text{per}^G A$$

$$\text{Def } \text{per}^G A := \text{thick} \{A(i) \mid i \in G\} \subset D^b(\text{mod}^G A)$$

- $i \in G$  is a degree shift functor

$$(i) : \text{Mod}^G A \longrightarrow \text{Mod}^G A \quad (X(i))_j = X_{i+j} \quad (j \in G)$$

$R$ :  $G$ -graded (commu) Gor. ring, of  $\dim R = d$

Def  $\bigoplus_{i \neq 0} R_i$  is  $R$ 's maximal ideal and assume  $\mathfrak{m} := R_0$

$$\exists a \in G \quad \text{Ext}_R^d(\mathfrak{m}, R) \simeq \mathfrak{m}(-a) \text{ in } \text{Mod}^G R$$

$a$  is  $a$ -invariant of  $R$  and  
( $-a$  is Gorenstein parameter)

$$CM^G R := \{X \in CM^G R \mid X \in CM_0 R \text{ as ungraded } R\text{-mod}\}$$

is Frobenius cat.

Thm [Auslander-Reiten 87]  $\underline{CM}_0^G R$  has Serre functor  
(a)[d-1]

# §8 Dimension 0

$R$ : field.

$A$ : fin. dim.  $R$ -alg.

(28)

$T(A) := A \oplus DA$  : trivial extension alg.

$$(a, f) \cdot (b, g) = (ab, ag + fb)$$

$$a, b \in A, \quad f, g \in DA$$

Fact •  $T(A) := B$  is fin. dim. symmetric  $R$ -alg.  $\left( \begin{array}{l} DB \simeq B \\ \text{as } B^e\text{-mod} \end{array} \right)$

•  $T(A)$  is  $\mathbb{Z}$ -graded alg.  $A \oplus DA$

•  $T(A) \twoheadrightarrow A$   $\deg 0 \quad 1$

Thm [Happel] 上の設定のもと

$$\underline{\text{mod}}^{\mathbb{Z}} T(A) = D_{\text{sg}}^{\mathbb{Z}}(T(A)) \ni A : \text{tilting obj.}$$

$$\forall i \in \mathbb{Z} \quad \underline{\text{mod}}^{\mathbb{Z}} T(A) \simeq \text{per } A$$

proof

!!

$$B := T(A) = \begin{array}{|c|} \hline A \\ \hline DA \\ \hline \end{array} \begin{array}{l} \textcircled{0} \\ \textcircled{1} \end{array}$$

①  $\mathcal{T}(A, A[i]) = 0 \quad \forall i \neq 0$  を示す

$$0 \rightarrow \Omega^i A \rightarrow P_{i-1} \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0 : \text{proj-resol}$$

$\parallel$  in  $\mathcal{T}$   $\searrow \nearrow$   $\parallel$   
 $AE[-i]$   $\textcircled{DA[-i]}$   $B$   
in  $\text{mod}^{\mathbb{Z}} B$

$$\forall i \geq 1 \quad (\Omega^i A)_f = 0 \quad \forall f \leq 0 \quad \mathcal{T}(A, A[i])$$

$\parallel$

$$\left. \begin{array}{l} \text{Hom}_{\mathbb{B}}^{\mathbb{Z}}(\Omega^i A, A) = 0 \\ \text{Hom}_{\mathbb{B}}^{\mathbb{Z}}(A, \Omega^i A) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \mathcal{T}(A[-i], A) = 0 \\ \mathcal{T}(A, A[i]) = 0 \end{array} \right.$$

②  $\text{thick}_{\mathcal{T}} A = \mathcal{T}$  を示す  $\mathcal{T} \ni \forall x = \begin{pmatrix} 0 \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \textcircled{1} \\ \textcircled{0} \\ \textcircled{0} \\ \textcircled{0} \end{array}$

$X_i \in \text{mod}^{\mathbb{Z}} B$  とみたと、 $X \in \text{thick} \{ X_i(-i) \mid i \in \mathbb{Z} \}$   
 $\text{gl-dim } A < \infty \Rightarrow X_i \in \text{thick} \{ A(i) \mid i \in \mathbb{Z} \} \Rightarrow \bigcap$   
 $(\exists 0 \rightarrow P_e \rightarrow \dots \rightarrow P_0 \rightarrow X_i \rightarrow 0) \quad \text{thick} \{ A(i) \mid i \in \mathbb{Z} \}$

示すべきこと  $\forall i \in \mathbb{Z}, \quad A(i) \in \text{thick}_{\mathcal{T}} A$

•  $i=0$  trivial

•  $i > 0$   $0 \rightarrow DA(i-1) \rightarrow B(i) \rightarrow A(i) \rightarrow 0$

よ)  $A(i) \simeq DA(i-1)[1] \in \text{thick}_{\mathcal{T}} A$   
induction

•  $i < 0$  同様  $\square$

Thm [Yamaya]  $A = \bigoplus_{i \geq 0} A_i : \mathbb{Z}\text{-graded selfinj } k\text{-alg}$

$\text{mod}^{\mathbb{Z}} A (= \text{Dsg}^{\mathbb{Z}}(A))$  has a tilting obj  $\iff \text{gl-dim } A_0 < \infty$

In this case,  $\bigoplus_{i \geq 1} A(i)_{\geq 0}$  is a tilting obj.

ただし  $X \in \text{mod}^{\mathbb{Z}} A$  に対し、 $X_{\geq i} := \bigoplus_{j \geq i} X_j \in \text{mod}^{\mathbb{Z}} A$

$\dim_k A < \infty$  より  $\bigoplus$  は有限直和

Sketch  $\Leftarrow$ ) Happel's Thm と同様

$\Rightarrow$ )  $\text{gl-dim } A_0 = \infty$  とすると  $S := A_0/JA_0$  に対し

$\forall i \geq 0, \text{Ext}_{A_0}^i(S, S) \neq 0$

$(\text{mod}^{\mathbb{Z}} A)(S, S[i]) = \text{Ext}_{\text{mod}^{\mathbb{Z}} A}^i(S, S) = \text{Ext}_{A_0}^i(S, S) \neq 0$

Exercise in (2b) に反する  $\square$

§3 Thm [Keller]  $A$  : dg alg.  $K_p(A) \simeq D(A)$   
Thm [Buchweitz]  $A$  : I-Gor.  $\underline{CM} A \simeq D_{sg}(A)$

右辺の(難しい) Verdier quot. が

$K_p(A)$  とし  $K(A)$  の 内部に実現されている  
 $\underline{CM} A$   $D^b(A)$

$$\underline{CM} A \subset D^b(A) \twoheadrightarrow D_{sg}(A)$$

$$\downarrow$$

$$\underline{CM} A \sim$$

以下.  $A = A_0 : \mathbb{Z}$ -graded I-Gor. ring

cotilting theory より

$$(\ )^* := R\mathrm{Hom}_A(\_, A) : D^b(A) \simeq D^b(A^{op}) \quad [\text{Miyachi}]$$

$i \in \mathbb{Z}$

$$D^b(\mathrm{mod}^{\mathbb{Z}} A) \simeq D^b(\mathrm{mod}^{\mathbb{Z}} A^{op})$$

$$\mathrm{mod}^{\geq i} A = \{ X \in \mathrm{mod}^{\mathbb{Z}} A \mid X = X_{\geq i} \} \quad \mathrm{mod}^i A = \mathrm{mod}^{\geq i} A \cap \mathrm{mod}^{\leq i} A$$

$$\mathrm{mod}^{\leq i} A = \{ \text{---} \mid 0 = X_{\geq i} \}$$

Thm [Orlov]

$$\textcircled{1} \text{ gl. dim } A_0 < \infty \Rightarrow D_{sg}^{\mathbb{Z}}(A) \simeq D^b(\mathrm{mod}^{\geq 0} A) \cap (D^b(\mathrm{mod}^{> 0} A^{op})^*)$$

②  $A_0$  : artinian ring

$$\text{Assume } \exists a \in \mathbb{Z} \text{ s.t. } (\ )^* : D^b(\mathrm{mod}^0 A) \simeq D^b(\mathrm{mod}^a A)$$

( Ex.  $A = R$  が  $a$ -invariant  $a \notin \mathbb{Z}$  は OK )

$$\Rightarrow D^b(\text{qgr } A) \simeq D^b(\mathrm{mod}^{\geq 0} A) \cap (D^b(\mathrm{mod}^{> a} A^{op})^*)$$

$$\left( \begin{array}{l} \text{ただし } \text{qgr } A := \mathrm{mod}^{\mathbb{Z}} A / \mathrm{mod}_0^{\mathbb{Z}} A : \text{Serre quotient} \\ \quad \hookrightarrow \text{長土有限の module 全体} \\ \text{Ex } A = R : \text{gen. in deg 1} \Rightarrow \text{qgr } R = \mathrm{coh}(\mathrm{Proj} R) \end{array} \right)$$

③ A が ① ② の仮定をともに満たせば"

③1

$$a=0 \Rightarrow \exists \text{ tri. eq. } D_{\text{sg}}^{\mathbb{Z}}(A) \simeq D^b(\text{qgr } A)$$

$$a < 0 \Rightarrow \begin{array}{c} \exists \text{ fully faithful} \\ \text{tri. functor} \end{array} \quad \text{---} \hookrightarrow \text{---}$$

$$a > 0 \Rightarrow \text{---} \leftarrow \text{---}$$

以下 Buchweitz's Thm と Orlov's Thm を一般的な

三角圏  $\mathcal{T}$  の Verdier quot.  $\mathcal{T}/\mathcal{S}$  を  $\mathcal{T}$  の内部に実現する 定理 から導く

Assumption  $\mathcal{T}$ : tri. cat.  $\mathcal{S}$ : thick subcat

(I)  $\mathcal{X}, \mathcal{Y}$ : full subcat. of  $\mathcal{S}$  s.t.  $\mathcal{X} = \text{add } \mathcal{X}$ ,  $\mathcal{Y} = \text{add } \mathcal{Y}$

$\text{add } \mathcal{X}$  は  $\mathcal{X}$  を含み 直和と直和因子で閉じた  $\mathcal{T}$  の最小の full subcat.

$$(II) \mathcal{S} = \mathcal{X} \perp \mathcal{Y} \iff \left\{ \begin{array}{l} \bullet \forall S \in \mathcal{S}, \exists \begin{array}{c} X \rightarrow S \rightarrow Y \rightarrow X[1] \\ \uparrow \quad \quad \uparrow \\ \mathcal{X} \quad \quad \mathcal{Y} \end{array} \text{ triangle} \\ \bullet \mathcal{S}(\mathcal{X}, \mathcal{Y}) = 0 \end{array} \right.$$

torsion pair

(  $\mathcal{X}, \mathcal{Y}$  がともに thick subcat. のときは semi-orthogonal decomp. とよぶ )

Ex  $A$ : dg alg  $K(A) = K_p(A) \perp K_a(A) = K_a(A) \perp K_i(A)$

$$(III) \mathcal{T} = \mathcal{X} \perp \mathcal{X}^{\perp} \quad \mathcal{X}^{\perp} := \{ T \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, T) = 0 \}$$

$$= {}^{\perp} \mathcal{Y} \perp \mathcal{Y} \quad {}^{\perp} \mathcal{Y} := \{ \text{---} (T, \mathcal{Y}) \text{---} \}$$

