C_0-Semigroups on Banach Space

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1 February 2021

On this course

Purpose: We learn the Hille–Yosida theorem for C_0-semigroups of linear operators on a Banach space, and its applications to PDEs.

References:

- H. Fujita, S.T. Kuroda, S. Ito, "Functional Analysis", Iwanami Shoten (in Japanese)
- K. Yosida, "Functional Analysis", Springer
- K. Masuda, "Evolution Equations", Kinokuniya (in Japanese)
- H. Tanabe, "Evolution Equations", Iwanami Shoten (in Japanese)

Section 1
Introduction: Matrix Exponential

§ 1.1 Linear ODEs

∘ **First order linear ODE**

We begin with a first order linear ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = au(t), \quad u(0) = u_0.$$

We can solve it as follows. Multiply e^{-at}, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{-at}u(t)\Big) = 0, \quad \text{so that} \quad \mathrm{e}^{-at}u(t) = \mathrm{e}^{-a0}u(0) = u_0.$$

Thus we obtain a solution

$$u(t) = \mathrm{e}^{at}u_0.$$

We can further generalize this argument.

○ **Second order linear ODE**

Next we consider a second order linear ODE:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) = au(t), \quad u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1$$

Let us set $\boldsymbol{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$, and then

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} u' \\ au \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \boldsymbol{u}.$$

Thus, if we set $A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$, $\boldsymbol{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$, the equation is rewritten as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(t) = A\boldsymbol{u}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0. \qquad (\clubsuit)$$

For a square matrix X define a **matrix exponential** e^X as

$$\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

It is well known that each component of e^X is convergent, and for $X = tA$ it satisfies

$$\mathrm{e}^{0A} = 1, \quad \mathrm{e}^{tA}\mathrm{e}^{sA} = \mathrm{e}^{(t+s)A}, \quad \frac{\mathrm{d}\mathrm{e}^{tA}}{\mathrm{d}t} = A\mathrm{e}^{tA} = \mathrm{e}^{tA}A.$$

Now multiply e^{-tA} on the equation ($\clubsuit$), and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-tA}\boldsymbol{u}\right) = \boldsymbol{0}, \quad \text{so that} \quad \mathrm{e}^{-tA}\boldsymbol{u}(t) = \mathrm{e}^{-0A}\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Hence we obtain a solution to ($\clubsuit$) as

$$\boldsymbol{u}(t) = \mathrm{e}^{tA}\boldsymbol{u}_0.$$

§ 1.2 Evolution Equations

We shall call a PDE that describes an evolution of a state function u an **evolution equation**. Examples are the following.

Heat (or diffusion) equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad u(0,\cdot) = u_0.$$

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad u(0,\cdot) = u_0, \quad \frac{\partial u}{\partial t}(0,\cdot) = u_1.$$

Schrödinger equation:

$$\mathrm{i}\frac{\partial u}{\partial t} = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + Vu, \quad u(0,\cdot) = u_0.$$

○ **Heat (or diffusion) equation**

The space of the temprature (or concentration) distritutions would be given by the space of the functions

$$X = \left\{u \colon \mathbb{R}^3 \to \mathbb{R}\right\}.$$

This is obviously a vector space. We define the **Laplacian** as a linear operator acting on X as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \colon X \to X.$$

Then we can regard the heat (or diffusion) equation as describing the evolution of the distribution $u(t) \in X$ by

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = \Delta u(t), \quad u(0) = u_0.$$

Hence we obtain a solution $u(t) = \mathrm{e}^{t\Delta}u_0$ (?)

◦ **Wave equation**

The space of the displacements of particles in a medium would be given by

$$X = \{ u \colon \mathbb{R}^3 \to \mathbb{R} \}.$$

Consider the Laplacian Δ a linear operator acting on X. Then we can regards the wave equation as describing the time-evolution of a displacement vector $u(t) \in X$ by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) = \Delta u(t), \quad u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1.$$

Let us further set

$$\tilde{X} = X \times X, \quad \boldsymbol{u} = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \boldsymbol{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix},$$

and then the wave equation is rewritten as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(t) = A\boldsymbol{u}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Hence we obtain a solution $\boldsymbol{u}(t) = \mathrm{e}^{tA}\boldsymbol{u}_0$ (?)

We can argue similarly for the Schrödinger equation.

Now the problem is "**How could and should we define the exponential function of a linear operator on a vector space of infinite dimension?**"

§ 1.3 Review of Matrix Exponentials

Let us recall a matrix exponential e^A for a square matrix A of order d. For simplicity first let us assume A is diagonalizable:

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

for some invertible matrix P. Then

$$\begin{aligned} \mathrm{e}^A &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n = P \Bigl[\sum_{n=0}^{\infty} \frac{1}{n!} \bigl(P^{-1}AP \bigr)^n \Bigr] P^{-1} \\ &= P \left[\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_d^n \end{pmatrix} \right] P^{-1} = P \begin{pmatrix} \mathrm{e}^{\lambda_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_d} \end{pmatrix} P^{-1}. \end{aligned}$$

Hence we obtain an expression of e^A.

In general we can always transform A into a Jordan normal form

$$P^{-1}AP = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix}.$$

for some invertible matrix P. Then, similarly to the above,

$$\begin{aligned} \mathrm{e}^A &= P \Bigl[\sum_{n=0}^{\infty} \frac{1}{n!} \bigl(P^{-1}AP \bigr)^n \Bigr] P^{-1} \\ &= P \left[\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} J_1^n & & \\ & \ddots & \\ & & J_d^n \end{pmatrix} \right] P^{-1} \\ &= P \begin{pmatrix} \mathrm{e}^{J_1} & & \\ & \ddots & \\ & & \mathrm{e}^{J_d} \end{pmatrix} P^{-1}. \end{aligned}$$

It reduces to the exponential function of a Jordan block J_i.

Let J be a Jordan block of order s, and we compute e^J. Let

$$J = \lambda I + N.$$

Then, noting that $N^s = 0$, we have

$$\begin{aligned}
e^J &= \sum_{n=0}^{\infty} \frac{1}{n!} J^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\min\{n,s-1\}} \frac{n!}{(n-k)!k!} \lambda^{n-k} N^k \\
&= \left(\sum_{n=0}^{s-1} \sum_{k=0}^{n} + \sum_{n=s}^{\infty} \sum_{k=0}^{s-1} \right) \frac{\lambda^{n-k}}{(n-k)!k!} N^k \\
&= \left(\sum_{k=0}^{s-1} \sum_{n=k}^{s-1} + \sum_{k=0}^{s-1} \sum_{n=s}^{\infty} \right) \frac{\lambda^{n-k}}{(n-k)!k!} N^k \\
&= \sum_{k=0}^{s-1} \sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{(n-k)!k!} N^k = \sum_{k=0}^{s-1} \frac{e^{\lambda}}{k!} N^k.
\end{aligned}$$

Thus we obtain

$$e^J = \begin{pmatrix} e^{\lambda} & e^{\lambda} & e^{\lambda}/2! & e^{\lambda}/3! & \cdots & e^{\lambda}/(s-1)! \\ 0 & e^{\lambda} & e^{\lambda} & e^{\lambda}/2! & \cdots & e^{\lambda}/(s-2)! \\ 0 & 0 & e^{\lambda} & e^{\lambda} & \cdots & e^{\lambda}/(s-3)! \\ 0 & 0 & 0 & e^{\lambda} & \cdots & e^{\lambda}/(s-4)! \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e^{\lambda} \end{pmatrix}.$$

Problem. 1. Show that, if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then so is e^{λ} for e^A. The converse is not true. Give a counterexapmle.

2. Similarly to the above, compute e^{tJ} for $t \in \mathbb{R}$.

Solution. 1. Omitted. 2. We can proceed as

$$\begin{aligned}
e^{tJ} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\min\{n,s-1\}} \frac{n!}{(n-k)!k!} \lambda^{n-k} N^k \\
&= \cdots \\
&= \sum_{k=0}^{s-1} \frac{t^k e^{t\lambda}}{k!} N^k \\
&= \begin{pmatrix} e^{t\lambda} & te^{t\lambda} & t^2 e^{t\lambda}/2! & t^3 e^{t\lambda}/3! & \cdots & t^{s-1} e^{t\lambda}/(s-1)! \\ 0 & e^{t\lambda} & te^{t\lambda} & t^2 e^{t\lambda}/2! & \cdots & t^{s-2} e^{t\lambda}/(s-2)! \\ 0 & 0 & e^{t\lambda} & te^{t\lambda} & \cdots & t^{s-3} e^{t\lambda}/(s-3)! \\ 0 & 0 & 0 & e^{t\lambda} & \cdots & t^{s-4} e^{t\lambda}/(s-4)! \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e^{t\lambda} \end{pmatrix}.
\end{aligned}$$

□

Section 2
Review of Banach Spaces

§ 2.1 Linear Operators on Banach Space

∘ **Banach space**

Definition. Let X be a complex vector space. We call a mapping $\|\cdot\|: X \to \mathbb{R}$ a **norm** if it satisfies

1. For any $u \in X$ one has $\|u\| \geq 0$;
2. $\|u\| = 0$ holds if and only if $u = 0$;
3. For any $c \in \mathbb{C}$ and $u \in X$ one has $\|cu\| = |c|\|u\|$;
4. For any $u, v \in X$ one has $\|u + v\| \leq \|u\| + \|v\|$.

In addition, we call a pair $(X, \|\cdot\|)$ of a vector space X and a norm $\|\cdot\|$ on X a **normed space**. We denote it simply by X.

Proposition 2.1. A normed space X is a metric space with respect to the **natural metric**

$$\mathrm{dist}(u, v) = \|u - v\|; \quad u, v \in X.$$

Proof. We leave it to the reader as **Problem**. □

Definition. A normed space is called a **Banach space** if it is complete with respect to the natural metric.

∘ **Linear operators**

For the rest of the section we let X be a Banach space.

Definition. Let $D \subset X$ be a linear subspace. A linear mapping $A: D \to X$ is called a **linear operator**, or simply an **operator**, on X. We denote the domain and the range of A by

$$D(A) = D \quad \text{and} \quad \mathrm{Ran}\, A,$$

respectively.

Remark. We shall NOT write

$$A: X \to X$$

since $D \neq X$ is often the case, but DO call it an operator on X. We distinguish them.

Definition. A densely defined operator A on X is said to be **bounded** if there exists $C \geq 0$ such that for any $u \in D(A)$

$$\|Au\| \leq C\|u\|.$$

We denote the set of all the bounded operators on X by $\mathcal{B}(X)$.

Proposition 2.2. A bounded operator on X extends uniquely as a continuous linear operator with domain X. Convesely, a continuous linear operator with domain X is bounded.

Proof. We leave it to the reader as **Problem**. □

Remarks. 1. In the following we may always assume that a bounded operator A has a domain $D(A) = X$.