このとき  $\mathcal{T} \supset \mathcal{Z} := \mathcal{X}^{\perp} \cap {}^{\perp} \mathcal{Y}[1]$  とおく

$$\bigcup \mathcal{S} \supset \mathcal{P} := \bigcup \mathcal{Z} \cap \mathcal{S} \stackrel{\text{easy}}{=} \mathcal{Y} \cap \mathcal{X}[1]$$

$\mathcal{Z}$  は Frobenius cat. の類似物. §1 の stable cat. の類似を考える

• ideal quotient  $Z/[P]$

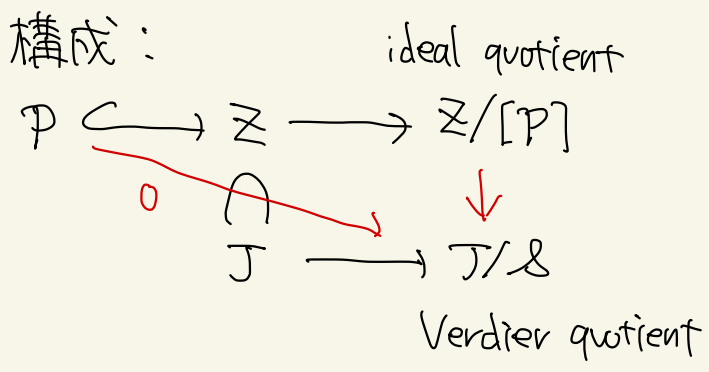
object は  $Z$  と同じ

$$(Z/[P])(X, Y) := Z(X, Y) / \{ X \xrightarrow{f} Y \mid \exists P \in P \}$$

Thm [I-Yang]

$$\exists Z/[P] \xrightarrow{\sim} T/\mathcal{S}$$

equivalence



(証明はかなり平易)

Rem •  $\mathcal{X} \subset \mathcal{X}[1] \Rightarrow \exists$  Happel-type triang. str. on  $Z/[P]$

• Related results by Nakaoka (Hovey's twin cotorsion pair)

Sketch of (IY's Thm)  $\Rightarrow$  (Buchweitz's Thm)

$$\begin{aligned} T &:= D^b(A) \supset \mathcal{S} := \text{per } A = K^b[\text{proj } A] \\ \mathcal{X} &:= K^{>0}[\text{proj } A] = \{ \dots \rightarrow 0 \rightarrow 0 \rightarrow P^1 \rightarrow P^2 \rightarrow \dots \} \\ \mathcal{Y} &:= K^{\leq 0}[\text{proj } A] = \{ \dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0 \rightarrow 0 \rightarrow \dots \} \end{aligned}$$

よおす  $\mathcal{S} = \mathcal{X} \perp \mathcal{Y}$  ,  $T = \mathcal{X} \perp D^{\leq 0}(A)$

2つ目の式の  $A \in A^{\text{op}}$  にして  $\{ X \in D^b(A) \mid H^i(X) = 0 \ \forall i > 0 \}$

$$D^b(A^{\text{op}}) = K^{>0}[\text{proj } A^{\text{op}}] \perp D^{\leq 0}(A^{\text{op}}) \quad ( )^* \text{ を apply して}$$

$$T = \underbrace{D^{\leq 0}(A^{\text{op}})^*}_{\perp \mathcal{Y}[1]} \perp \underbrace{K^{>0}[\text{proj } A^{\text{op}}]^*}_{\mathcal{X}[1]}$$

以上より (I) ~ (III) 成立

easy

$$Z = D^{\leq 0}(A) \cap (D^{\leq 0}(A^{\text{op}})^*) = \text{CMA}$$

IY's Thm

$$\begin{aligned} \cup \\ P = \mathcal{Y} \cap \mathcal{X}[1] = \text{proj } A \therefore \underline{\text{CMA}} = Z/[P] \simeq T/\mathcal{S} = D_{\text{sg}}(A) \end{aligned}$$



# Sketch of (IY's Thm) $\Rightarrow$ (Orlov's Thm)

$$\textcircled{1} \quad \mathcal{T} := D^b(\text{mod}^{\mathbb{Z}} A) \supset \mathcal{S} := \text{per}^{\mathbb{Z}} A$$

$$\text{proj}^{\geq 0} A := \{P \in \text{proj}^{\mathbb{Z}} A \mid P \text{ is generated in degrees } \geq 0\}$$

$$\text{proj}^{< 0} A := \{ \text{---} < 0 \}$$

$$\text{よおし} \quad \mathcal{S} = K^b[\text{proj}^{< 0} A] \perp K^b[\text{proj}^{\geq 0} A]$$

!!  
 $\mathcal{X}$                       !!  
 $\mathcal{Y}$

$$\mathcal{T} = \mathcal{X} \perp D^b(\text{mod}^{\geq 0} A)$$

!!  
 $\mathcal{X}^\perp$

2つ目の式に  $A$  を  $A^{op}$  にして degree shift  $(-1)$  を apply して

$$D^b(\text{mod}^{\mathbb{Z}} A^{op}) = K^b[\text{proj}^{\leq 0} A^{op}] \perp D^b(\text{mod}^{> 0} A^{op})$$

これに  $( )^*$  を apply して

$$\mathcal{T} = D^b(\text{mod}^{> 0} A^{op})^* \perp K^b[\text{proj}^{\geq 0} A]$$

!!                      !!  
 $\perp \mathcal{Y}[1]$                        $\mathcal{Y}[1]$

以上より (I) ~ (III) 成立

$$\mathcal{Z} = D^b(\text{mod}^{\geq 0} A) \cap (D^b(\text{mod}^{> 0} A)^*) \supset \mathcal{P} = 0 \text{ より}$$

IY's Thm より  $\mathcal{Z} \simeq \mathcal{T}/\mathcal{S} = D_{sg}^{\mathbb{Z}}(A)$  をうる

② の証明も同様. ③ は ①② よりすぐ  $\square$

Assumption  $R = \bigoplus_{i \geq 0} R_i$  :  $\mathbb{Z}$ -graded Gorenstein ring.  $\dim R = 1$

$R_0 = k$  : field.  $a := a$ -invariant of  $R$   
(i.e.  $\text{Ext}_R^1(k, R) \simeq k(-a)$ )

- $\text{CM}_0^{\mathbb{Z}} R$  is a triang. cat. with Serre functor  $(a)$  and AR triangles

Thm  $a \geq 0 \Rightarrow \text{CM}_0^{\mathbb{Z}} R \ni V := \bigoplus_{i=1}^{a+p} R(i)_{\geq 0}$  is a tilting obj  
( $\exists p \geq 1$ )

$\forall i \geq 2 \text{ CM}_0^{\mathbb{Z}} R \simeq \text{per } A$

$A := \text{End}_R^{\mathbb{Z}}(V) =$

$$\left[ \begin{array}{c|c} \begin{array}{ccc} R_0 & & \\ R_1 & \ddots & \\ & R_{a-1} & R_1 R_0 \end{array} & 0 \\ \hline \begin{array}{ccc} K_a & \cdots & K_1 \\ \vdots & & \vdots \\ K_{a+p-1} & \cdots & K_p \end{array} & \begin{array}{ccc} K_0 & \cdots & K_{1-p} \\ K_1 & \ddots & \\ K_{p-1} & \cdots & K_1 K_0 \end{array} \end{array} \right] \begin{array}{l} \left. \vphantom{\begin{array}{c} R_0 \\ R_1 \\ \vdots \\ R_{a-1} \end{array}} \right\} a \\ \left. \vphantom{\begin{array}{c} K_a \\ \vdots \\ K_{a+p-1} \end{array}} \right\} p \end{array}$$

$S := \{r \in R \mid \text{零因子でない} \}$  齊次元

$R \subset K := R_S$  :  $\mathbb{Z}$ -graded ring

$p := \gcd \{ \deg r \mid r \in S \} \geq 1$

Fact ①  $K(p) \simeq K$  in  $\text{Mod}^{\mathbb{Z}} R$   $K_{i+p} \simeq K_i$

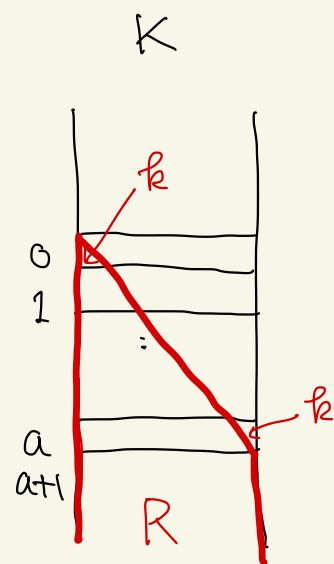
②  $\forall r \in S \setminus k \quad K = R[r^{-1}]$

$\dim (R/Rr) = 0 \quad \forall i \gg 0, \quad R_i \simeq K_i$

③  $R_{\geq a+1} = K_{\geq a+1} \quad \dim_k (K_a/R_a) = 1$   
( $\because \text{Ext}_R^1(k(a), R) \simeq k$ )

④  $\text{CM}^{\mathbb{Z}} R = \{X \in \text{mod}^{\mathbb{Z}} R \mid \text{socle } X = 0\}$

$\text{CM}_0^{\mathbb{Z}} R = \{X \in \text{CM}^{\mathbb{Z}} R \mid X \otimes_R K \in \text{proj } K\}$





⑤  $R$ : reduced (中零元が0のみ) なS  $K \simeq \prod_{i=1}^m \mathbb{R}[t_i^{\pm 1}]$   
 $\text{CM}^{\mathbb{Z}} R = \text{CM}_0^{\mathbb{Z}} R$  ( $\mathbb{R}^i$ :  $\mathbb{R}$  の拡大体)  
 さらに  $[a \geq 0 \text{ or } R = \mathbb{R}[t]]$   
 $\hookrightarrow a = -\deg t < 0$

$a$ -invariant の公式 ( $\dim$  は任意)

①  $\mathbb{R}[x_1, \dots, x_d]$  ( $x_i$ : homog.)  $\Rightarrow a = -\sum_{i=1}^d \deg x_i$

②  $R \ni r$ : homog. regular elem.  $\Rightarrow a(R/Rr) = a(R) + \deg r$

Ex  $R = \mathbb{R}[x, y] / (x^{2n+1} - y^2) \simeq \mathbb{R}[t^2, t^{2n+1}] \subset \mathbb{R}[t]$   
 $\begin{matrix} x & \longmapsto & t^2 \\ y & \longmapsto & t^{2n+1} \end{matrix}$   $\subset K = \mathbb{R}[t^{\pm 1}]$   
 $P = 1$

$$a = -2 - (2n+1) + (4n+2) = 2n-1$$

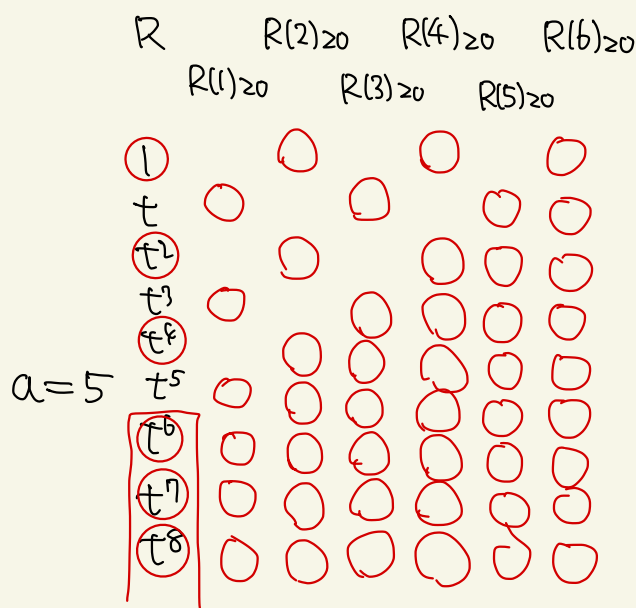
$$V = \left[ \begin{array}{ccccccc} x & & x & & x & & \\ R(1)_{\geq 0} & \subset & R(3)_{\geq 0} & \subset & \cdots & \subset & R(2n+1)_{\geq 0} \\ & & & & & \cap & t \\ & x & & x & & x & \\ R(2)_{\geq 0} & \subset & R(4)_{\geq 0} & \subset & \cdots & \subset & R(2n)_{\geq 0} \end{array} \right]$$

$A_{2n}$  型 quiver  $Q$

$$\text{CM}^{\mathbb{Z}} R = \text{CM}_0^{\mathbb{Z}} R = \text{per } \mathbb{R}Q$$

同様にして示される

$n=3$  のとき



Ex  $R = \mathbb{R}[x, y] / (f)$   $f$  は以下  $\Rightarrow \text{CM}^{\mathbb{Z}} R \simeq \text{per } \mathbb{R}Q$  CM-finite

| $f$ | $A_n$<br>$x^{n+1} - y^2$                          | $D_n$<br>$x^{n-1} - xy^2$                | $E_6$<br>$x^4 - y^3$ | $E_7$<br>$x^3y - y^3$ | $E_8$<br>$x^5 - y^3$ |
|-----|---------------------------------------------------|------------------------------------------|----------------------|-----------------------|----------------------|
| $Q$ | $D_{\frac{n+3}{2}}$ $n$ : odd<br>$A_n$ $n$ : even | $A_{2n-3}$ $n$ : odd<br>$D_n$ $n$ : even | $E_6$                | $E_7$                 | $E_8$                |

ただし  $(\deg x, \deg y)$  は  $f$  が斉次 となる中で 最小のもの

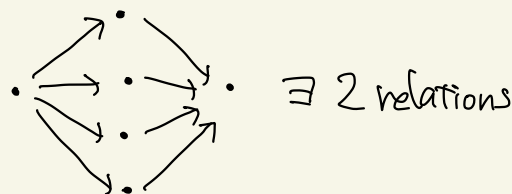
Ex  $T_{\text{ps}} \text{ sing. } x^p + \lambda x^2 y^2 + y^8$  CM-fame [Drozd-Greuel]

$\mathbb{Z}$ -graded に注意 (p,q) = (4,4), (3,6)

(deg x, deg y) = (1,1), (2,1)

$\Rightarrow \text{CM}^{\mathbb{Z}} R \simeq \text{per } A$

A : canonical algebra of type (2,2,2,2)



$\exists 2 \text{ relations}$

proof of Thm for reduced case

$\text{CM}_0^{\mathbb{Z}} R = \text{CM}^{\mathbb{Z}} R = D_{\text{sg}}^{\mathbb{Z}}(R)$

(reduced でないときは  $D_{\text{sg}}^{\mathbb{Z}}(R)$  を適切な subcat. に置き換える)

Use Orlov's Thm

$D^b(\text{mod}^{\mathbb{Z}} R) \hookrightarrow ( )^* = R\text{Hom}_R(-, R)$

$\text{mod}^{\geq i} R = \{ X \in \text{mod}^{\mathbb{Z}} R \mid X_{\geq i} = X \}$   $D^{\geq i} := D^b(\text{mod}^{\geq i} R)$

$D^{\geq -a} \cap (D^{\geq 0})^* \simeq D_{\text{sg}}^{\mathbb{Z}}(R) \supset D^{\geq 0} \cap (D^{\geq a})^* \simeq D^b(\text{qgr} R)$

さらに 次の semi-orthogonal decomposition がある

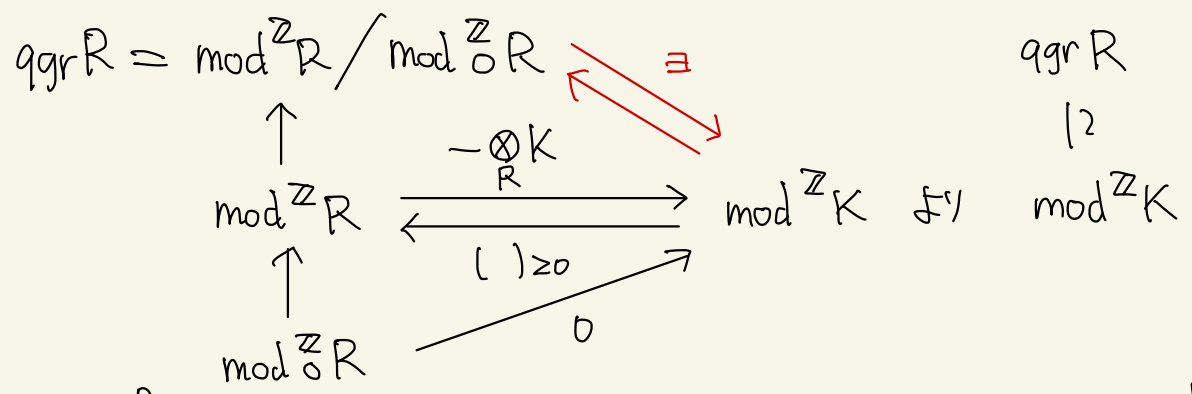
$$D^{\geq -a} \cap (D^{\geq a})^* = \underbrace{(D^{\geq 0} \cap (D^{\geq a})^*)}_{\substack{\downarrow \\ D_{\text{sg}}^{\mathbb{Z}}(R)}} \perp \underbrace{D^{[-a, -1]}}_{\substack{\downarrow \\ D^b(\text{mod}^{[-a, -1]} R)}} \quad \begin{array}{l} R\text{-mod } \mathbb{Z} \\ -a \sim -1 \text{ 次} \\ \text{以外は } 0 \end{array}$$

( $\odot$ )  $\forall X \in \text{mod}^{\geq -a} R, \exists 0 \rightarrow X_{\geq 0} \rightarrow X \rightarrow X/X_{\geq 0} \rightarrow 0$   
 $\downarrow \quad \quad \quad \downarrow$   
 $\text{複体で同じことをして} \quad \text{mod}^{\geq 0} A \quad \text{mod}^{[-a, -1]} R$   
 $D^{\geq -a} = D^{\geq 0} \perp D^{[-a, -1]} \leftarrow (D^{[a+1, 2a]})^* \subset (D^{\geq a})^*$   
 $\leftarrow R^* = R(-a)[-1]$   
 $\cap (D^{\geq a})^*$  をとて与式をうる

(右側)  $\text{mod}^{[-a, -1]} R \ni P := \bigoplus_{1 \leq i \leq a} (R/Rz_i)(i)$  は progenerator  
 $\downarrow$   
 $\text{mod } \text{End}_R^{\mathbb{Z}}(P)$   
 $\text{gl.dim} < \infty$   
 $\ll 12$   
 $D^{[-a, -1]} \ni P : \text{tilting obj.}$

|           |          |
|-----------|----------|
| $\vdots$  | $\vdots$ |
| 0         | $-a$     |
| 0         | $\vdots$ |
| $R_0$     | $-i$     |
| $R_1$     | $1-i$    |
| $\vdots$  | $\vdots$ |
| $R_{i-1}$ | $-1$     |
| 0         | 0        |
| $\vdots$  | $\vdots$ |

左側



$\text{mod}^{\mathbb{Z}} K \ni \bigoplus_{i=1}^p K(i)$  は progenerator  $\therefore$  tilting obj in  $\text{per}^{\mathbb{Z}} K = D^b(\text{mod}^{\mathbb{Z}} K)$

$\mathcal{U} \subset \mathcal{D}^b(\text{agr} R) \ni Q := \bigoplus_{i=1}^p K(i)_{\geq 0} : \text{tilting obj.}$

$$D^{\geq -a} \cap (D^{> a})^* = (D^{\geq 0} \cap (D^{> a})^*) \perp D^{[-a, -1]}$$

$\downarrow \cup$

主張  $Q \oplus P[-1]$  は tilting obj となる

- ①  $Q \in D^{\geq 0} \cap (D^{> a})^*$  であること : 少し計算 (略)
- ② 生成性 : 明らか
- ③ Ext 消滅 :  $D^b(\text{mod}^{\mathbb{Z}} A)(P[-1], Q[l]) = 0 \ (\forall l \neq 0)$  以外は明らか
  - $l \leq -2$   $P, Q \in \text{mod}^{\mathbb{Z}} R$  より 明らか
  - $l = -1$   $P \in \text{mod}^{\mathbb{Z}}_0 R, Q \in \text{CM}^{\mathbb{Z}} R$  より 明らか
  - $l > 0$   $K(j)_{\geq 0}$  が  $\text{mod}^{\geq 0} R$  で injective であることは容易

$$\text{Ext}_{\text{mod}^{\mathbb{Z}} R}^{l+1}((R/R_{\geq i})(i), K(j)_{\geq 0}) = \text{Ext}_{\text{mod}^{\mathbb{Z}} R}^l(R_{\geq i}(i), K(j)_{\geq 0}) = 0$$

$D_{\text{sg}}^{\mathbb{Z}}(R)$  で  $V \simeq Q \oplus P[-1]$  なので Thm が従う  $\square$

•  $a < 0$  のときを考える

Thm  $a < 0$  とする

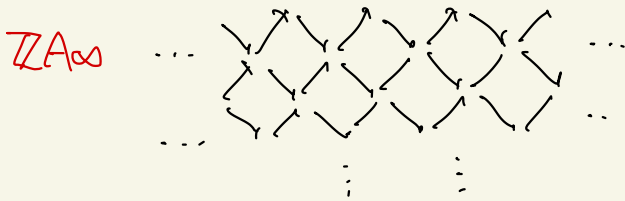
- ①  $\text{CM}^{\mathbb{Z}}_0 R$  has a tilt. obj  $\iff R$  : regular (i.e.  $\text{CM}^{\mathbb{Z}}_0 R = 0$ )
- ②  $\text{CM}^{\mathbb{Z}}_0 R$  has a tilting object.  $\bigoplus_{i=1}^{a+p} R(i)_{\geq 0}$

$$\textcircled{3} \quad \underline{\text{CM}}^{\mathbb{Z}}_0 R \simeq \text{per } A / \text{thick } P$$

$$A = \begin{bmatrix} K_0 & \cdots & K_{1-p} \\ K_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ K_{p-1} & \cdots & K_1 & K_0 \end{bmatrix} \quad \begin{array}{l} P = (\text{first } (-a) \text{ rows of } A) \\ n \\ \text{proj } A \end{array}$$

Ex  $R = \mathbb{k}[x, y] / (y^2) \quad a = 1-n : \underline{\text{CM countable}}$

The AR quiver of  $\underline{\text{CM}}^{\mathbb{Z}}_0 R$  has precisely  $n$   $\mathbb{Z}A_{\infty}$  components



$$\underline{\text{CM}}^{\mathbb{Z}}_0 R \simeq \text{per } A / \text{thick } (P \oplus \cdots \oplus P_{n-1}) \quad A = \mathbb{k} \begin{bmatrix} \overset{\mathbb{Z}}{\nearrow} 1 & \overset{\mathbb{Z}}{\nearrow} 2 \\ n & \downarrow \mathbb{Z} \\ \underset{\mathbb{Z}}{\nwarrow} & \cdots \end{bmatrix} / (\mathbb{Z}^2)$$

$$\simeq \text{per } B$$

$$B := \mathbb{k}[\varepsilon] / (\varepsilon^2) \quad \text{dg algebra} \quad d=0$$

$$\deg \varepsilon = 1-n$$

Sketch of proof of Thm ①

もし  $\underline{\text{CM}}^{\mathbb{Z}}_0 R \simeq \text{per } A$  であれば Serre functor も一致する

$$\begin{array}{ccc} \cup & \cup & \\ (a) \simeq & \vee & \end{array} \quad i \gg 0 \text{ に対し } (\text{per } A)(A, v^i(A)[i]) \text{ を考える}$$

$$v = -\bigoplus_A^L DA \text{ より, } i > 0 \text{ では } 0.$$

一方, これを  $\underline{\text{CM}}^{\mathbb{Z}}_0 R$  で考えると,  $a < 0$  と  $[-1] = \Omega$  より,  $i \leq 0$  でも 0.

$$v^i(A) = 0 \text{ となり矛盾} \quad \square$$

- Ongoing jt. work with Yuta Kimura, Ueyama for non-commu. version  
of above Thms (non-connected) Artin-Schelter Gorenstein rings

Summary 0次元, 1次元では一般的な (非)存在定理があった  
しかし 2次元 以上にはそのようなものはなく, 環毎の個別の理論となる.