2. A general operator on X is sometimes called an **unbounded** operator in contrast to a bounded operator.

Proposition 2.3. $\mathcal{B}(X)$ is a Banach space with respect to the **operator norm**

$$\|A\| := \sup_{\|u\|=1} \|Au\| = \inf\Big\{C \geq 0;\ \forall u \in X\ \|Au\| \leq C\|u\|\Big\}$$

Proof. We leave it to the reader as **Problem**. □

Definition. A linear operator A on X is said to be **closed** if for any sequence $(u_n)_{n\in\mathbb{N}}$ on $D(A)$ with limits

$$\lim_{n\to\infty} u_n =: u, \quad \lim_{n\to\infty} Au_n =: v$$

these limits satisfy

$$u \in D(A), \quad Au = v.$$

Proposition 2.4. A linear operator A on X is closed if and only if its **graph**

$$\mathcal{G}(A) = \Big\{(u, Au) \in X \times X;\ u \in D(A)\Big\}$$

is a closed subspace of $X \times X$. Here $X \times X$ is a Banach space with the norm

$$\|(u, v)\|_{X\times X} = \|u\|_X + \|v\|_X.$$

Proof. It is straightforward from the definition. □

Definition. Let A, B linear operators on X. We say B is an **extension** of A, or A is a **restriction** of B, if

$$D(A) \subset D(B), \quad \forall u \in D(A)\ \ Au = Bu,$$

and we denote it by $A \subset B$.

Definition. A linear operator A on X is said to be **closable** if it has a closed extension. The minimum closed extension of a closable operator A is called a **closure**, and is denoted by $\bar{A}$.

Proposition 2.5. A linear operator A on X is closable if and only if for any sequence $(u_n)_{n\in\mathbb{N}}$ on $D(A)$ with limits

$$\lim_{n\to\infty} u_n = 0, \quad \lim_{n\to\infty} Au_n =: v$$

the latter limit satisfies $v = 0$.

Proof. We leave it to the reader as **Problem**. □

Theorem 2.6 (Closed graph theorem). Let A be a closed operator on X. If $D(A) = X$, then A is bounded.

Proof. The proof depends on the Baire category theorem, and we omit it. □

§ 2.2 Calculus for Vector-Valued Functions

In this section we continue to let X be a Banach space.

∘ **Continuity and differetiability**

Definition. Let $I \subset \mathbb{R}$ be an inverval.

1. An X-valued function $u: I \to X$ is said to be **continuous** on I if for any $t \in I$
$$u(t) = \lim_{h\to 0} u(t+h) \quad \text{in (the topology of) } X.$$
2. An X-valued function $u: I \to X$ is said to be **differentiable** on I if for any $t \in I$ there exists the limit
$$\frac{\mathrm{d}u}{\mathrm{d}t} = u'(t) := \lim_{h\to 0} h^{-1}\big(u(t+h) - u(t)\big) \quad \text{in } X.$$

3. Similarly, we extend terminologies for scalar-valued functions to X-valued ones. For each $k \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $C^k(I;X)$ the set of all the X-valued C^k functions on I.

Problem. Let $u \in C^1(\mathbb{R};X)$. Show, if $u'(t) \equiv 0$, then $u(t) \equiv u(0)$.

Solution. Let $v(t) = \|u(t) - u(0)\|$, and we show $v(t) \equiv 0$. By the triangle inequality and the assumption we have, as $h \to 0$,
$$h^{-1}|v(t+h) - v(t)| \le h^{-1}\|u(t+h) - u(t)\| \to 0,$$
hence $v'(t) \equiv 0$. Then by the mean value theorem for real-valued functions we obtain $v(t) \equiv v(0) = 0$. □

∘ **Riemann integral**

Let $u \in C^0([a,b];X)$ with $a < b$. Let $\Delta = \{t_0, t_1, \dots, t_n\}$ be a **partition** of the interval $[a,b]$, i.e.,
$$a = t_0 < t_1 < \cdots < t_n = b,$$
and let $\tau_j \in [t_{j-1}, t_j]$. The sum
$$\sum_{j=1}^{n} u(\tau_j)(t_j - t_{j-1}) \qquad (\heartsuit)$$
is called a **Riemann sum**. The Riemann sum ($\heartsuit$) is known to converges as $|\Delta| := \max_j(t_j - t_{j-1}) \to 0$. We denote the limit by
$$\int_a^b u(t)\,\mathrm{d}t = \lim_{|\Delta|\to 0} \sum_{j=1}^{n} u(\tau_j)(t_j - t_{j-1}),$$
and call it the **Riemann integral** of u on $[a,b]$.

Remark. The fundamental theorem of calculus extends to the X-valued continuous functions. We omit the arguments.

∘ **Holomorphy**

For the rest of the section we let $D \subset \mathbb{C}$ be a domain.

Definition. An X-valued function $u: D \to X$ is said to be **holomorphic** on D if for any $z \in D$ there exists the limit
$$\frac{\mathrm{d}u}{\mathrm{d}z} = u'(z) := \lim_{h\to 0} h^{-1}\big(u(z+h) - u(z)\big)$$

We omit the definition of a line integral of an X-valued function along a path, which is completely parallel to the $\mathbb{C}$-valued case.

Theorem 2.7 (**Cauchy's integral theorem**)**.** Let D be simply connected, and $u\colon D \to X$ holomorphic. Then for any closed C^1 path $\Gamma \subset D$

$$\int_\Gamma u(z)\,\mathrm{d}z = 0.$$

Proof. It is the same as the $\mathbb{C}$-valued case, and we omit it. □

Theorem 2.8 (**Cauchy's integral formula**)**.** Let D be simply connected, and $u\colon D \to X$ holomorphic. Then for any $a \in D$ and any simple closed C^1 path $\Gamma \subset D$ encircling a

$$u(a) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z-a)^{-1} u(z)\,\mathrm{d}z.$$

Proof. We omit it by the same reason as above. □

Corollary 2.9. An X-valued holomorphic function $u\colon D \to X$ is **analytic** on D, i.e., u is infinitely complex-differentiable on D, and for any $a \in D$ there exists a neighborhood $U \subset D$ of a such that for any $z \in U$

$$u(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} u^{(n)}(a).$$

Proof. We omit it by the same reason as above. □

◦ **Strong operator topology**

Definition. A sequence $(A_n)_{n\in\mathbb{N}}$ on $\mathcal{B}(X)$ is said to **converge in norm** to $A \in \mathcal{B}(X)$ if

$$\lim_{n\to\infty} \|A - A_n\|_{\mathcal{B}(X)} = 0.$$

We denote it by

$$\lim_{n\to\infty} A_n = A.$$

The corresponding topology of $\mathcal{B}(X)$ is called the **norm topology**, or the **uniform** (**operator**) **topology**.

Remark. The above topology obviously coincides with that of $\mathcal{B}(X)$ as a Banach space equipped with the operator norm.

Definition. A sequence $(A_n)_{n\in\mathbb{N}}$ on $\mathcal{B}(X)$ is said to **converge strongly** to $A \in \mathcal{B}(X)$ if for any $u \in X$

$$\lim_{n\to\infty} \|Au - A_n u\|_X = 0, \quad \text{or} \quad \lim_{n\to\infty} A_n u = Au.$$

We denote it by

$$\operatorname*{s\text{-}lim}_{n\to\infty} A_n = A.$$

The corresponding topology of $\mathcal{B}(X)$ is called the **strong** (**operator**) **topology**.

Remark. More precisely, the strong topology is a locally convex topology induced by the family of seminorms $A \mapsto \|Au\|_X$ indexed by u running over X. We do not discuss the detail.

Theorem 2.10 (Uniform boundedness principle). Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of elements in $\mathcal{B}(X)$. If for each $u \in X$

$$\sup_{\lambda\in\Lambda} \|A_\lambda u\|_X < \infty,$$

then

$$\sup_{\lambda\in\Lambda} \|A_\lambda\|_{\mathcal{B}(X)} < \infty.$$

Proof. The proof depends on the Baire category theorem. We omit it. □

Corollary 2.11. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence on $\mathcal{B}(X)$, and assume that for each $u \in X$ there exists the limit

$$Au := \lim_{n\to\infty} A_n u.$$

Then A is a bounded operator on X, or $A \in \mathcal{B}(X)$.

Remark. This says completeness of the strong topology of $\mathcal{B}(X)$.

Proof. The mapping $A\colon X \to X$ is obviously linear, and it suffices to show the boundedness. For any $u \in X$ we have

$$\|Au\| = \lim_{n\to\infty} \|A_n u\| \le \sup_{n\in\mathbb{N}} \|A_n u\| \le \Big(\sup_{n\in\mathbb{N}} \|A_n\|\Big)\|u\|.$$

By the uniform boundedness principle we can see

$$\sup_{n\in\mathbb{N}} \|A_n\| < \infty,$$

and thus the assertion is verified. □

◦ **Operator-valued functions**

Definition. Let $I \subset \mathbb{R}$ be an interval.

1. An operator-valued function $A\colon I \to \mathcal{B}(X)$ is **continuous in norm** if for any $t \in I$

$$A(t) = \lim_{h\to 0} A(t+h).$$

2. An operator-valued function $A\colon I \to \mathcal{B}(X)$ is **strongly continuous** if for any $t \in I$

$$A(t) = \operatorname{s-lim}_{h\to 0} A(t+h).$$

3. .... (We define other terminologies similarly.)

Cauchy's integral theorem and consequences derived from it hold for operator-valued strongly holomorphic functions as well as those in norm. We do not present their precise statements.

Theorem 2.12. Let $D \subset \mathbb{C}$ be a domain. An operator-valued function $A\colon D \to \mathcal{B}(X)$ is strongly holomorphic on D if and only if it is holomorphic in norm on D.

Remark. Hence we do not need to distinguish the strong holomorphy and the holomorphy in norm. We shall simply say A is **holomorphic** (or **analytic**).

Proof. If A is holomorphic in norm, then obviously it is strongly holomorphic. Thus it suffices to prove the converse. Let $z \in D$, and take a sufficiently small, simple closed path $\Gamma \subset D$ encircling z. Then by the assumption for any $u \in X$

$$A(z)u = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - z)^{-1} A(\zeta) u \, \mathrm{d}\zeta. \qquad (\diamondsuit)$$

Since $A(z)$ is strongly continuous and Γ is compact, we have for any $u \in X$

$$\sup_{\zeta \in \Gamma} \|A(\zeta)u\| < \infty,$$

and this implies by the uniform boundedness principle

$$\sup_{\zeta \in \Gamma} \|A(\zeta)\| < \infty. \qquad (\clubsuit)$$

Then it follows that A is continuous in norm. In fact, for any w close to z and any $u \in X$ by $(\diamondsuit)$

$$\|A(z)u - A(w)u\| \le \frac{|z-w|\|u\|}{2\pi} \int_\Gamma \frac{\|A(\zeta)\|}{|\zeta - z||\zeta - w|} \, \mathrm{d}\zeta,$$

which with $(\clubsuit)$ implies

$$\|A(z) - A(w)\| \le C|z-w|.$$

Therefore

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - z)^{-1} A(\zeta) \, \mathrm{d}\zeta$$

is convergent in norm, and again by $(\diamondsuit)$ we obtain

$$A(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - z)^{-1} A(\zeta) \, \mathrm{d}\zeta.$$

The last expression implies A is holomorphic in norm. □

§ 2.3 Resolvent

In this section we let X be a Banach space.