# §11. Geigle-Lenzing complete intersections

(39)  
 (it Herschend.  
 Minamoto, Oppermann)  
 to appear AMS Memoir

$d=1$  is classical: GL's weighted  
 projection line [1987]

- Fix •  $C := \mathbb{k}[T_0, T_1, \dots, T_d]$  : poly alg of  $d+1$  variables / field  $\mathbb{k}$  ( $d \geq 0$ )
- $C \ni l_1 \sim l_n$  : linear form ( $n \geq 0$ )
  - $p_1 \sim p_n \geq 2$  : integers

$R := C[X_1, \dots, X_n] / (X_i^{p_i} - l_i \mid 1 \leq i \leq n)$  : commu- $\mathbb{k}$ -alg

$\mathbb{L} := \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle / \langle \vec{c} - p_i \vec{x}_i \mid 1 \leq i \leq n \rangle$  : abelian gp.

Regard  $R$  as an  $\mathbb{L}$ -graded ring by  $\deg X_j = \vec{x}_j$ ,  $\deg T_j = \vec{c}$  ( $\forall i, j$ )

Prop ①  $X_1^{p_1} - l_1, \dots, X_n^{p_n} - l_n$  is a regular sequence in  $C[X_1, \dots, X_n]$

$\therefore R$  is a Gorenstein ring of dim  $d+1$

②  $R$  has an  $\mathbb{L}$ -invariant  $\vec{\omega} := (n-d-1)\vec{c} - \sum_{i=1}^n \vec{x}_i$

Def Call  $(R, \mathbb{L})$  : **GL complete intersection** if  $\forall$  at most  $d+1$   
 elements from  $l_1 \sim l_n$  are linearly independent

Rem By linear transformation of  $T_0 \sim T_d$ , we can assume

$$l_i = T_{i-1} \quad (1 \leq i \leq \min\{n, d+1\})$$

•  $n \leq d+1 \Rightarrow R = \mathbb{k}[X_1, \dots, X_n, T_n, \dots, T_d]$  : regular

$$n = d+2 \Rightarrow R = \mathbb{k}[X_1, \dots, X_n] / (\alpha_1 X_1^{p_1} + \dots + \alpha_n X_n^{p_n})$$

Prop  $(R, \mathbb{L})$  : GL compl. int.  $\Rightarrow CM^{\mathbb{L}} R = CM_0^{\mathbb{L}} R$

$\therefore \underline{CM}^{\mathbb{L}} R = D_{Sg}^{\mathbb{L}}(R)$  has a Serre functor.  $(\vec{\omega})[d]$

Thm [H(MD)]  $\exists$  fin. dim.  $k$ -alg  $A^{cm}, A^d$  with  $gl.\dim < \infty$

①  $D_{sg}^{\mathbb{L}}(R) = \underline{CM}^{\mathbb{L}} R \simeq \text{per } A^{cm}$     ②  $D^b(\text{qgr } R) \simeq \text{per } A^d$

②  $n=0$      $\text{qgr } R = \text{coh } \mathbb{P}^d$  [Beilinson]     $\text{qgr } R := \frac{\text{mod}^{\mathbb{L}} R}{\text{mod}_0^{\mathbb{L}} R}$   
 $d=1$     weighted projective line [GL]  
 $n \leq d+1$  [Baer]     $n = d+2$  [Ishii-Ueda]

①  $n = d+2$     Futaki-Ueda, Kussin-Lenzing-Meltzer. ( $d=1$ )

$A^{cm}, A^d$  の def     $\mathbb{L}_+ := (\text{submonoid of } \mathbb{L} \text{ gen. by } \vec{c}, \vec{x}_i) \subset \mathbb{L}$

Regard  $\mathbb{L}$  as a poset by  $(\vec{x} \leq \vec{y} \iff \vec{y} - \vec{x} \in \mathbb{L}_+)$

$\mathbb{L} \supset I$  : finite subset

$A^I := (R_{\vec{x}-\vec{y}})_{\vec{x}, \vec{y} \in I}$  : fin. dim.  $k$ -alg by matrix multi.

$(r_{\vec{x}, \vec{y}}) \cdot (s_{\vec{x}, \vec{y}}) := (\sum_{\vec{z} \in I} r_{\vec{x}, \vec{z}} s_{\vec{z}, \vec{y}})$

$A^d := A^{[0, d\vec{c}]} \xleftarrow{\text{interval}} \xrightarrow{\text{interval}} A^{cm} := A^{[0, \vec{\delta}]} \quad \vec{\delta} := d\vec{c} + 2\vec{w}$   
 $d$ -canonical alg.     $cm$ -canonical alg.

Rem  $gl.\dim A^I < \infty$  ( $\because A^I$  の quiver は acyclic)

Ex ①  $n \leq d+1 \Rightarrow \vec{\delta} \notin \mathbb{L}_+ \Rightarrow A^{cm} = 0$

レポート ①  $\mathbb{L}$  の元は一意的に  $l\vec{c} + \sum_{i=1}^n l_i \vec{x}_i$  ( $l \in \mathbb{Z}, 0 \leq l_i \leq p_i - 1$ )  
と表されることを示せ. ②  $\vec{x} \in \mathbb{L}_+ \iff l \geq 0$   
 $\iff \vec{x} \leq d\vec{c} + \vec{w}$  を示せ

②  $n = d+2 \Rightarrow \vec{\delta} = \sum_{i=1}^n (p_i - 2) \vec{x}_i$      $\nexists$  が正しい

$\Rightarrow [0, \vec{\delta}] = \left\{ \sum_{i=1}^n l_i \vec{x}_i \mid 0 \leq l_i \leq p_i - 2 \right\}$

41

③  $n = d+3$  ,  $p_1 = p_2 = \dots = p_n = 2 \Rightarrow \vec{s} = \vec{c}$

$$A^{CM} = R \left[ \begin{array}{c} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_n \end{array} \right] \bigg/ (2 \text{ relations})$$

④  $d=1$  [GL]  $A^d$ : canonical algebra of type  $(p_1, \dots, p_n)$

$$\begin{array}{c}
 \nearrow \overline{x_1} \rightarrow 2\overline{x_1} \rightarrow \dots \rightarrow (p_1-1)\overline{x_1} \\
 \leftarrow \overline{x_2} \rightarrow 2\overline{x_2} \rightarrow \dots \rightarrow (p_2-1)\overline{x_2} \\
 \vdots \\
 \searrow \overline{x_n} \rightarrow 2\overline{x_n} \rightarrow \dots \rightarrow (p_n-1)\overline{x_n}
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \leftarrow \overline{c} \\
 \nearrow
 \end{array}
 \equiv n-2 \text{ relations}$$
 [Ringel]

$$D^b(\text{qgr } R) \simeq \text{per } A^d = D^b(A^d)$$

$$D^b(\text{qgr } R) \simeq \text{per } A^d = D^b(A^d)$$

|             |               |                                                                                                                                                                                                        |
|-------------|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| type        | $\mathcal{Q}$ |                                                                                                                                                                                                        |
| $n \leq 2$  | $A$           | $\left. \begin{array}{c} \\ \\ \\ \\ \end{array} \right\} \text{domestic} \Rightarrow \text{per } A^d \simeq \text{per } k\tilde{\mathcal{Q}}$<br>$(\tilde{\mathcal{Q}}: \text{finite Dynkin quiver})$ |
| $(2, 2, p)$ | $D_{p+2}$     |                                                                                                                                                                                                        |
| $(2, 3, 3)$ | $E_6$         |                                                                                                                                                                                                        |
| $(2, 3, 4)$ | $E_7$         |                                                                                                                                                                                                        |
| $(2, 3, 5)$ | $E_8$         |                                                                                                                                                                                                        |

Rem  $R$  と 2次元 simple singularity の関係 [GL, 91]

$$d=1, n=3 \quad R^{(\vec{w})} := \bigoplus_{i \in \mathbb{Z}} R i \vec{w} \simeq \frac{R[x,y,z]}{(f)} \quad \left( \begin{array}{l} \simeq R[u,v]^G \\ G \subset \mathrm{SL}_2(k) \end{array} \right)$$

Veronese subring simple sing.

さらに  $\text{qgr } R \simeq \text{qgr } R^{(\bar{w})}$  と  $\text{CM}^{\perp} R \simeq \text{CM}^{\perp} R^{(\bar{w})}$  が成立  
(このような equiv. は  $d \geq 2$  では不成立)

Rem  $\text{qgr } R \cong \text{stack } \mathbb{X} = [(\text{Spec } R) \setminus \{R_+\}] / \text{Spec } R[[L]]$  上の  
(GL projective space)

連接層の圏  $\text{coh } X$  と見ると,  $\text{equiv. } D^b(\text{coh } X) \simeq \text{per } A^d$  で

$A^d$  は line bundle の直和  $\bigoplus_{\vec{x} \in [0, d\vec{c}]} \mathcal{O}(\vec{x})$  に対応する



Orlov's Thm の  $\mathbb{L}$  バージョンを用いる

$$\mathbb{L} \supsetneq I \neq \emptyset \quad I + \mathbb{L}^+ \subset I \quad I^c := \mathbb{L} \setminus I$$

$$\text{mod}^I R \subset \text{mod}^{\mathbb{L}} R \quad D^I := D^b(\text{mod}^I R) \subset D^b(\text{mod}^{\mathbb{L}} R)$$

$$(X \text{ s.t. } X_i = 0 \ \forall i \in I^c)$$

$$(\ )^* = \text{RHom}_R(\_, R) \hookrightarrow D^b(\text{mod}^{\mathbb{L}} R)$$

$$D^I \cap (D^{-I^c})^* \simeq D_{\text{sg}}^{\mathbb{L}}(R) \simeq \underline{\text{CM}}^{\mathbb{L}} R$$

$$D^I \supset \mathcal{S}^I := D^b(\text{mod}_{\mathbb{O}}^I R)$$

$\hookrightarrow \text{mod}^I R$  中 長さ有限なもの全体

$$D^I \cap (D^{-I^c})^* \supset \mathcal{S}^I \cap (\mathcal{S}^{-I^c})^* \underset{\uparrow}{=} \mathcal{S}^I \cap \mathcal{S}^{\vec{w} + I^c} = \mathcal{S}^{I \cap (\vec{w} + I^c)}$$

$$\mathcal{R}^* = \mathcal{R}(-\vec{w})[-d-1]$$

$$\text{ここで } I := \mathbb{L}^+ \text{ とおく } \text{ p40 } \underline{\text{レポート ②}} \text{ より } \mathbb{L}^+ \cap (\vec{w} + \mathbb{L}^+)^c = [0, \vec{s}']$$

$$D_{\text{sg}}^{\mathbb{L}}(R) \simeq D^{\mathbb{L}^+} \cap (D^{-\mathbb{L}^+})^* \supset D^b(\text{mod}^{[0, \vec{s}']} R) = D^b(A^{\text{CM}}) = \text{per } A^{\text{CM}}$$

等号が成立する。

$$\textcircled{\text{⊙}} \ D_{\text{sg}}^{\mathbb{L}}(R) = \text{thick}(\text{mod}^{[0, \vec{s}']} R) \text{ を示せばよい.}$$

これは以下のように示される。

- GL complete intersection では  $D_{\text{sg}}^{\mathbb{L}}(R) = \text{thick}(\text{mod}_{\mathbb{O}}^{\mathbb{L}} R)$
- §8 Happel's Thm の生成性の証明と同様の induction により.

$$\forall \vec{x} \in \mathbb{L}, \ \mathcal{R}(\vec{x}) \in \text{thick}_{D_{\text{sg}}^{\mathbb{L}}(R)}(\text{mod}^{[0, \vec{s}']} R)$$

を regular sequence を用いて示すことができる。





# § 12 Cluster tilting

$$\text{Ext}_{\mathcal{A}}^i(X, Y) := \mathcal{A}(X, Y(i))$$

(43)

$\mathcal{A}$  : exact category or triang. cat.  $d \geq 1$  : integer

Def •  $\mathcal{A} \supset \mathcal{C}$  : full subcat.  $\mathcal{A}$  is  $d$ -cluster tilting ( $d$ -CT)

$$\Leftrightarrow \textcircled{1} \text{Ext}_{\mathcal{A}}^i(\mathcal{C}, \mathcal{C}) = 0 \quad \forall 1 \leq i \leq d-1$$

$$\textcircled{2} X \in \mathcal{A} \text{ s.t. } \text{Ext}_{\mathcal{A}}^i(\mathcal{C}, X) = 0 \quad \forall 1 \leq i \leq d-1 \Rightarrow X \in \mathcal{C}$$


---

$(X, \mathcal{C})$

$\textcircled{3}$   $\mathcal{C}$  is functorially finite i.e.

$$\left[ \begin{array}{l} \cdot \forall X \in \mathcal{A} \cdot \exists f: C \rightarrow X \text{ s.t. } \forall g: C' \rightarrow X \text{ factors through } f \\ \cdot \text{Dual} \end{array} \right]$$

$\begin{array}{c} C' \\ \uparrow \\ \mathcal{C} \end{array}$

$\xrightarrow{f}$

$\begin{array}{c} C' \\ \downarrow \\ X \end{array}$

$\begin{array}{c} C' \\ \swarrow \searrow \\ C \xrightarrow{f} X \end{array}$

•  $\mathcal{A} \ni M$  is  $d$ -CT  $\Leftrightarrow \mathcal{A} \supset \text{add } M$  ( $:= \{ M \oplus \dots \oplus M \text{ の直和因子} \}$ )  
 is  $d$ -CT subcat

Ex  $d$ -cluster category (see p47 for general def.)

$Q$  : acyclic quiver

$$\mathcal{C}_d(\mathbb{K}Q) := D^b(\mathbb{K}Q) / \nu_d \quad \mathcal{C}_d(\mathbb{K}Q)(X, Y) := \bigoplus_{i \in \mathbb{Z}} D^b(\mathbb{K}Q)(X, \nu_d^i(Y))$$

$$\nu = - \bigoplus_A^L DA \quad \nu_d := \nu \circ [-d]$$

$\mathbb{K}Q \in \mathcal{C}_d(\mathbb{K}Q)$  is  $d$ -CT object.

Rem  $d=2$  のとき Cluster algebra の categorification に 対応する

Def  $\mathcal{T}$  : triang. cat. with Serre functor  $S$   $S_d := S \circ [-d]$

Prop  $\mathcal{T} \supset \mathcal{C}$  :  $d$ -CT subcat  $\Rightarrow S_d^{\pm 1}(\mathcal{C}) = \mathcal{C}$

レポート問題 これをたしかめよ ( $\textcircled{1} \textcircled{2}$  のみか3行)

Def  $A: \text{fin. dim } \mathcal{R}\text{-alg}$   $\text{gl. dim } A \leq d$

Recall  $D^b(A) = \text{per } A$  has a Serre functor  $V = -\bigotimes_A^L DA$  (§5, §6)

$$V_d := V \circ [-d] \hookrightarrow D^b(A)$$

Prop  $A: V_d\text{-finite}$  ( $\Leftrightarrow H^i(V_d^{-i}(A)) = 0$  for  $i \gg 0$ )

$\Rightarrow D^b(A) \supset \mathcal{U}_d(A) := \text{add} \{ V_d^{-i}(A) \mid i \in \mathbb{Z} \}$  は  $d$ -CT subcat

Ex  $\text{gl. dim } A < d \Rightarrow V_d\text{-finite}$

- $\mathcal{A}$  の 1-CT subcat. は  $\mathcal{A}$  自身 : exact / triang. cat
- $\mathcal{A}$  の  $d$ -CT subcat. は exact / tri. cat に類似した公理系を満たす  $d\text{-exact cat.}$  [Jasso 16]
- $(d+2)\text{-angulated cat}$  [Geiss - Keller - Oppermann 13]

- $d$ -CT subcat. には AR theory の高次元類似を展開できる

(I) AR seq. / triang. の類似物の存在.

Thm [I-Yoshino]  $\mathcal{T}: \text{KS tri. cat. with Serre functor } S$

$$S_d := S \circ [-d]$$

$\mathcal{T} \supset \mathcal{C}: d\text{-CT subcat.}$

$\Rightarrow \forall X \in \mathcal{C}: \text{indec. obj.} \quad \exists \text{ AR } (d+2)\text{-angles}$

$$\begin{array}{ccccccc}
 S_d(X) & \xrightarrow{f_{d+1}} & C_d & \xrightarrow{f_d} & \cdots & \rightarrow & C_2 & \xrightarrow{f_2} & C_1 & \xrightarrow{f_1} & X \\
 & & \searrow & & & & \nearrow & \searrow & \nearrow & & \\
 & & X_{d-1} & & & & X_2 & & X_1 & & \\
 \underbrace{\hspace{10em}}_{\text{tri.}} & & & \cdots & & \underbrace{\hspace{10em}}_{\text{tri.}} & & \underbrace{\hspace{10em}}_{\text{tri.}} & & & 
 \end{array}$$

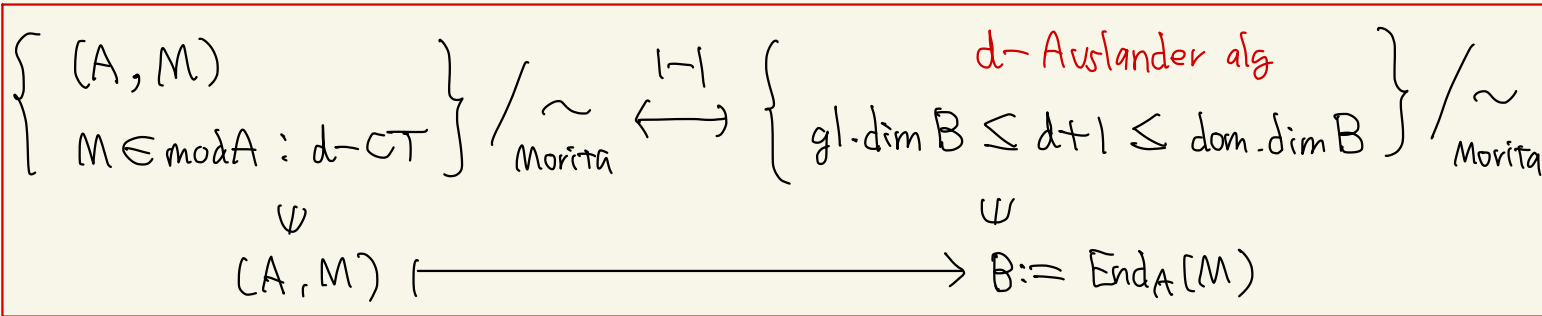
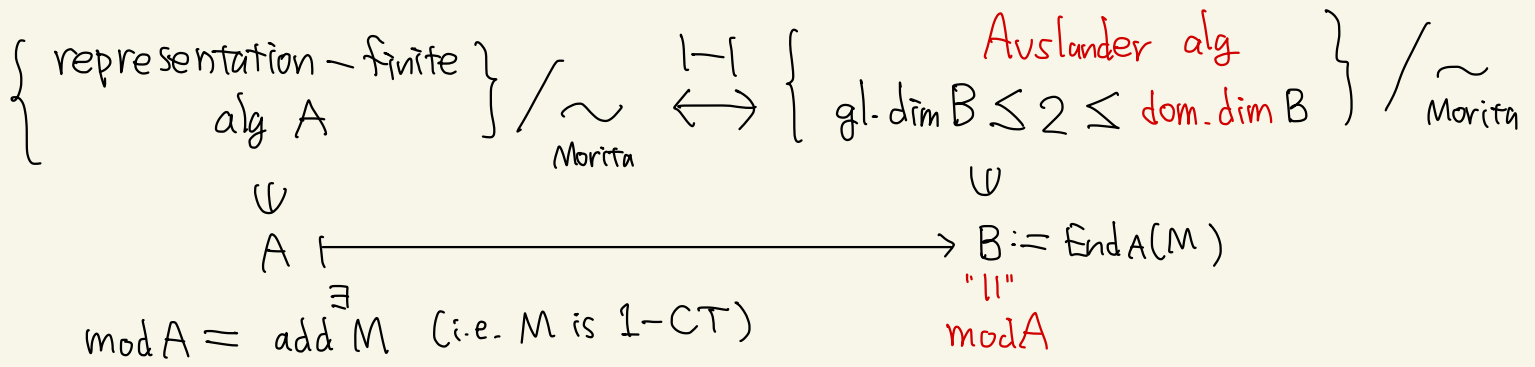
$C_i \in \mathcal{C}$   
 $f_1: \text{sink mor. in } \mathcal{C}$   
 $f_{d+1}: \text{source mor. in } \mathcal{C}$   
 $f_i \in \mathcal{T}_{\mathcal{C}}$

$$X \rightarrow C^1 \rightarrow C^2 \rightarrow \cdots \rightarrow C^d \rightarrow S_d^{-1}(X)$$

同様のことは KS exact cat. の  $d$ -CT subcat. でも成立

## II Auslander 対応の類似が成立

① fin. dim.  $R$ -alg / field  $R$  のとき



②  $R$ : Gorenstein local ring of dim  $d+1$  のとき

$M \in \text{CMR} : d\text{-CT} \implies B := \text{End}_R(M)$  は non-singular  $R$ -order  
(i.e.  $B \in \text{CMR}$ ,  $\text{gl-dim } B = d+1$ )