Definition. Let A be an injective linear operator on X. Then the inverse mapping of A defined on $\operatorname{Ran} A$ is called the **inverse operator** of A. We denote it by A^{-1}.

Remarks. 1. The inverse operator may not be defined on all of X, but we do say A^{-1} exists if A is injective.

2. Obviously, if A^{-1} exists, then

$$D(A^{-1}) = \operatorname{Ran} A, \quad \operatorname{Ran} A^{-1} = D(A).$$

3. A linear operator A on X is injective if and only if

$$\operatorname{Ker} A := \{u \in D(A);\ Au = 0\} = \{0\}.$$

Let A be a closed linear operator on X, and $z \in \mathbb{C}$. Then one of the following holds:

1. $(z-A)^{-1}$ does not exist;
2. $(z-A)^{-1}$ exists, but does not belong to $\mathcal{B}(X)$;
3. $(z-A)^{-1}$ exists, and belong to $\mathcal{B}(X)$,

Here z denotes a multiplication operator by the scalar z, or zI.

Problem. Under the above notation show the following.

1. $z - A$ is closed.
2. If $(z-A)^{-1}$ exists, it is closed as well.
3. If $(z-A)^{-1}$ exists, $\operatorname{Ran}(z-A) \subset X$ is dense, and $(z-A)^{-1}$ is bounded (in the original sense), then $\operatorname{Ran}(z-A) = X$.

Definition. Let A be a closed linear operator on X. We call

$$\rho(A) = \left\{z \in \mathbb{C};\ \exists (z - A)^{-1} \in \mathcal{B}(X)\right\},$$

the **resolvent set** of A, and

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

the **spectrum** of A. For each $z \in \rho(A)$ we denote

$$R(z) = R_A(z) = (z - A)^{-1},$$

and call it the **resolvent** of A.

Remark. The spectrum is a generalization of eigenvalues.

Proposition 2.13 (Neumann series). Let $A \in \mathcal{B}(X)$ satisfy

$$\|A\| < 1.$$

Then $(1 - A)^{-1}$ exists and belongs to $\mathcal{B}(X)$. Moreover, it is expressed by the **Neumann series** as

$$(1 - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

Remark. The Neumann series is analogous to a geometric series: For any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$

$$\frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \cdots .$$

Proof. We have, as $\nu > \mu \to \infty$,

$$\left\| \sum_{n=0}^{\nu} A^n - \sum_{n=0}^{\mu} A^n \right\| \le \sum_{n=\mu+1}^{\nu} \|A\|^n = \|A\|^{\mu+1} \frac{1 - \|A\|^{\nu-\mu}}{1 - \|A\|} \to 0,$$

and thus the Neumann series is convergent and bounded:

$$\sum_{n=0}^{\infty} A^n \in \mathcal{B}(X).$$

In addition, we can compute the compositions as

$$(1 - A) \sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = 1,$$
$$\left(\sum_{n=0}^{\infty} A^n\right)(1 - A) = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = 1,$$

which implies the assertion. □

Corollary 2.14. Let $A \in \mathcal{B}(X)$. Then

$$\rho(A) \supset \{z \in \mathbb{C};\ |z| > \|A\|\}, \quad \sigma(A) \subset \{z \in \mathbb{C};\ |z| \le \|A\|\}.$$

Proof. Let $z \in \mathbb{C}$ with $|z| > \|A\|$. Then, since $\|z^{-1}A\| < 1$, we have by Proposition 2.13

$$1 \in \rho(z^{-1}A).$$

This implies $z \in \rho(A)$, hence the assertion. □

Theorem 2.15 ((First) resolvent identity). Let A be a closed linear operator on X. Then for any $z, w \in \rho(A)$

$$R(z) - R(w) = (w-z)R(z)R(w) = (w-z)R(w)R(z).$$

Remark. Formally we can write it as

$$\frac{1}{z-A} - \frac{1}{w-A} = \frac{w-z}{(z-A)(w-A)} = \frac{w-z}{(w-A)(z-A)}.$$

Proof. Noting $\mathrm{Ran}(R(w)) \subset D(A)$, we can compute

$$\begin{aligned} R(z) - R(w) &= R(z)(w-A)R(w) - R(z)(z-A)R(w) \\ &= (w-z)R(z)R(w). \end{aligned}$$

The second identity can be verified similarly. □

Theorem 2.16. Let A be a closed linear operator on X. Then $\rho(A)$ is an open subset of $\mathbb{C}$, and $R(z)$ is holomorphic on $\rho(A)$. Moreover,

$$R'(z) = -R(z)^2.$$

Proof. Let $z \in \rho(A)$, and take any $\zeta \in \mathbb{C}$ with $|\zeta - z| < \|R(z)\|^{-1}$. Then

$$\mathcal{R} := \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^n R(z)^{n+1}$$

is convergent in norm in $\mathcal{B}(X)$. This operator satisfies

$$\begin{aligned} \mathcal{R}(\zeta - A) &= \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^n R(z)^n \\ &\quad + \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^{n+1} R(z)^{n+1} = 1, \end{aligned}$$

and

$$\begin{aligned} (\zeta - A)\mathcal{R} &= \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^n R(z)^n \\ &\quad + \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^{n+1} R(z)^{n+1} = 1. \end{aligned}$$

It follows that $\zeta \in \rho(A)$, and hence $\rho(A) \subset \mathbb{C}$ is open. In addition, we obtain

$$R(\zeta) = \mathcal{R} = \sum_{n=0}^{\infty} (-1)^n (\zeta - z)^n R(z)^{n+1},$$

implying that $R(z)$ is analytic, or holomorphic, on $\rho(A)$. Finally by the resolvent identity we obtain

$$\lim_{w \to z} (w-z)^{-1}(R(w) - R(z)) = -\lim_{w \to z} R(w)R(z) = -R(z)^2.$$

Thus we are done. □

∘ **Resolvent of matrix**

As an example, let us compute $R(z) = (z-A)^{-1}$ for a square matrix A of order d. Let us first assume A is diagonalizable, i.e.,

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

for some invertible matrix P. Then obviously

$$\rho(A) = \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_d\}, \quad \sigma(A) = \{\lambda_1, \dots, \lambda_d\},$$

and for any $z \in \rho(A)$

$$\begin{aligned} R(z) &= P\left(z - P^{-1}AP\right)^{-1} P^{-1} \\ &= P \begin{pmatrix} (z-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (z-\lambda_d)^{-1} \end{pmatrix} P^{-1}. \end{aligned}$$

In a general case consider a Jordan normal form

$$P^{-1}AP = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix}$$

for some invertible matrix P. Then, similarly to the above,

$$\rho(A) = \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_d\}, \quad \sigma(A) = \{\lambda_1, \dots, \lambda_d\},$$

and for any $z \in \rho(A)$

$$\begin{aligned} R(z) &= P\big(z - P^{-1}AP\big)^{-1}P^{-1} \\ &= P \begin{pmatrix} (z-J_1)^{-1} & & \\ & \ddots & \\ & & (z-J_p)^{-1} \end{pmatrix} P^{-1}. \end{aligned}$$

Thus it reduces to the resolvent of a Jordan block J_i.

Let $J = \lambda I + N$ be a Jordan block of order s, and let $z \neq \lambda$. Similarly to the Neumann series, we can compute, noting $N^s = 0$,

$$\begin{aligned} &(z-J)^{-1} \\ &= (z-\lambda)^{-1}\big(1 - (z-\lambda)^{-1}N\big)^{-1} \\ &= (z-\lambda)^{-1}\sum_{k=0}^{s-1}(z-\lambda)^{-k}N^k \\ &= \begin{pmatrix} (z-\lambda)^{-1} & (z-\lambda)^{-2} & (z-\lambda)^{-3} & \cdots & (z-\lambda)^{-s} \\ 0 & (z-\lambda)^{-1} & (z-\lambda)^{-2} & \cdots & (z-\lambda)^{-s+1} \\ 0 & 0 & (z-\lambda)^{-1} & \cdots & (z-\lambda)^{-s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (z-\lambda)^{-1} \end{pmatrix}. \end{aligned}$$

Section 3
Semigroups and Hille–Yosida Theorem

§ 3.1 One-Parameter Semigroups

Let X be a Banach space.

Theorem 3.1. Let $A \in \mathcal{B}(X)$. For any $t \in \mathbb{C}$ the series

$$\mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{t^n}{n!}A^n \qquad (\diamond)$$

converges in norm in $\mathcal{B}(X)$, and satisfies the following.

1. $\mathrm{e}^{0A} = 1$.
2. For any $t, s \in \mathbb{C}$ one has $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$.
3. e^{tA} is analytic in $t \in \mathbb{C}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA} = \mathrm{e}^{tA}A.$$

Proof. The proof is similar to that for $e^{\alpha t}$ with $\alpha \in \mathbb{C}$, and we omit it. □

Corollary 3.2. Let $A \in \mathcal{B}(X)$, and $u_0 \in X$. Then an **abstract evolution equation**

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) \quad \text{for } t > 0, \quad u(0) = u_0 \qquad (\clubsuit)$$

has a unique solution in $C([0,\infty); X) \cap C^1((0,\infty); X)$, which is given by

$$u(t) = \mathrm{e}^{tA}u_0. \qquad (\diamondsuit)$$

Proof. By Theorem 3.1 ($\diamondsuit$) obviously solves ($\clubsuit$), and it suffices to show the uniqueness. Let

$$u, v \in C([0,\infty); X) \cap C^1((0,\infty); X)$$

be solutions to ($\clubsuit$). Set $w = u - v$, and then it follows that

$$w'(t) = Aw(t) \quad \text{for } t > 0, \quad w(0) = 0.$$

Multiplying e^{-tA} to the above equation, we obtain

$$(\mathrm{e}^{-tA}w)' = 0,$$

so that for any $t > 0$

$$\mathrm{e}^{-tA}w(t) = \lim_{s\to+0} \mathrm{e}^{-sA}w(s) = 0.$$

Thus $w \equiv 0$, and this implies the asserted uniqueness. □

Definition. An operator-valued function $U\colon [0,\infty) \to \mathcal{B}(X)$ is called a **one-parameter semigroup** on X if

1. $U(0) = 1$;
2. For any $t, s \geq 0$ it satisfies $U(t+s) = U(t)U(s)$.

In addition, if U is strongly continuous on $[0,\infty)$, i.e., for any $u \in X$ and $t \geq 0$

$$\lim_{s\to t} U(s)u = U(t)u,$$

then U is called a C_0-**semigroup** on X.

Proposition 3.3. A one-parameter semigroup $U\colon [0,\infty) \to \mathcal{B}(X)$ is a C_0-semigroup if (and only if)

$$\operatorname*{s-lim}_{t\to+0} U(t) = 1.$$

Proof. Step 1. Here we claim that there exist $M \geq 1$ and $\beta \geq 0$ such that for any $t \geq 0$

$$\|U(t)\| \leq M\mathrm{e}^{\beta t}.$$

For that we first show that there exists $\delta > 0$ such that

$$\sup_{t\in[0,\delta]} \|U(t)\| < \infty.$$

In fact, otherwise, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ on $(0,\infty)$ such that as $n \to \infty$

$$t_n \to 0, \quad \|U(t_n)\| \to \infty.$$

However this contradicts the uniform boundedness principle since by the assumption for any u

$$U(t_n)u \to u.$$

Now we choose $M \geq 1$ and $\beta \geq 0$ such that

$$M = \mathrm{e}^{\beta\delta} = \sup_{t\in[0,\delta]} \|U(t)\| \geq 1.$$

Then for any $t \geq 0$ we can find $k \in \mathbb{N}_0$ such that $k\delta \leq t < (k+1)\delta$, and it follows that

$$\|U(t)\| \leq \|U(t-k\delta)\|\|U(\delta)\|^k \leq M\mathrm{e}^{\beta k\delta} \leq M\mathrm{e}^{\beta t}.$$

Step 2. Let $u \in X$. It suffices to show the continuity of $U(t)u$ at $t > 0$. Due to Step 1 and the assumption, as $h \to +0$,

$$\begin{aligned}\Big\|U(t+h)u - U(t)u\Big\| &\leq \|U(t)\|\|U(h)u - u\| \\ &\leq M\mathrm{e}^{\beta t}\|U(h)u - u\| \\ &\to 0,\end{aligned}$$

and

$$\begin{aligned}\Big\|U(t-h)u - U(t)u\Big\| &\leq \|U(t-h)\|\|u - U(h)u\| \\ &\leq M\mathrm{e}^{\beta(t-h)}\|U(h)u - u\| \\ &\to 0.\end{aligned}$$

These prove the assertion. □

Corollary 3.4. Let $U\colon [0,\infty) \to \mathcal{B}(X)$ be a C_0-semigroup. Tnen there exist $M \geq 1$ and $\beta \in \mathbb{R}$ such that for any $t \geq 0$

$$\|U(t)\| \leq M\mathrm{e}^{\beta t}.$$

Proof. It is clear from Step 1 of the proof of Proposition 3.3. □

Definition. A C_0-semigroup $U\colon [0,\infty) \to \mathcal{B}(X)$ is called a **contraction semigroup** if one can take $\beta \leq 0$ and $M = 1$ in Corollary 3.4, or

$$\|U(t)\| \leq 1 \quad \text{for all } t \geq 0.$$

§ 3.2 Infinitesimal Generator

Let X be a Banach space.

Definition. Let $U\colon [0,\infty) \to \mathcal{B}(X)$ be a C_0-semigroup. An **infinitesimal generator**, or simply a **generator**, of U is a linear operator A on X defined as

$$\begin{aligned}D(A) &= \Big\{u \in X;\ \exists \lim_{h\to+0} h^{-1}\big(U(h)u - u\big)\Big\}, \\ Au &= \lim_{h\to+0} h^{-1}\big(U(h)u - u\big) \quad \text{for } u \in D(A).\end{aligned}$$

If A is the generator of U, we say A **generates** U, and denote

$$U(t) = \mathrm{e}^{tA} \quad \text{for } t \geq 0.$$

Remark. The last notation is well-defined due to Proposition 3.8.

Proposition 3.5. Let $U\colon [0,\infty) \to \mathcal{B}(X)$ be a C_0-semigroup with generator A. Suppose $M \geq 1$ and $\beta \in \mathbb{R}$ satisfy that for any $t \geq 0$

$$\|U(t)\| \leq M\mathrm{e}^{\beta t}.$$

1. The generator A is a densely defined closed operator X.
2. One has

$$\sigma(A) \subset \{z \in \mathbb{C};\ \mathrm{Re}\, z \leq \beta\}, \quad \{z \in \mathbb{C};\ \mathrm{Re}\, z > \beta\} \subset \rho(A),$$

and for any $z \in \mathbb{C}$ with $\mathrm{Re}\, z > \beta$ and any $n \in \mathbb{N}$

$$\|(z - A)^{-n}\| \leq M(\mathrm{Re}\, z - \beta)^{-n}.$$

Proof. The proof reduces to Lemmas 3.6 and 3.7 below. □

Lemma 3.6. For any $\mathrm{Re}\, z > \beta$ the improper integral

$$R_z := \int_0^\infty \mathrm{e}^{-zt} U(t)\, \mathrm{d}t$$

converges srongly in $\mathcal{B}(X)$, and it satisfies

$$\operatorname*{s-lim}_{\mathbb{R}\ni\lambda\to\infty} \lambda R_\lambda = 1, \quad R_z = (z - A)^{-1}.$$

In particular, A is a densely defined closed operator on X, and

$$\sigma(A) \subset \{z \in \mathbb{C};\ \mathrm{Re}\, z \leq \beta\}, \quad \{z \in \mathbb{C};\ \mathrm{Re}\, z > \beta\} \subset \rho(A).$$

Lemma 3.7. For any $z \in \mathbb{C}$ with $\mathrm{Re}\, z > \beta$ and any $n \in \mathbb{N}$

$$\|(z - A)^{-n}\| = M(\mathrm{Re}\, z - \beta)^{-n}.$$

Proof of Lemma 3.6. Step 1. Let $u \in X$ and $\mathrm{Re}\, z > \beta$. The mapping

$$[0,\infty) \to X, \quad t \mapsto \mathrm{e}^{-zt} U(t) u$$

is continuous, and satisfies

$$\|\mathrm{e}^{-zt} U(t) u\| \leq M\mathrm{e}^{-(\mathrm{Re}\, z - \beta)t} \|u\|.$$

Therefore the improper integral

$$\int_0^\infty \mathrm{e}^{-zt} U(t) u\, \mathrm{d}t$$

converges absolutely in X, which in turn implies the strong convergence of the improper integral R_z.

Step 2. Let $u \in X$. For any $\lambda > \max\{2\beta, 0\}$ and $K \geq 0$ let us decompose

$$\begin{aligned}\|\lambda R_\lambda u - u\| &= \left\| \int_0^\infty \mathrm{e}^{-t} \big[U(t/\lambda) u - u \big]\, \mathrm{d}t \right\| \\ &\leq \int_0^K \big\| U(t/\lambda) u - u \big\|\, \mathrm{d}t + \int_K^\infty \Big(M\mathrm{e}^{-t/2} + \mathrm{e}^{-t} \Big) \|u\|\, \mathrm{d}t.\end{aligned}$$

Now for any $\epsilon > 0$ we can find $K \geq 0$ such that

$$\int_K^\infty \Big(M\mathrm{e}^{-t/2} + \mathrm{e}^{-t} \Big) \|u\|\, \mathrm{d}t < \epsilon.$$

We then let $\lambda \to \infty$, and obtain

$$\limsup_{\mathbb{R}\ni\lambda\to\infty} \|\lambda R_\lambda u - u\| < \epsilon.$$

Hence

$$\operatorname*{s-lim}_{\mathbb{R}\ni\lambda\to\infty} \lambda R_\lambda = 1.$$

Step 3. Let $u \in X$ and $\mathrm{Re}\, z > \beta$. For any $h > 0$ we can compute

$$\begin{aligned} U(h)R_z u &= U(h)\int_0^\infty \mathrm{e}^{-zt}U(t)u\,\mathrm{d}t \\ &= \mathrm{e}^{zh}\int_h^\infty \mathrm{e}^{-zs}U(s)u\,\mathrm{d}s \\ &= \mathrm{e}^{zh}R_z u - \mathrm{e}^{zh}\int_0^h \mathrm{e}^{-zs}U(s)u\,\mathrm{d}s, \end{aligned}$$

and this implies

$$h^{-1}\big(U(h)-1\big)R_z u = h^{-1}\big(\mathrm{e}^{zh}-1\big)R_z u - \mathrm{e}^{zh}h^{-1}\int_0^h \mathrm{e}^{-zs}U(s)u\,\mathrm{d}s.$$

Now by letting $h \to +0$ we obtain

$$R_z u \in D(A), \quad AR_z u = zR_z u - u.$$

Particularly with Step 2, A is densely defined operator on X, and

$$(z-A)R_z = 1.$$

Next let $u \in D(A)$. Write

$$h^{-1}U(t)\big(U(h)-1\big)u = h^{-1}\big(U(h)-1\big)U(t)u,$$

multiply e^{-zt}, and integrate it in $t \in [0,\infty)$. Then we have

$$h^{-1}R_z\big(U(h)-1\big)u = h^{-1}\big(U(h)-1\big)R_z u,$$

so that by letting $h \to +0$

$$R_z A u = AR_z u.$$

This and the result above imply

$$R_z(z-A) = (z-A)R_z|_{D(A)} = \mathrm{id}_{D(A)}.$$

Hence

$$R_z = (z-A)^{-1}.$$

Since R_z is bounded, we have $z \in \rho(A)$. In addition, since R_z is closed, so is its inverse $z - A$. Thus A is closed as well. □

Proof of Lemma 3.7. Let $u \in X$. By Lemma 3.7 for any $z \in \mathbb{C}$ with $\mathrm{Re}\, z > \beta$ we have

$$(z-A)^{-1}u = \int_0^\infty \mathrm{e}^{-zt}U(t)u\,\mathrm{d}t.$$

We differentiate both sides $(n-1)$-times in z, to obtain

$$(-1)^{n-1}(n-1)!(z-A)^{-n}u = \int_0^\infty (-t)^{n-1}\mathrm{e}^{-zt}U(t)u\,\mathrm{d}t.$$

Then it follows that

$$\begin{aligned} \|(z-A)^{-n}u\| &= \frac{1}{(n-1)!}\int_0^\infty t^{n-1}|\mathrm{e}^{-zt}|\|U(t)u\|\,\mathrm{d}t \\ &\le \frac{M}{(n-1)!}\|u\|\int_0^\infty t^{n-1}\mathrm{e}^{-(\mathrm{Re}\, z-\beta)t}\,\mathrm{d}t \\ &= M(\mathrm{Re}\, z-\beta)^{-n}\|u\|, \end{aligned}$$

which implies the assertion. □

Proposition 3.8. Let $U\colon [0,\infty) \to \mathcal{B}(X)$ be a C_0-semigroup with generator A.