$(d+1)$ -Calabi-Yau alg

非可換 ヲルベント 特異点 解消

# §13 (Classical and higher) preprojective algebras

Def-Prop  $k$ : field,  $d \geq 1$ ,  $A$ : fin. dim  $k$ -alg,  $\text{gl. dim } A \leq d$

$$\text{mod } A^e \ni E := \text{Ext}_{A^e}^d(A, A^e) \simeq \text{Ext}_A^d(DA, A) \simeq \text{Ext}_{A^{op}}^d(DA, A)$$

$$\Pi = \Pi_{d+1}(A) := T_A(E) : \begin{array}{l} (d+1)\text{-preprojective alg. of } A \\ \text{tensor alg} \end{array} \quad \mathbb{Z}\text{-graded } k\text{-alg.}$$

proof  $R\text{Hom}_{A^e}(A, A^e) \simeq R\text{Hom}_{A^e}(A, \text{Hom}_k(DA, A)) \xrightarrow{\uparrow} R\text{Hom}_A(A \overset{L}{\otimes}_A DA, A)$   
 $H^d$  をとて. 同型をうる. Hom-tensor adjunction  $\square$

$$\text{Def } \textcircled{1} \quad \left. \begin{array}{l} \tau_d := \text{Tor}_d^A(-, DA) \simeq \text{Hom}_A(E, -) \\ \bar{\tau}_d := \text{Ext}_A^d(DA, -) \simeq - \otimes_A E \end{array} \right\} \begin{array}{l} \text{mod } A \longrightarrow \text{mod } A \\ d\text{-AR translation} \end{array}$$

$$\textcircled{2} \quad X \in \text{mod } A : \text{indec. が } \begin{array}{l} d\text{-preprojective} \\ d\text{-preinjective} \end{array} \iff \exists i \geq 0, X \text{ は } \tau_d^{-i}(A) \text{ の直和因子}$$

$\tau_d^i(DA)$

Prop  $\textcircled{1}$  ( $\Pi$  の名前の由来)  $\Pi = \bigoplus_{i \geq 0} \tau_d^{-i}(A)$  as  $A$ -modules

$\textcircled{2}$  ( $\tau_d$  と  $\bar{\tau}_d$ ,  $\Pi$  と  $\mathcal{U}_d(A)$  の関係)

$$\tau_d = ( \text{mod } A \subset D^b(A) \xrightarrow{V_d} D^b(A) \xrightarrow{H^0} \text{mod } A )$$

$$\bar{\tau}_d = ( \text{mod } A \subset D^b(A) \xrightarrow{V_d^{-1}} D^b(A) \xrightarrow{H^0} \text{mod } A )$$

$$\therefore \exists \text{ equiv. } \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D(A)}(A, V_d^{-i}(-)) : \mathcal{U}_d(A) \xrightarrow{\sim} \text{proj}^{\mathbb{Z}} \Pi$$

• 例を挙げる前に derived version を考える.  $k$ : field,  $d \geq 1$  とす

Def [Keller]  $A$ : dg  $k$ -algebra,  $A^e = A \otimes_k A^{op}$

$$D(A^e) \ni R\text{Hom}_{A^e}(A, A^e) \simeq R\text{Hom}_A(DA, A) \simeq R\text{Hom}_{A^{op}}(DA, A)$$

$$C(A^e) \ni \textcircled{H} := [ R\text{Hom}_{A^e}(A, A^e) \text{ の projective resol. } ], \textcircled{H}_d := \textcircled{H}[d]$$

$$\Pi = \Pi_{d+1}(A) := T_A(\textcircled{H}_d) \quad \begin{array}{l} \text{derived } (d+1)\text{-preproj. alg of } A \\ \text{dg algebra } ((d+1)\text{-CY completion}) \end{array}$$

- $\Pi$  と  $\Pi$  の関係は以下で与えられる

Prop  $A$ : fin. dim.  $k$ -algebra,  $\text{gl. dim } A \leq d$

① ( $V_d$  と  $\Pi$  の関係)  $V_d^{-1} \simeq -\bigotimes_A^L \oplus_d : D(A) \rightarrow D(A)$

とくに  $\Pi \simeq \bigoplus_{i \geq 0} V_d^{-i}(A)$  in  $D(A)$

② ( $\Pi$  と  $\Pi$  の関係)  $H^0(\Pi) = \Pi$

$H^i(\Pi) = 0$  ( $\forall i > 0$ ) i.e.  $\Pi \in \text{per } \Pi$  は *silting object*

③ [Amiot, Guo]  $\mathcal{C}_d(A) := \text{per } \Pi / \bigcap_{ii} D^{\text{fd}}(\Pi) : d\text{-cluster cat.}$   
 $\{X \in D(\Pi) \mid \dim_k H(X) < \infty\}$

$A: V_d\text{-finite} \Rightarrow \mathcal{C}_d(A): d\text{-CY triang. cat. with } d\text{-CT obj } A$

Def  $A$ : dg  $k$ -algebra が *homologically smooth*  $\iff A \in \text{per}(A^e)$

$A: d\text{-CY} \iff \text{homlog-sm.} \iff R\text{Hom}_{A^e}(A, A^e) \simeq A[-d]$  in  $D(A^e)$

Prop  $A$ : homol. smooth  $\Rightarrow D^{\text{fd}}(A)$  has a Serre functor  $(-\bigotimes_A^L \oplus)^{-1}$

とくに  $A: d\text{-CY} \Rightarrow D^{\text{fd}}(A)$  is a  $d\text{-CY triangulated category}$

Ex fin. dim.  $k$ -alg  $A$  が homol. smooth  $\iff \text{gl. dim } A < \infty$  かつ

$A/J_A$  は separable  $k$ -alg

Thm [Keller]  $A$ : homol. sm. dg  $k$ -alg  $\Rightarrow \Pi: (d+1)\text{-CY}$

Sketch (  $\vee := R\text{Hom}_{A^e}(-, A^e): \text{per } A^e \xrightarrow{\sim} \text{per } A^e$  とおく.)

$\oplus = A^\vee$ ,  $\oplus^\vee \simeq A$  (\*) 一方  $B := \Pi = T_A(\oplus_d)$  なので p20 Thm と同様に

$\exists B \bigotimes_A^L \oplus_d \bigotimes_A^L B \rightarrow B \bigotimes_A^L A \bigotimes_A^L B \rightarrow B \rightarrow B \bigotimes_A^L \oplus_d \bigotimes_A^L B[1] : \text{triangle in } D(B^e)$

$A, \oplus \in \text{per } A^e$  より  $B \in \text{per } B^e$ . さらに  $R\text{Hom}_{B^e}(-, B^e)[d]$  を apply して

$R\text{Hom}_{B^e}(B \bigotimes_A^L \bigotimes_A^L B, B^e) \simeq B \bigotimes_A^L (-)^\vee \bigotimes_A^L B$  を用いると

$R\text{Hom}_{B^e}(B, B^e)[d] \rightarrow B \bigotimes_A^L A^\vee[d] \bigotimes_A^L B \rightarrow B \bigotimes_A^L \oplus_d^\vee[d] \bigotimes_A^L B \rightarrow R\text{Hom}_{B^e}(B, B^e)[d+1]$

: triangle in  $D(B^e)$  をうる.

上の  $\begin{array}{c} \xrightarrow{\quad} \\ \parallel \\ \xrightarrow{\quad} \end{array}$  は可換なので,  $B = R\text{Hom}_{B^e}(B, B^e)[d+1]$  をうる  $\square$

• 以下では  $\Pi$  に焦点を当てる, まず. 古典的な  $d=1$  の場合.

Ex  $d=1$  のときは  $\Pi$  を単に **preproj. alg** とよぶ  $\left[ \begin{array}{l} \text{Gelfand-Ponomarev} \\ \text{Dlab-Ringel} \\ \text{Baer-Gergle-Lenzing} \end{array} \right]$

$Q$ : acyclic quiver  $A = kQ$

$\overline{Q}$ :  $Q$  のダブル  $\overline{Q} \begin{array}{c} \xrightarrow{a} \quad \xrightarrow{\quad} \\ \xleftarrow{a^*} \quad \xleftarrow{\quad} \end{array}$

Prop  $\Pi_2(kQ) \simeq k\overline{Q} / (\sum_{a \in Q_1} (aa^* - a^*a))$

Sketch §6 より  $A := kQ = T_S(kQ_1) \ (S := kQ_0)$

$$0 \rightarrow A \otimes_S kQ_1 \otimes_S A \rightarrow A \otimes_S A \rightarrow A \rightarrow 0 : \text{proj. resol. of } A \in \text{mod } A^e$$

これに  $\text{Hom}_{A^e}(-, A^e)$  を apply して

$$A \otimes_S A \rightarrow A \otimes_S \text{Hom}_{S^e}(kQ_1, S^e) \otimes_S A \rightarrow E \rightarrow 0 : \text{exact} \quad (*)$$

$(E \in \text{mod } A^e \text{ の最小生成系}) = [\text{Hom}_{S^e}(kQ_1, S^e) \in \text{mod } S^e \text{ の最小生成系}]$

$$= (kQ_1 \text{ の dual basis}) = \{a^* \mid a \in Q_1\}$$

このことより tensor alg.  $T_A(E)$  の quiver は,  $Q$  に  $\{a^* \mid a \in Q_1\}$  をつけ加えて得られる. 関係式も  $(*)$  より分かる.  $\square$

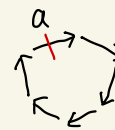
Ex  $Q$ : acyclic quiver  $A = kQ$

| $Q$    | Dynkin                                      | non-Dynkin                                    |
|--------|---------------------------------------------|-----------------------------------------------|
| $kQ$   | representation - finite                     | representation - infinite                     |
| $\Pi$  | finite dim. selfinjective alg               | infin. dim. 2-CY alg.                         |
| CY 三角圏 | $\text{mod } \Pi = \text{Dsg}(\Pi)$ is 2-CY | $\text{Dfd}(\Pi)$ is 2-CY<br>[Crawley-Boevey] |

• 次に  $d=2$  の場合

Def quiver with potential  $(Q, W)$  とは  $\begin{cases} Q: \text{quiver} \\ W \in \mathbb{k}Q \text{ は サイクルの線形和} \end{cases}$   
(QP)

$$\mathcal{P}(Q, W) := \mathbb{k}Q / (\partial_a W \mid a \in Q_1)$$



Jacobi alg

$$\partial_a (\underbrace{a_1 a_2 \cdots a_n}_{\text{サイクル}}) = \sum_{a_\ell = a} a_{\ell+1} \cdots a_n a_1 \cdots a_{\ell-1}$$

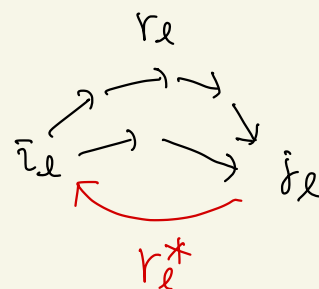
Ex  $Q = \begin{array}{ccc} & a \nearrow & b \searrow \\ & \swarrow c & \searrow \\ & & \end{array}$   $W = abc$   $\mathcal{P}(Q, W) = \mathbb{k}Q / (\partial_a W, \partial_b W, \partial_c W)$

Def [Keller]  $A = \mathbb{k}Q / (r_1, \dots, r_n) : \text{fin. dim } \mathbb{k}\text{-alg.}$

$$r_\ell \in e_{i_\ell}(\mathbb{k}Q)e_{j_\ell} \quad (\ell=1 \sim n)$$

このとき  $Q' := Q \sqcup \{ r_\ell^* : j_\ell \rightarrow i_\ell \}$

$$\mathbb{k}Q' \ni W := \sum_{\ell=1}^n r_\ell^* r_\ell \quad (Q', W) : \text{QP}$$



Prop [Keller]  $\text{gl. dim } A \leq 2 \Rightarrow \Pi_3(A) \simeq \mathcal{P}(Q', W)$

Rem より強く.  $\Pi_3(A)$  は  $(Q', W)$  の Ginzburg dg alg. と quasi-iso. となる

Ex  $A = \mathbb{k} \left[ \begin{array}{ccccc} & & b & & d \\ & a \nearrow & & \searrow & \\ & & c & & e \\ & & & & f \end{array} \right] / \begin{pmatrix} ac & bd-ce & ef \\ \parallel & \parallel & \parallel \\ r_1 & r_2 & r_3 \end{pmatrix} : \text{Auslander alg. of } \mathbb{k}A_3$

$\Rightarrow Q' = \begin{array}{ccccc} & & b & & d \\ & a \nearrow & & \searrow & \\ & & c & & e \\ & & & & f \end{array}$   $g=r_1^*, h=r_2^*, i=r_3^*$   
 $W = acg + bdh - ce h + ef i$

$\Pi_3(A) = \mathbb{k}Q' / (\underbrace{cg}_{\partial_a W}, \underbrace{dh}_{\partial_b W}, \underbrace{ga-eh}_{\partial_c W}, \underbrace{hb}_{\partial_d W}, \underbrace{fi-hc}_{\partial_e W}, \underbrace{ie}_{\partial_f W}, r_1, r_2, r_3)$



- $d=1$  では. Dynkin, non-Dynkin とともに.  $\Pi$  は 2-CY 性を持っていた (50)

$A$  に次の条件を課すことで. 同じことが  $d \geq 2$  でも成立する

Def [Herschend - I - Oppermann]  $A$  fin. dim alg.  $gl\text{-dim } A \leq d$

$A$ :  $d$ -hereditary  $\iff \forall i \in \mathbb{Z}, \forall_d^i(A)$  の cohomology  $H^j(\forall_d^i(A))$  は  $j$  が  $d$  の倍数でなければ 0

Rem  $d=1$  なら automatic.  $\therefore$  1-hereditary  $\equiv$  hereditary

以下  $A$ :  $d$ -hered.  $\Rightarrow \Pi$  に付随する  $\mathbb{Z}$ -graded sing. cat. に tilt. obj. がある  
ungraded CT obj

ことを見る. ただし考える sing. cat. は  $A$  が以下のどちらのクラスに属するかによって変わる

Def-Prop (dichotomy)  $A$ : ring-indecomp fin. dim.  $\mathbb{k}$ -alg.  $gl\text{-dim } A \leq d$

このとき  $A$ :  $d$ -hered.  $\iff$  以下のいずれか片方 (のみ) が成立

①  $\exists M \in \text{mod } A$ :  $d$ -CT obj (このとき  $A$  を  $d$ -represent.-finite ( $d$ -RF) とよぶ)

②  $\forall i \geq 0, \forall_d^i(A) \in \text{mod } A$  (infin. ( $d$ -RI))

Rem ①  $d=1, A = \mathbb{k}Q$   $1\text{-RF} \iff Q$ : Dynkin  
 $1\text{-RI} \iff Q$ : non-Dynkin

②  $A$ :  $d$ -RI  $\iff \Pi_{d+1}(A) \xrightarrow[q\text{-iso.}]{\simeq} \Pi_{d+1}(A) \Rightarrow \Pi$ :  $(d+1)$ -CY alg.

- Results for  $d$ -RF alg.

Thm [IO]  $A$ :  $d$ -RF alg.

①  $\text{mod } A \ni \Pi$  は  $d$ -CT obj.

②  $\Pi$  は  $\mathbb{Z}$ -graded fin. dim. selfing. alg.

③  $\text{mod}^{\mathbb{Z}} \Pi = D_{\text{sg}}^{\mathbb{Z}}(\Pi)$  has a Serre functor  $(-)[d+1]$

$\text{mod } \Pi = D_{\text{sg}}(\Pi)$  is  $(d+1)$ -CY

④  $\exists$  triangle equiv.  $\text{mod}^{\mathbb{Z}} \Pi \simeq \text{per } B, \text{mod } \Pi \simeq \mathcal{C}_{d+1}(B)$

$B := \text{End}_A(\Pi)$ : stable  $d$ -Auslander alg has  $gl\text{-dim } B \leq d+1$



Rem •  $d=1$  のとき ④ の前半は [Amiot]

(51)

•  $A: d\text{-RF} \Rightarrow T_d$  により 以下の bijection が与えられる

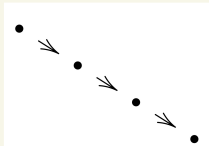
$$\{ \text{non-proj. indec. obj in add } T_A \} / \text{iso} \xrightarrow{\sim} \{ \text{non-inf. indec. obj in add } T_A \} / \text{iso}$$

$\pm S$  に全ての  $T_d^\pm$ -orbit in  $\text{add } T_A$  は indec. proj  $A\text{-mod}$  と indec. inf.  $A\text{-mod}$  を  
1つずつ含む

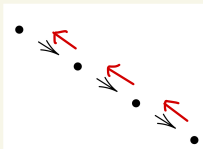
Ex  $A$ : higher Aus. alg of type  $A_n$

以下  $n=4$

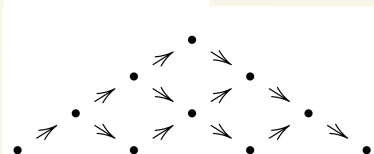
$d=1$   $A$



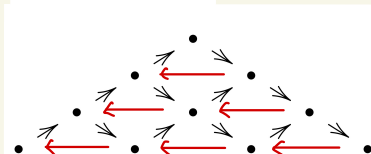
$\pi$



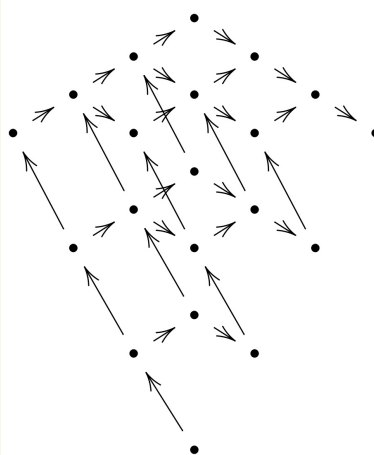
$d=2$   $A$



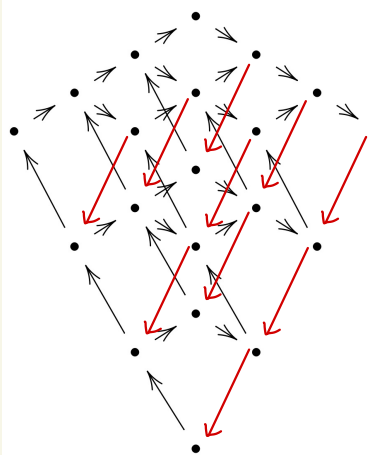
$\pi$



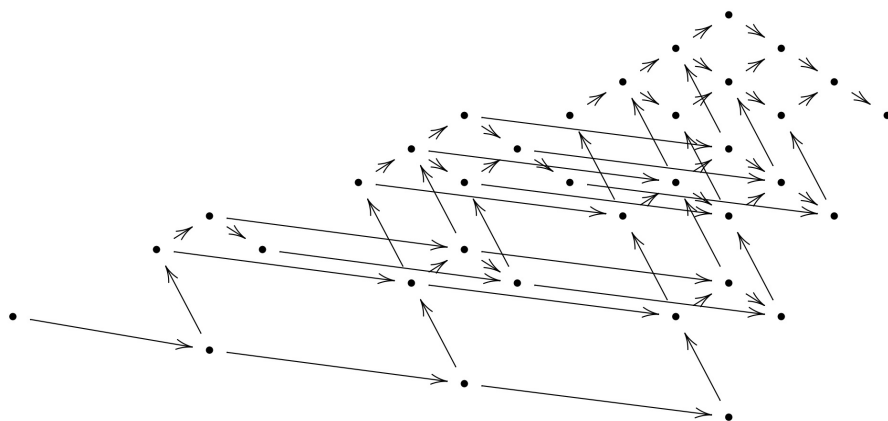
$d=3$   $A$



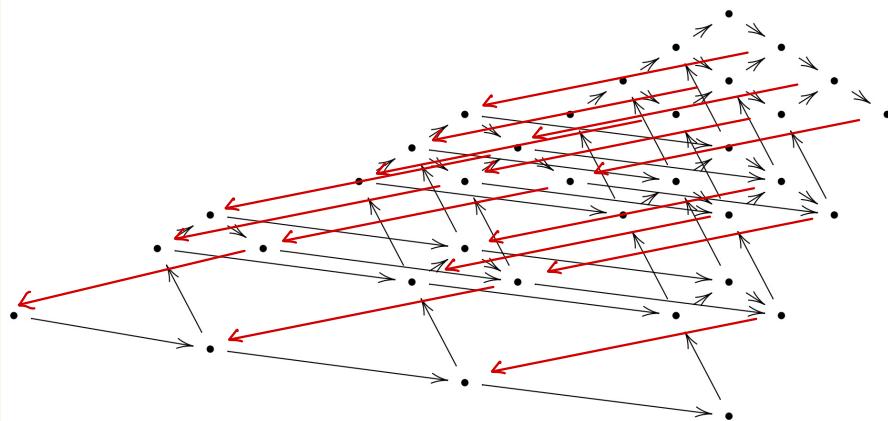
$\pi$



$d=4$   $A$



$\pi$

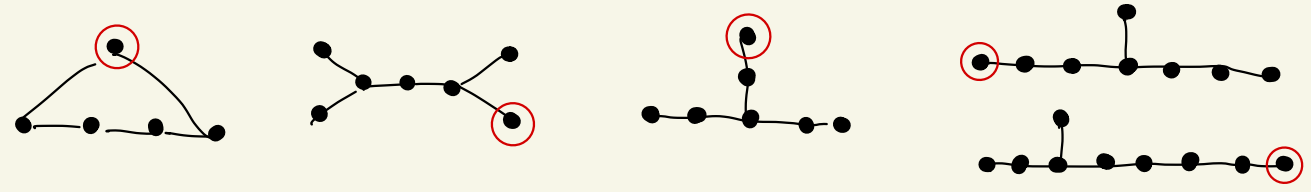


proof of ④ Use Yamana or Happel's Thm in §7  $\square$

• Results for d-RI alg.

Ex [Geigle-Lenzing 91]  $d=1$

$k$ : alg. closed of char 0       $Q$ : 非大 Dynkin quiver



$A := kQ \ni e$ :  $\bigcirc$  に対応する idempotent

$\pi := \pi_2(A) \Rightarrow$  ①  $R := e\pi e$  は simple sing. of dim 2

$\pi$  は tensor alg  $T_A(E)$  なので  $\mathbb{Z}$ -graded であり.  $R$  もそう.