1. Let $u \in D(A)$. Then for any $t \ge 0$ one has

   $$U(t)u \in D(A).$$

   Moreover, $U(\cdot)u \in C^1((0,\infty);X)$, and

   $$\frac{\mathrm{d}}{\mathrm{d}t}(U(t)u) = AU(t)u = U(t)Au.$$

2. If $V\colon [0,\infty) \to \mathcal{B}(X)$ is a C_0-semigroup with the same generator A, then

   $$U \equiv V.$$

Proof. 1. Let $u \in D(A)$. For any $t \geq 0$ write

$$h^{-1}\Big(U(h)-1\Big)U(t)u = h^{-1}U(t)\Big(U(h)-1\Big)u,$$

and let $h \to +0$. Then the above right-hand side converge to $U(t)Au$, from which it follows that

$$U(t)u \in D(A).$$

It also follows that we have the right derivative

$$\lim_{h\to+0} h^{-1}\Big(U(t+h)u - U(t)u\Big) = AU(t)u = U(t)Au.$$

To examine the left derivative let $t > 0$. Then for small $h > 0$

$$\begin{aligned}&\Big\|(-h)^{-1}\Big(U(t-h)u - U(t)u\Big) - U(t)Au\Big\| \\ &\leq \|U(t-h)\|\Big\|h^{-1}\Big(U(h)u - u\Big) - U(h)Au\Big\|.\end{aligned}$$

Recalling $\|U(t-h)\| \leq M\mathrm{e}^{\beta(t-h)}$, we obtain

$$\lim_{h\to+0}(-h)^{-1}\Big(U(t-h)u - U(t)u\Big) = AU(t)u = U(t)Au.$$

Hence the assertion 1 is verified.

2. Let $u \in D(A)$ and $T > 0$. Then for any $t \in [0, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(U(T-t)V(t)u\Big) = -U(T-t)AV(t)u + U(T-t)AV(t)u = 0,$$

and therefore

$$U(T-t)V(t)u = U(T)u = V(T)u.$$

This certainly implies $U \equiv V$. □

Problem. Let A be a generator of a C_0-semigroup on X. Show that e^{tA} extends analytically in $t \in \mathbb{C}$ if and only if $A \in \mathcal{B}(X)$.

§ 3.3 Hille–Yosida Theorem

Let X be a Banach space

Theorem 3.9 (Hille–Yosida). A linear operator A on X is a generator of a C_0-semigroup $U\colon [0,\infty) \to \mathcal{B}(X)$ with constants $M \geq 1$ and $\beta \in \mathbb{R}$ such that for any $t \geq 0$

$$\|U(t)\| \leq M\mathrm{e}^{\beta t}$$

if and only if both of the following hold:

1. A is closed and densely defined on X;
2. One has $(\beta, \infty) \subset \rho(A)$, and for any $\lambda > \beta$ and $n \in \mathbb{N}$

$$\|(\lambda - A)^{-n}\| \leq M(\lambda - \beta)^{-n}.$$

Theorem 3.9 (continued). In addition, in the affirmative case, one has

$$\{z \in \mathbb{C};\ \mathrm{Re}\, z > \beta\} \subset \rho(A),$$

and for any $\mathrm{Re}\, z > \beta$ and $n \in \mathbb{N}$

$$\|(z - A)^{-n}\| \leq M(\mathrm{Re}\, z - \beta)^{-n}.$$

Remarks.
1. Theorem 3.9 was proved by E. Hille and K. Yosida independently at almost the same time. Their proofs are different from each other, and we shall present both of them.
2. The necessity and the last part of the assertion is already done in Proposition 3.5. We will prove only the sufficiency.

Yosida's idea. If A were bounded, we could apply Theorem 3.1 to construct the C_0-semigroup U with generator A. In fact, it would suffice to set

$$U(t) = \mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

However, this construction fails when A is unbounded. Yosida's idea is to approximate A by a squence $(A_\lambda)_{\lambda>0}$ of bounded operators defined as

$$A_\lambda = AJ_\lambda \in \mathcal{B}(X); \quad J_\lambda = \lambda(\lambda - A)^{-1}, \quad \operatorname*{s-lim}_{\lambda\to\infty} J_\lambda = 1.$$

Then we could construct the desired C_0-semigroup as

$$U(t) = \mathrm{e}^{tA} = \operatorname*{s-lim}_{\lambda\to\infty} \mathrm{e}^{tA_\lambda}.$$

The operator A_λ is called the **Yosida approximation** to A.

Yosida's proof. It suffices to prove the sufficiency.

Step 1. For $\lambda > \beta$ we set

$$J_\lambda = \lambda(\lambda - A)^{-1}, \quad A_\lambda = AJ_\lambda = \lambda J_\lambda - \lambda \in \mathcal{B}(X).$$

Here we prove that for any $u \in D(A)$

$$\lim_{\lambda\to\infty} A_\lambda u = Au.$$

In fact, for any $u \in D(A)$ we have

$$A_\lambda u = AJ_\lambda u = \lambda J_\lambda u - \lambda u = J_\lambda A u,$$

and thus it suffices to show

$$\operatorname*{s-lim}_{\lambda\to\infty} J_\lambda = 1.$$

To prove it let $v \in X$. For any $\epsilon > 0$ take $w \in D(A)$ such that

$$\|v - w\| < \epsilon.$$

Then, as $\lambda \to \infty$,

$$\begin{aligned}\|J_\lambda v - v\| &\le \|J_\lambda(v-w)\| + \|J_\lambda w - w\| + \|w - v\| \\ &\le \lambda\|(\lambda - A)^{-1}(v-w)\| + \|(\lambda - A)^{-1}Aw\| + \epsilon \\ &\le \lambda M(\lambda-\beta)^{-1}\epsilon + (\lambda-\beta)^{-1}\|Aw\| + \epsilon \\ &\to M\epsilon + \epsilon,\end{aligned}$$

hence

$$J_\lambda v \to v.$$

The claim is verified.

Step 2. Next we prove for any $\lambda > \beta$ and $t \ge 0$

$$\|\mathrm{e}^{tA_\lambda}\| \le M \exp\left(\frac{\lambda\beta}{\lambda-\beta} t\right),$$

but this is straightforward. In fact, noting that

$$A_\lambda + \lambda = \lambda^2(\lambda - A)^{-1},$$

we can bound the operator norm by the assumptions as

$$\begin{aligned}\|\mathrm{e}^{tA_\lambda}\| = \mathrm{e}^{-t\lambda}\|\mathrm{e}^{t(A_\lambda+\lambda)}\| &\le \mathrm{e}^{-t\lambda}\sum_{n=0}^{\infty}\frac{t^n}{n!}\|(A_\lambda+\lambda)^n\| \\ &\le M\mathrm{e}^{-t\lambda}\sum_{n=0}^{\infty}\frac{(t\lambda^2)^n}{n!}(\lambda-\beta)^{-n} \\ &= M\exp\left(\frac{\lambda\beta}{\lambda-\beta}t\right).\end{aligned}$$

Step 3. Now we prove there exists a strong limit

$$U(t) := \operatorname*{s\text{-}lim}_{\lambda\to\infty} e^{tA_\lambda} \in \mathcal{B}(X)$$

locally uniformly in $t \geq 0$. First let $u \in D(A)$. By the fundamental theorem of calculus for any $\lambda > \mu > \beta$ and $t \geq 0$

$$\begin{aligned}\left\| e^{tA_\lambda}u - e^{tA_\mu}u \right\| &= \left\| \int_0^t e^{(t-s)A_\mu} e^{sA_\lambda}(A_\lambda - A_\mu)u \, ds \right\| \\ &\leq \|A_\lambda u - A_\mu u\| \int_0^t \|e^{(t-s)A_\mu}\| \|e^{sA_\lambda}\| \, ds.\end{aligned}$$

Let us show that the the last integral is bounded locally uniformly in $t \geq 0$. After some computations employing Step 2 we obtain

$$\begin{aligned}&\int_0^t \|e^{(t-s)A_\mu}\| \|e^{sA_\lambda}\| \, ds \\ &\leq M^2 \frac{(\lambda-\beta)(\mu-\beta)}{(\lambda-\mu)\beta^2} \left[\exp\left(\frac{\mu\beta}{\mu-\beta} t \right) - \exp\left(\frac{\lambda\beta}{\lambda-\beta} t \right) \right].\end{aligned}$$

Then by the mean value theorem there exists $\theta = \theta_{\lambda,\mu,t} \in (0,1)$ such that

$$\int_0^t \|e^{(t-s)A_\mu}\| \|e^{sA_\lambda}\| \, ds \leq tM^2 \exp\left[(1-\theta) \frac{\mu\beta}{\mu-\beta} t + \theta \frac{\lambda\beta}{\lambda-\beta} t \right].$$

From the above estimates it follows that $(e^{tA_\lambda}u)_{\lambda>\beta}$ is a Cauchy sequence on X locally uniformly in $t \geq 0$, and hence has a limit locally uniformly in $t \geq 0$ as $\lambda \to \infty$.

Problem. Let $u \in X$. Show by using the denseness of $D(A) \subset X$ that there exists the limit

$$\lim_{\lambda\to\infty} e^{tA_\lambda}u$$

locally uniformly in $t \geq 0$.

Thus the claim is done.

Step 4. Here we prove that U is a C_0-semigroup on X satisfying for any $t \geq 0$

$$\|U(t)\| \leq Me^{\beta t}. \tag{♣}$$

By definition we can immediately see $U(0) = 1$. Let $t, s \geq 0$. Then by Steps 2 and 3 for any $u \in X$

$$\begin{aligned}\left\| U(t+s)u - U(t)U(s)u \right\| &= \lim_{\lambda\to\infty} \left\| e^{(t+s)A_\lambda}u - e^{tA_\lambda}U(s)u \right\| \\ &\leq \lim_{\lambda\to\infty} M \exp\left(\frac{\lambda\beta}{\lambda-\beta} t \right) \left\| e^{sA_\lambda}u - U(s)u \right\| \\ &= 0,\end{aligned}$$

so that

$$U(t+s) = U(t)U(s).$$

Hence U is certainly a one-parameter semigroup on X. In addition, since the strong limit in Step 3 is locally uniform in $t \geq 0$, U is a C_0-semigroup. The estimate (♣) follows from Step 2.

Step 5. Lastly we prove the generator of U, denoted by B, coincides with A. For any $u \in D(A)$, $\lambda > \beta$ and $t \geq 0$ by the fundamental theorem of calculus

$$e^{tA_\lambda}u - u = \int_0^t e^{sA_\lambda} A_\lambda u \, ds,$$

so that by taking a limit as $\lambda \to \infty$

$$U(t)u - u = \int_0^t U(s)Au \, ds.$$

Therefore by the fundamental theorem of calculus again

$$\lim_{t\to+0} t^{-1}\big(U(t)u - u\big) = t^{-1} \int_0^t U(s)Au \, ds = Au.$$

This implies $A \subset B$. However, note that for any $\lambda > \beta$ both $A - \lambda$ and $B - \lambda$ are injective, and

$$X = \mathrm{Ran}(A-\lambda) \subset \mathrm{Ran}(B-\lambda).$$

Then it follows that $A - \lambda = B - \lambda$, or $A = B$. □

Hille's idea. Let us **discretize** the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = AU(t), \quad U(0) = 1,$$

replacing the differentiation in t by the **backward difference** of step size $h > 0$. Then we have

$$h^{-1}(U_n - U_{n-1}) = AU_n, \quad U_0 = 1,$$

which in fact has an explicit solution: $U_n = (1 - hA)^{-n}$. In the **continuum limit** as $h \to +0$ and $n \to \infty$ with $nh \to t$ we expect

$$U_n = (1 - hA)^{-n} \to U(t).$$

Now, letting $h = t/n$, we adopt

$$U_n(t) = \Big(1 - \frac{t}{n}A\Big)^{-n}$$

as an approximation of the desired C_0-semigroup as $n \to \infty$.

Hille's proof. It suffices to prove the sufficiency.

Step 1. We first define the approximate operator $U_n(t)$, and state its basic properties. For any $n \in \mathbb{N}$ we let

$$T_n = \begin{cases} n/\beta & \text{if } \beta > 0, \\ \infty & \text{if } \beta \le 0, \end{cases}$$

and define for $n \in \mathbb{N}$ and $t \in [0, T_n)$

$$U_n(t) = \Big(1 - \frac{t}{n}A\Big)^{-n} \in \mathcal{B}(X).$$

For $t \neq 0$ we may write it also as

$$U_n(t) = \Big(\frac{n}{t}\Big)^n \Big(\frac{n}{t} - A\Big)^{-n}.$$

Now it is straightforward from the assumptions that

$$\|U_n(t)\| \le M\Big(1 - \frac{\beta t}{n}\Big)^{-n}. \qquad (\spadesuit)$$

In addition, for any $u \in D(A)$ the vector-valued function $U_n(\cdot)u$ is differentiable on $[0, T_n)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}(U_n(t)u) = U_n(t)\Big(1 - \frac{t}{n}A\Big)^{-1}Au. \qquad (\heartsuit)$$

Here we omit a verification of ($\heartsuit$).

Problem. Verify the claimed identity ($\heartsuit$) based on the definition of differentiation. (Except at $t = 0$ we may verify it by the holomorphy of resolvent as well.)

Step 2. Here we prove existence of a strong limit

$$U(t) := \operatorname*{s-lim}_{n\to\infty} U_n(t)$$

locally uniformly in $t \ge 0$. First let $u \in D(A)$. For any $T > 0$ take $n \ge m$ large enough that $T_n \ge T_m > T$. Then for any $t \in [0, T]$ by the fundamental theorem of calculus and Step 1

$$\begin{aligned} &\Big\|U_n(t)u - U_m(t)u\Big\| \\ &= \left\|\int_0^t U_m(t-s)U_n(s)\left[\Big(1 - \frac{s}{n}A\Big)^{-1}Au - \Big(1 - \frac{t-s}{m}A\Big)^{-1}Au\right]\mathrm{d}s\right\| \\ &\le M^2\Big(1 - \frac{\beta T}{m}\Big)^{-m}\Big(1 - \frac{\beta T}{n}\Big)^{-n} \\ &\quad \cdot \int_0^T \left\|\Big(1 - \frac{s}{n}A\Big)^{-1}Au - \Big(1 - \frac{t-s}{m}A\Big)^{-1}Au\right\|\mathrm{d}s. \end{aligned}$$

Here, similarly to Step 1 of Yosida's proof, we have

$$\lim_{n\to\infty}\Big(1-\frac{t}{n}A\Big)^{-1}Au \to Au$$

uniformly in $t \in [0, T]$. Therefore by the above aruguments $(U_n(t)u)_{n\in\mathbb{N}}$ is a Cauchy sequence on X uniformly in $t \in [0, T]$, hence has a uniform limit in $t \in [0, T]$ as $n \to \infty$.

For general $u \in X$ we can argue similarly to Step 3 of Yosida's proof, using the denseness of $D(A) \subset X$ and the bound (♠) from Step 1. Thus the claimed strong limit exists.

Step 3. Here we prove U from Step 2 is a C_0-semigroup on X satisfying for any $t \geq 0$

$$\|U(t)\| \leq M e^{\beta t}.$$

Obviously we have $U(0) = 1$. For any $t, s \geq 0$ let $n \in \mathbb{N}$ be sufficiently large. Then for any $u \in D(A)$ by the fundamental theorem of calculus

$$U_n(t+s)u - U_n(t)U_n(s)u$$
$$= \int_0^s U_n(t+r)U_n(s-r)\Big[\Big(1-\frac{t+r}{n}A\Big)^{-1} - \Big(1-\frac{s-r}{n}A\Big)^{-1}\Big]Au\,\mathrm{d}r.$$

Letting $n \to \infty$, we obtain for any $u \in D(A)$

$$U(t+s)u = U(t)U(s)u.$$

By the denseness of $D(A) \subset X$ it follows that U is certainly a one-parameter semigroup on X.

Since the strong limit in Step 2 is locally uniform in $t \geq 0$, the one-parameter semigroup U is strongly continuous, and hence is a C_0-semigroup on X.

By (♠) from Step 1 we obtain the claimed estimate for $U(t)$.

Step 4. Lastly we prove the generator of U, denoted here by B, coincides with A. For any $u \in D(A)$ by the fundamental theorem of calculus and (♡) from Step 1

$$U_n(t)u - u = \int_0^t U_n(s)\Big(1-\frac{s}{n}A\Big)^{-1}Au\,\mathrm{d}s,$$

so that by letting $n \to \infty$

$$U(t)u - u = \int_0^t U(s)Au\,\mathrm{d}s.$$

This implies by the fundamental theorem of calculus again

$$\lim_{t\to+0} t^{-1}\big(U(t)u - u\big) = t^{-1}\int_0^t U(s)Au\,\mathrm{d}s = Au.$$

Thus we have $A \subset B$. Now, repeating the same argument as in Step 5 of Yosida's proof, we obtain $A = B$. We are done. □

Corollary 3.10. Let A be a generator of a C_0-semigroup on X. Then e^{tA} for any $t \geq 0$ has the expressions

$$\mathrm{e}^{tA} = \operatorname*{s-lim}_{\lambda\to\infty} \exp\Big(t\lambda A(\lambda - A)^{-1}\Big)$$

and

$$\mathrm{e}^{tA} = \operatorname*{s-lim}_{n\to\infty} \Big(1 - \frac{t}{n}A\Big)^{-n}$$

Proof. These expressions are straightforward from the proofs of Theorem 3.9. □

Corollary 3.11. Let A be a generator of a C_0-semigroup on X, and let $u_0 \in D(A)$. Then an abstract evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) \quad \text{for } t > 0, \qquad u(0) = u_0 \qquad (\heartsuit)$$

has a unique solution in

$$\Big\{u \in C([0,\infty); X) \cap C^1((0,\infty); X);\ \forall t > 0\ u(t) \in D(A)\Big\}, \qquad (\diamondsuit)$$

which is given by

$$u(t) = \mathrm{e}^{tA}u_0.$$

Remark. Sometimes, even for general $u_0 \in X$, the vector-valued function

$$u(t) = \mathrm{e}^{tA}u_0$$

is called a solution to (♡), though it is not differentiable in $t > 0$.

Proof. By Proposition 3.8 $u(t) = \mathrm{e}^{tA}u_0$ certainly solves (♡). On the other hand, let v be a solution to (♡) belonging to (◇). Then for any $T > 0$ and any $t \in (0, T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{(T-t)A}v(t)\Big) = -\mathrm{e}^{(T-t)A}Av(t) + \mathrm{e}^{(T-t)A}Av(t) = 0.$$

Hence by continuity of v at $t = 0, T$ we obtain

$$\mathrm{e}^{(T-t)A}v(t) = \mathrm{e}^{TA}u_0 = v(T).$$

This implies $v(t) = \mathrm{e}^{tA}u_0$ for any $t \geq 0$. We are done. □

§ 3.4 Analytic Semigroups

Let X be a Banach space. In this course we denote for any $\theta > 0$

$$\mathbb{C}_\theta = \{z \in \mathbb{C} \setminus \{0\};\ |\arg z| < \theta\}.$$

◦ **Analytic semigroup on closed sector**

Definition. An operator-valued function $U\colon \overline{\mathbb{C}}_\theta \to \mathcal{B}(X)$ with $\theta \in (0, \pi/2]$ is called an **analytic semigroup** on X (defined on $\overline{\mathbb{C}}_\theta$) if

1. $U(0) = 1$;
2. For any $z, w \in \overline{\mathbb{C}}_\theta$ one has $U(z + w) = U(z)U(w)$;
3. U is strongly continuous on $\overline{\mathbb{C}}_\theta$, and analytic on $\mathbb{C}_\theta$.

Definition. A **generator** of an analytic semigroup $U\colon \overline{\mathbb{C}}_\theta \to \mathcal{B}(X)$ with $\theta \in (0, \pi/2]$ is the generator of the C_0-semigroup $U|_{[0,\infty)}$. If A is the generator of U, we say A **generates** U, and denote

$$U(z) = \mathrm{e}^{zA} \quad \text{for } z \in \overline{\mathbb{C}}_\theta.$$

Remark. Due to the analytic continuation U is uniquely determined by its restriction $U|_{[0,\infty)}$, which in turn is uniquely determined by the generator A. Therefore the last notation is well-defined.

Problem. Let A be a generator of an analytic semigroup on X. Show that, if e^{zA} extends analytically in $z \in \mathbb{C}_\theta$ for some $\theta > \pi/2$, then e^{zA} extends entirely in $z \in \mathbb{C}$, and in particular $A \in \mathcal{B}(X)$.

Proposition 3.12. Let A be a generator of an analytic semigroup defined on $\overline{\mathbb{C}}_\theta$ with $\theta \in (0, \pi/2]$.

1. For any $u \in X$, $z \in \mathbb{C}_\theta$ and $n \in \mathbb{N}$ one has $\mathrm{e}^{zA}u \in D(A^n)$, and
$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\mathrm{e}^{zA}u) = A^n \mathrm{e}^{zA}u.$$
2. Let $u_0 \in X$. Then an abstract evolution equation
$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) \quad \text{for } t > 0, \qquad u(0) = u_0 \qquad (\heartsuit)$$
has a unique solution in
$$\left\{u \in C([0,\infty); X) \cap C^1((0,\infty); X);\ \forall t > 0\ u(t) \in D(A)\right\},$$
which is given by
$$u(t) = \mathrm{e}^{tA}u_0.$$

Remark. See also Proposition 3.8 and Corollary 3.11. The above assertions hold for all $u, u_0 \in X$.

Proof. 1. Since e^{zA} is analytic in $z \in \mathbb{C}_\theta$, $\mathrm{e}^{zA}u$ for any $u \in X$ is infinitely differentialble in $z \in \mathbb{C}_\theta$. Then it is straightforward to see $\mathrm{e}^{zA}u \in D(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{zA}) = \lim_{h \to 0} h^{-1}(\mathrm{e}^{hA} - 1)\mathrm{e}^{zA}u = A\mathrm{e}^{zA}u.$$

We can discuss the higher derivatives similarly, noting that A and e^{hA} commute. The detail is omitted.