Recall from §7 ②  $R$  は CM-finite

③ (CMR の AR quiver)  $= \overline{Q} = (\pi$  の quiver)

(正確には CMR は Krull-Schmidt でない) ので.  $R = \bigoplus_{i \in \mathbb{Z}} R_i$  を completion  $\widehat{R} := \varprojlim_{i \in \mathbb{Z}} R_i$  とし  $\pi R_i$  が ②③ を満たす

実は ③ より 強く. 次の成立.

④  $\exists$  equivalence  $(-)e: \text{proj } \pi \xrightarrow{\sim} \text{CMR}$   
 $X \longmapsto Xe$

よく  $\pi e \in \text{CMR}$  は  $\text{add } \pi e = \text{CMR}$  を満たす (つまり 1-CT object)

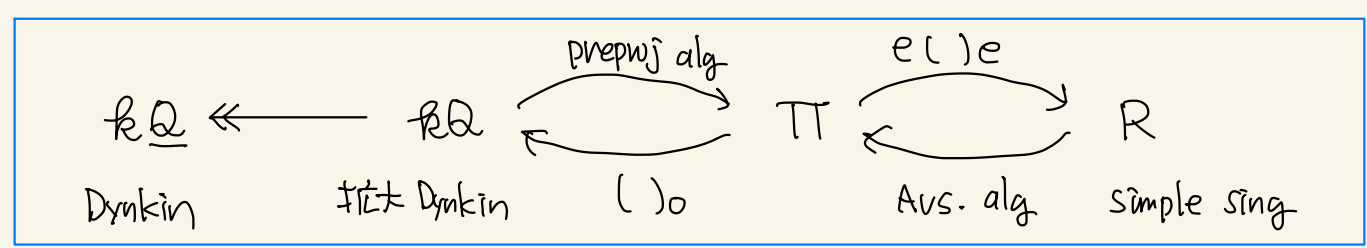
$\pi = \text{End}_R(\pi e)$  が成立 (つまり  $\pi$  は  $R$  の Auslander alg.)

⑤  $\underline{Q} := Q \setminus \bigcirc$ : Dynkin quiver に対し.  $\exists$  triangle equiv.

$$\underline{\text{CM}}^{\mathbb{Z}} R = D_{\text{sg}}^{\mathbb{Z}}(R) \simeq D^b(k\underline{Q}) \quad [\text{Geigle-Lenzing, Kajiwara-Saito-Takahashi}]$$

$$\underline{\text{CM}} R = D_{\text{sg}}(R) \simeq \mathcal{C}_1(k\underline{Q}) \quad [\text{Amiot}]$$

以上まとめると



- 以上の  $d=1$  の事実を  $d \geq 1$  に拡張する

Assume •  $A$ :  $d$ -RI alg. で  $\Pi := \Pi_{d+1}(A)$  は Noether ring

- $A \ni e$ : idempotent s.t.  $\dim_R(\Pi/(e)) < \infty$

- $eA(1-e) = 0$  (i.e.  $A = \begin{pmatrix} eAe & 0 \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$ )

このとき  $R := eAe$  (一般に非可換),  $\underline{A} := A/(e)$  とおく

Thm [Amiot-I-Reiten] 以上の仮定のもと

①  $R$  は Iwanaga-Gorenstein ring

②  $\Pi e$  は CMR の  $d$ -CT object

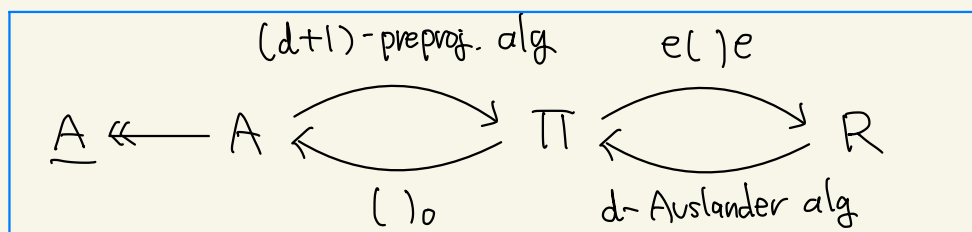
$\text{End}_R(\Pi e) = \Pi$  が成立 (つまり  $\Pi$  は  $R$  の  $d$ -Auslander alg.)

③  $\text{gl-dim } \underline{A} \leq d$  で  $\exists$  triangle equivalences

$$\underline{\text{CM}}^{\mathbb{Z}} R = D_{\text{sg}}^{\mathbb{Z}}(R) \simeq D^b(\underline{A})$$

$$\underline{\text{CM}} R = D_{\text{sg}}(R) \simeq \mathcal{C}_d(\underline{A})$$

以上まとめると



Idea of Proof ① は 「symmetric alg.  $B$  とその idempotent  $e$  に対して.

$eBe$  も symmetric alg. である」ことの類似

② は 高次 Auslander 対応 (§12) の類似

③ 前半は  $D^b(\underline{A}) \xrightarrow{\text{restriction}} D^b(A) \xrightarrow{-\otimes_A^L \Pi e} D^b(\text{mod } \mathbb{Z} R) \rightarrow D_{\text{sg}}^{\mathbb{Z}}(R)$  が equivalence を与える. 後半は  $\mathcal{C}_d(\underline{A})$  の universality を用いる.  $\square$

Rem ①  $\exists$  similar results by [de Volcsey-Van den Bergh], [Kalck-Yang]

②  $\left\{ \begin{array}{c} d\text{-RI alg.} \\ \downarrow \\ A \end{array} \right\} / \sim_{\text{Morita}} \xrightarrow{|\cdot|} \left\{ \begin{array}{c} (d+1)\text{-CY alg. of } d\text{-inv. } -1 \\ \downarrow \text{ (non-dg.)} \\ \Pi_{d+1}(A) \end{array} \right\} / \sim_{\text{gr. Morita}}$   
 [Keller, Minamoto-Mori]

これより  $(d+1)$ -CY alg. of  $a$ -inv.  $-1$  から始めて 11313 本例を作る

Ex  $\mathbb{k}$ : field,  $S = \mathbb{k}[x_0, \dots, x_d]$   $\zeta \in \mathbb{k}$ : primitive  $n$ -th root of 1

$$SL_{d+1}(\mathbb{k}) \supset G := \langle \text{diag}(\zeta^{a_0}, \dots, \zeta^{a_d}) \rangle \quad (a_i \in \mathbb{Z}/n\mathbb{Z})$$

$G$  acts on  $S$  naturally.  $S^G$ : invariant subring

Cor Assume  $1 \leq a_i \leq n-1$  ( $i=0 \sim d$ ) and  $\sum_{i=0}^d a_i = n$

Then  $\exists$  fin. dim  $\mathbb{k}$ -alg  $A$ ,  $\exists \mathbb{Z}$ -grading on  $S^G$  and triangle equivalences

$$\underline{\text{CM}}^{\mathbb{Z}} S^G \simeq D^b(A), \quad \underline{\text{CM}} S^G \simeq \mathcal{C}_d(A)$$

Rem 1つ目の equivalence は [Ueda 08] で Orlov's Thm により示された

$A$  と  $(S^G \text{ の } \mathbb{Z}\text{-grading})$  の定め方:

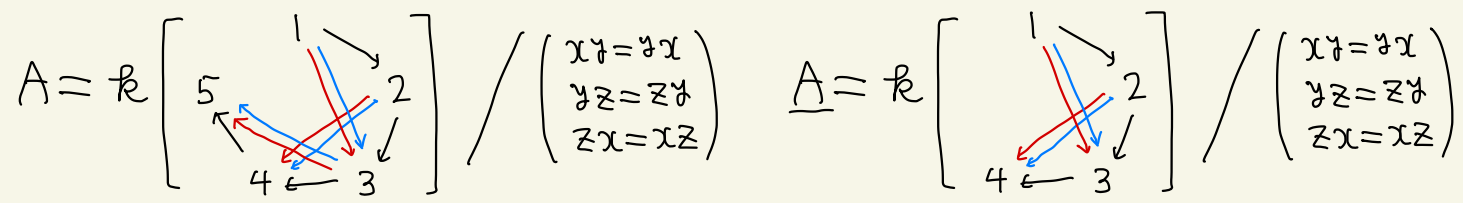
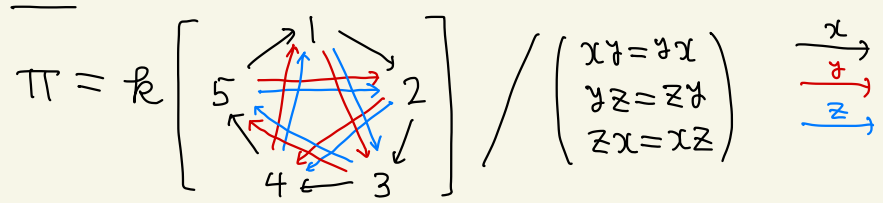
- $\Pi := S * G$ : skew group ring
  - $\Pi = S \otimes_{\mathbb{k}} \mathbb{k}G$  as a  $\mathbb{k}$ -vector space
  - $\forall f, f' \in S, g, g' \in G, (f \otimes g)(f' \otimes g') := f g(f') \otimes g g'$
- $\Pi$  は  $G$  の McKay quiver で表示される
  - vertices:  $\{1, 2, \dots, n\} \simeq \mathbb{Z}/n\mathbb{Z}$
  - arrows:  $x_j: i \rightarrow i + a_j \quad (i \in \mathbb{Z}/n\mathbb{Z}, 0 \leq j \leq d)$
  - relations:  $x_j x_{j'} = x_{j'} x_j$
- $\Pi$  の  $\mathbb{Z}$ -grading を以下で定める  $\Rightarrow S^G \simeq e_n \Pi e_n$  の  $\mathbb{Z}$ -grading も定まる

$$\deg(x_j: i \rightarrow i + a_j) := \begin{cases} 0 & i < i + a_j \\ 1 & \text{else} \end{cases} \quad [\text{Ueda}]$$

•  $A := \Pi_0$ ,  $\underline{A} := A/(e_n)$

すると AIR's Thm の仮定が満たされ. Cor が従う

Ex  $d=2, n=5, (a_0, a_1, a_2) = (1, 2, 2)$



他にも  $e$  のとり方はいくつかある。例えば  $e = e_4 + e_5$

## §14 他の結果

特異圏の同理論に関する他の結果をいくつか挙げる。

① [Kajiwara-Saito-Takahashi 07-09, Lenzing-de la Pena 11]

良い 2次元 hypersurface

② [Ueda 12, I-Takahashi 13, Mori-Veyama 16]

商特異点とその非可換版 (p54 Cor とは別の  $\mathbb{Z}$ -grading)

③ [Smith-Van den Bergh 13, Mori-Veyama, Higashitani-Veyama]

非可換 hypersurface

④ [Yuta Kimura 18, 19]

$\Pi = \Pi_2(\mathbb{R}\mathbb{Q})$  と Coxeter 群の元  $w \in W$  から定まる  $\Pi_w$

⑤ [Minamoto-Yamaura 21] homologically well-graded I-Gorenstein alg.

⑥ [Lu-Zhu] fin-dim. Iwanaga-Gorenstein algebra with inj-dim  $\leq 1$

⑦ [Hirano-Ouchi]

chain 型の invertible polynomial

⑧ [Hanihara] Yoneda algebra

etc.