2. The proof is almost the same as that of Corollary 3.11, and is omitted. □

Proposition 3.13. Let $U\colon \overline{\mathbb{C}}_\theta \to \mathcal{B}(X)$, $\theta \in (0, \pi/2]$, be an analytic semigroup. Then there exist $M \geq 1$ and $\beta \in \mathbb{R}$ such that for any $z \in \overline{\mathbb{C}}_\theta$

$$\|U(z)\| \leq M\mathrm{e}^{\beta|z|}.$$

Proof. Fix any $T > 0$. For any $u \in X$, since $U(\cdot)u$ is continuous,

$$\sup_{|z|\le T} \|U(z)u\| < \infty,$$

so that by the uniform boundedness principle we can find $M \ge 1$ and $\beta \ge 0$ such that

$$M = \mathrm{e}^{\beta T} = \sup_{|z|\le T} \|U(z)\| \in [1, \infty).$$

Now for any $z \in \overline{\mathbb{C}}_\theta \setminus \{0\}$, choosing $k \in \mathbb{N}_0$ such that $kT \le |z| < (k+1)T$, we obtain

$$\|U(z)\| \le \Big\|U\Big(z - kzT/|z|\Big)\Big\| \Big\|U\Big(zT/|z|\Big)\Big\|^k \le M\mathrm{e}^{\beta kT} \le M\mathrm{e}^{\beta|z|}.$$

Hence we are done. □

Theorem 3.14. A linear operator A on X is a generator of an analytic semigroup $U\colon \overline{\mathbb{C}}_\theta \to \mathcal{B}(X)$, $\theta \in (0, \pi/2]$, with constants $M \ge 1$ and $\beta \in \mathbb{R}$ such that for any $z \in \overline{\mathbb{C}}_\theta$

$$\|U(z)\| \le M\mathrm{e}^{\beta|z|}$$

if and only if both of the following hold:

1. A is closed and densely defined on X;
2. One has
$$\Big\{\mathrm{e}^{\mathrm{i}\omega}\lambda \in \overline{\mathbb{C}}_\theta;\ \lambda > \beta,\ |\omega| \le \theta\Big\} \subset \rho(A),$$
and for any $\lambda > \beta$, $|\omega| \le \theta$ and $n \in \mathbb{N}$
$$\|(\mathrm{e}^{\mathrm{i}\omega}\lambda - A)^{-n}\| \le M(\lambda - \beta)^{-n}.$$

Theorem 3.14 (continued). In addition, in the affirmative case, one has

$$\Big\{\mathrm{e}^{\mathrm{i}\omega}z \in \mathbb{C};\ \mathrm{Re}\, z > \beta,\ |\omega| \le \theta\Big\} \subset \rho(A),$$

and for any $\mathrm{Re}\, z > \beta$, $|\omega| \le \theta$ and $n \in \mathbb{N}$

$$\|(\mathrm{e}^{\mathrm{i}\omega}z - A)^{-n}\| \le M(\mathrm{Re}\, z - \beta)^{-n}.$$

Proof. Necessity. Let A be a generator of an analytic semigroup U with constants θ, M, β as in the assertion. For any $|\omega| \le \theta$ we let A_ω be a generator of a C_0-semigroup U_ω defined as

$$U_\omega(t) = U(\mathrm{e}^{-\mathrm{i}\omega}t) \quad \text{for } t \ge 0.$$

Then by the Hille–Yosida theorem A_ω is a densely defined closed operator on X with $(\beta, \infty) \subset \rho(A_\omega)$, and for any $\lambda > \beta$ and $n \in \mathbb{N}$

$$\|(\lambda - A_\omega)^{-n}\| \le M(\lambda - \beta)^{-n}.$$

Therefore it suffices to show that $A_\omega = \mathrm{e}^{-\mathrm{i}\omega}A$, from which we remark also the last assertion follows.

For that first let $u \in D(A)$. Since $U(\cdot)u$ is analytic on $\mathbb{C}_\theta$, we have for any $z \in \mathbb{C}_\theta$

$$U(z)u \in D(A) \cap D(A_\omega), \quad A_\omega U(z)u = \mathrm{e}^{-\mathrm{i}\omega}AU(z)u.$$

By $u \in D(A)$ it follows $AU(z)u = U(z)Au$, so that

$$A_\omega U(z)u = U(z)\mathrm{e}^{-\mathrm{i}\omega}Au.$$

Now let $z \to 0$. Then, since A_ω is closed, we have $u \in D(A_\omega)$ and $A_\omega u = \mathrm{e}^{-\mathrm{i}\omega}Au$, or $\mathrm{e}^{-\mathrm{i}\omega}A \subset A_\omega$. The converse is proved similarly.

Sufficeincy. Next assume conditions 1 and 2 of the assertion, and set for any $|\omega| \le \theta$

$$A_\omega = \mathrm{e}^{-\mathrm{i}\omega} A.$$

This A_ω satisfies the conditions of the Hille–Yosida theorem, and thus the strong limit

$$\mathrm{e}^{tA_\omega} = \operatorname*{s\text{-}lim}_{n\to\infty} \Big(1 - \frac{t}{n} A_\omega\Big)^{-n} \quad \text{for } t \ge 0$$

exists, and it gives a C_0-semigroup. Now we set

$$U(z) = \operatorname*{s\text{-}lim}_{n\to\infty} \Big(1 - \frac{z}{n} A\Big)^{-n} \quad \text{for } z = \mathrm{e}^{-\mathrm{i}\omega} t \in \overline{\mathbb{C}}_\theta.$$

For each $n \in \mathbb{N}$ the operator on the right-hand side above is analytic (where it is defined). In addition, by repeating the agrguments of Hille's proof the above strong limit is locally uniform in $z \in \overline{\mathbb{C}}_\theta$. Thus it follows that U is strongly continuous on $\overline{\mathbb{C}}_\theta$, and analytic on $\mathbb{C}_\theta$.

Moreover, for any $t, s \ge 0$ we have

$$U(t+s) = U(t)U(s),$$

and hence by the identity theorem for any $z, w \in \overline{\mathbb{C}}_\theta$

$$U(z+w) = U(z)U(w).$$

Thus U is an analytic semigroup on X, and by the construction its generator coincides with A. □

Corollary 3.15. Let A be a generator of an analytic semigroup on X defined on $\overline{\mathbb{C}}_\theta$, $\theta \in (0, \pi/2]$. Then e^{zA} for any $z \in \overline{\mathbb{C}}_\theta$ has expressions

$$\mathrm{e}^{zA} = \operatorname*{s\text{-}lim}_{z\lambda\in\mathbb{R},\, \lambda\to\infty} \exp\Big(z\lambda A(\lambda - A)^{-1}\Big)$$

and

$$\mathrm{e}^{zA} = \operatorname*{s\text{-}lim}_{n\to\infty} \Big(1 - \frac{z}{n} A\Big)^{-n}.$$

Proof. From the proof of Theorem 3.14 we obtain

$$\mathrm{e}^{zA} = \exp\Big[t(\mathrm{e}^{-\mathrm{i}\omega} A)\Big] \quad \text{for } z = \mathrm{e}^{-\mathrm{i}\omega} t \in \overline{\mathbb{C}}_\theta.$$

Hence the asserted expression follows by Corollary 3.10. □

Remark. There is yet another expression for an analytic semigroup, see Theorem 3.16 below.

∘ Analytic semigroup on open sector

Definition. An operator-valued function $U\colon \mathbb{C}_\theta \to \mathcal{B}(X)$ with $\theta \in (0, \pi/2]$ is called an **analytic semigroup** on X (defined on $\mathbb{C}_\theta$) if

1. For any $\omega \in (0, \theta)$ one has

$$\operatorname*{s\text{-}lim}_{\mathbb{C}_\omega \ni z \to 0} U(z) = 1;$$

2. For any $z, w \in \mathbb{C}_\theta$ one has $U(z+w) = U(z)U(w)$;
3. U is analytic on $\mathbb{C}_\theta$.

Remarks. 1. In this course we shall always extend an analytic semigroup U defined on an open sector to $z = 0$ as $U(0) = 1$.

2. Usually analytic semigroups on open and closed sectors are not really distinguished. (Essentially the distinction is not really needed, either.) It is only in this course.

Definition. A **generator** of an analytic semigroup $U : \mathbb{C}_\theta \to \mathcal{B}(X)$ with $\theta \in (0, \pi/2]$ is the generator of the C_0-semigroup $U|_{[0,\infty)}$. If A is the generator of U, we say A **generates** U, and denote

$$U(z) = \mathrm{e}^{zA} \quad \text{for } z \in \mathbb{C}_\theta.$$

Theorem 3.16. A linear operator A on X is a generator of an analytic semigroup defined on $\mathbb{C}_\theta$ with $\theta \in (0, \pi/2]$ if and only if

1. A is closed and densely defined on X;
2. For any $\omega \in (0, \theta)$ there exist $R_\omega, M_\omega > 0$ such that
$$\left\{ z \in \overline{\mathbb{C}}_{\pi/2+\omega};\ |z| \geq R_\omega \right\} \subset \rho(A),$$
and for any $z \in \overline{\mathbb{C}}_{\pi/2+\omega}$ with $|z| \geq R_\omega$
$$\|(z - A)^{-1}\| \leq M_\omega |z|^{-1}.$$

In addition, in the affirmative case, e^{zA} for any $z \in \mathbb{C}_\theta$ has an integral expression

$$\mathrm{e}^{zA} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta} (\zeta - A)^{-1} \, \mathrm{d}\zeta. \qquad (\spadesuit)$$

Theorem 3.16 (continued). Here for any $\omega \in (|\arg z|, \theta)$ a piecewise C^1 path $\Gamma = \{\zeta(t) \in \mathbb{C};\ t \in \mathbb{R}\}$ is chosen such that

$$\Gamma \subset \left\{ z \in \overline{\mathbb{C}}_{\pi/2+\omega};\ |z| \geq R_\omega \right\},$$

and further that there exists $T > 0$ such that for any $|t| \geq T$

$$\zeta(t) = |t| \mathrm{e}^{\pm \mathrm{i}(\pi/2+\omega)}.$$

Proof. Necessity. Let A be a generator of an analytic semigroup U defined on $\mathbb{C}_\theta$ with $\theta \in (0, \pi/2]$, and take any $\omega \in (0, \theta)$. Then for any $\tau \in (\omega, \theta)$ the restriction $U|_{\overline{\mathbb{C}}_\tau}$ is an analytic semigroup, and thus the necessity follows immediately from Theorem 3.14.

Sufficiency. Suppose A satisfies conditions 1 and 2 of the assertion. In the following we are going to show that the integral on the right-hand side of ($\spadesuit$) provides an analytic semigroup defined on $\mathbb{C}_\theta$, and that its generator coincides with A.

Step 1. Fix $z \in \mathbb{C}_\theta$, and take any $\Gamma = \{\zeta(t) \in \mathbb{C};\ t \in \mathbb{R}\}$ with $\omega \in (|\arg z|, \theta)$ and $T > 0$ as in the assertion. We first show that

$$U_\Gamma(z) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta}(\zeta - A)^{-1}\,\mathrm{d}\zeta$$

is absolutely convergent. In fact, by condition 2

$$\begin{aligned}
&\int_\Gamma \left\| \mathrm{e}^{z\zeta}(\zeta - A)^{-1} \right\| |\mathrm{d}\zeta| \\
&\le M_\omega \Big[\int_{-\infty}^{-T} |t|^{-1} \left| \exp\left(|zt| \mathrm{e}^{\mathrm{i}(\arg z - \pi/2 - \omega)} \right) \right| \mathrm{d}t \\
&\qquad + \int_{-T}^{T} \left| \mathrm{e}^{z\zeta(t)} \right| |\zeta(t)|^{-1} |\zeta'(t)|\,\mathrm{d}t \\
&\qquad + \int_T^\infty |t|^{-1} \left| \exp\left(|zt| \mathrm{e}^{\mathrm{i}(\arg z + \pi/2 + \omega)} \right) \right| \mathrm{d}t \Big] \\
&\le C + 2M_\omega \int_T^\infty t^{-1} \mathrm{e}^{-t|z|\sin(\omega - |\arg z|)}\,\mathrm{d}t < \infty,
\end{aligned}$$

and this implies the claim.

Step 2. Fix $z \in \mathbb{C}_\theta$ again, and we prove $U_\Gamma(z)$ is independent of choice of Γ. Take any paths Γ_1 and Γ_2 as in the assertion, and let $|\arg z| < \omega_1 \le \omega_2 < \theta$ be the associated angles. By Cauchy's integral theorem we can estimate

$$\begin{aligned}
&\left\| U_{\Gamma_1}(z) - U_{\Gamma_2}(z) \right\| \\
&\le \frac{1}{2\pi} \limsup_{r \to \infty} \int_{|\zeta| = r,\, \pi/2 + \omega_1 \le |\arg \zeta| \le \pi/2 + \omega_2} \left\| \mathrm{e}^{z\zeta}(\zeta - A)^{-1} \right\| |\mathrm{d}\zeta|.
\end{aligned}$$

Then by the condition 2 we can proceed as

$$\begin{aligned}
&\left\| U_{\Gamma_1}(z) - U_{\Gamma_2}(z) \right\| \\
&\le \frac{M_\omega}{2\pi} \limsup_{r \to \infty} \left(\int_{\pi/2 + \omega_1}^{\pi/2 + \omega_2} + \int_{-\pi/2 - \omega_2}^{-\pi/2 - \omega_1} \right) \left| \exp\left(|z| r \mathrm{e}^{\mathrm{i}(\arg z + \tau)} \right) \right| \mathrm{d}\tau \\
&\le \frac{M_\omega(\omega_2 - \omega_1)}{\pi} \limsup_{r \to \infty} \mathrm{e}^{-r|z|\sin(\omega_1 - |\arg z|)} = 0.
\end{aligned}$$

Thus $U_\Gamma(z)$ is independent of Γ, and we may write it simply as

$$U(z) := U_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta}(\zeta - A)^{-1}\,\mathrm{d}\zeta.$$

Note that it also follows that $U(z)$ is analytic in $z \in \mathbb{C}_\theta$.

Step 3. Here, take any $z, w \in \mathbb{C}_\theta$, and we prove

$$U(z + w) = U(z)U(w).$$

Choose paths Γ_1 and Γ_2 as in the assertion, and let

$$|\arg z| < \omega_1 < \theta, \quad |\arg w| < \omega_2 < \theta$$

be the associated angles, respectively. We may assume that Γ_1

lies in a region to the left of Γ_2. By the resolvent identity

$$\begin{aligned}
U(z)U(w) &= \frac{1}{(2\pi \mathrm{i})^2} \int_{\Gamma_1} \mathrm{e}^{z\zeta_1}(\zeta_1 - A)^{-1} \left[\int_{\Gamma_2} \mathrm{e}^{w\zeta_2}(\zeta_2 - A)^{-1}\,\mathrm{d}\zeta_2 \right] \mathrm{d}\zeta_1 \\
&= \frac{1}{(2\pi \mathrm{i})^2} \int_{\Gamma_1} \left[\int_{\Gamma_2} \frac{\mathrm{e}^{z\zeta_1}\mathrm{e}^{w\zeta_2}}{\zeta_2 - \zeta_1} (\zeta_1 - A)^{-1}\,\mathrm{d}\zeta_2 \right] \mathrm{d}\zeta_1 \\
&\quad - \frac{1}{(2\pi \mathrm{i})^2} \int_{\Gamma_2} \left[\int_{\Gamma_1} \frac{\mathrm{e}^{z\zeta_1}\mathrm{e}^{w\zeta_2}}{\zeta_2 - \zeta_1} (\zeta_2 - A)^{-1}\,\mathrm{d}\zeta_1 \right] \mathrm{d}\zeta_2 \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \mathrm{e}^{z\zeta_1}\mathrm{e}^{w\zeta_1}(\zeta_1 - A)^{-1}\,\mathrm{d}\zeta_1 \\
&= U(z + w).
\end{aligned}$$

In the above third equality we have used the identities

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} \frac{\mathrm{e}^{w\zeta_2}}{\zeta_2 - \zeta_1}\,\mathrm{d}\zeta_2 = \mathrm{e}^{w\zeta_1}, \quad \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \frac{\mathrm{e}^{z\zeta_1}}{\zeta_2 - \zeta_1}\,\mathrm{d}\zeta_1 = 0,$$

the verification of which is left to the reader as **Problem**.

Step 4. Now in order to see that U is an analytic semigroup it remains to show that for any $\omega \in (0,\theta)$

$$\operatorname*{s-lim}_{\mathbb{C}_\omega \ni z \to 0} U(z) = 1.$$

Fix any $\omega \in (0,\theta)$. We first let $u \in D(A)$. Choose a path Γ as in the assertion with the associated angle $\omega \in (0,\theta)$. Noting that $0 \in \mathbb{C}$ is in a region to the left of Γ, we have as $\mathbb{C}_\omega \ni z \to 0$

$$\begin{aligned} U(z)u - u &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta}(\zeta - A)^{-1} u \, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta}\zeta^{-1} u \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{z\zeta}\zeta^{-1}(\zeta - A)^{-1} Au \, \mathrm{d}\zeta \\ &\to \frac{1}{2\pi \mathrm{i}} \int_\Gamma \zeta^{-1}(\zeta - A)^{-1} Au \, \mathrm{d}\zeta. \end{aligned}$$

However, the last integral vanishes. In fact, by Cauchy's integral theorem and the condition 2 of the assertion

$$\begin{aligned} &\Big\| \int_\Gamma \zeta^{-1}(\zeta - A)^{-1} Au \, \mathrm{d}\zeta \Big\| \\ &= \Big\| \lim_{r\to\infty} \int_{|\zeta| = r,\, |\arg \zeta| \le \pi/2 + \omega} \zeta^{-1}(\zeta - A)^{-1} Au \, \mathrm{d}\zeta \Big\| \\ &\le \lim_{r\to\infty} r^{-1} M_\omega(\pi + 2\omega)\|Au\| = 0. \end{aligned}$$

Hence we obtain for any $u \in D(A)$

$$\lim_{\mathbb{C}_\omega \ni z \to 0} U(z)u = u.$$

To verify the same limit for general $u \in X$, due to denseness of $D(A) \subset X$, it suffices to show that $U(z)$ is bounded uniformly in small $z \in \mathbb{C}_\omega$. Choose Γ' along with $\omega' \in (\omega, \theta)$ as

$$\Gamma' = \big\{ |z|^{-1}\mathrm{e}^{\mathrm{i}\tau};\ |\tau| \le \pi/2 + \omega' \big\} \cup \big\{ t\mathrm{e}^{\pm \mathrm{i}(\pi/2 + \omega')};\ t \ge |z|^{-1} \big\}.$$

For any sufficiently small $z \in \mathbb{C}_\omega$ the above path Γ' certainly satisfies the properies required to define $U(z)$. Then by the condition 2 of the assertion

$$\begin{aligned} \|U(z)\| &\le \frac{M_{\omega'}}{2\pi} \int_{\Gamma'} |\mathrm{e}^{z\zeta}||\zeta|^{-1}\,|\mathrm{d}\zeta| \\ &= \frac{M_{\omega'}}{2\pi} \Big[\int_{-\infty}^{-1/|z|} |t|^{-1} \Big| \exp\Big(|zt| \mathrm{e}^{\mathrm{i}(\arg z - \pi/2 - \omega')} \Big) \Big| \, \mathrm{d}t \\ &\qquad + \int_{-\pi/2-\omega'}^{\pi/2+\omega'} \Big| \exp\Big(\mathrm{e}^{\mathrm{i}(\arg z + \tau)} \Big) \Big| \, \mathrm{d}\tau \\ &\qquad + \int_{1/|z|}^{\infty} |t|^{-1} \Big| \exp\Big(|zt| \mathrm{e}^{\mathrm{i}(\arg z + \pi/2 + \omega')} \Big) \Big| \, \mathrm{d}t \Big] \\ &\le \frac{M_{\omega'}}{2\pi} \Big[\mathrm{e}(\pi + 2\omega') + 2 \int_1^\infty s^{-1} \mathrm{e}^{-s \sin(\omega' - |\arg z|)} \, \mathrm{d}s \Big]. \end{aligned}$$

The last formula is obviously bounded uniformly in $z \in \mathbb{C}_\omega$. Thus we can conclude that U is an analytic semigroup.

Step 5. Finally we prove that the generator of U, denoted by B, coincides with A. For sufficiently large $\lambda > 0$ by Lemma 3.6

$$\begin{aligned} (\lambda - B)^{-1} &= \int_0^\infty \mathrm{e}^{-\lambda t} U(t) \, \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-\lambda t} \Big\{ \int_\Gamma \mathrm{e}^{t\zeta}(\zeta - A)^{-1} \, \mathrm{d}\zeta \Big\} \, \mathrm{d}t \end{aligned}$$

with an appropriate path Γ. If we choose Γ to be inside of the half-plane $\{\zeta \in \mathbb{C};\ \mathrm{Re}\,\zeta < \lambda\}$, we can change the order of the integrations, so that

$$\begin{aligned} (\lambda - B)^{-1} &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \Big\{ \int_0^\infty \mathrm{e}^{t(\zeta - \lambda)} \, \mathrm{d}t \Big\} (\zeta - A)^{-1} \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - \zeta)^{-1}(\zeta - A)^{-1} \, \mathrm{d}\zeta. \end{aligned}$$

The last integral coincides with $(\lambda - A)^{-1}$ (**Problem**), and hence $(\lambda - B)^{-1} = (\lambda - A)^{-1}$. This implies $B = A$. □

Theorem 3.17. A generator A of a C_0-semigroup on X is a generator of an analytic semigroup defined on $\mathbb{C}_\theta$ with $\theta \in (0, \pi/2]$ if and only if both of the following hold:

1. e^{tA} is differentiable in norm in $\mathcal{B}(X)$ with respect to $t > 0$, and therefore for any $t > 0$
$$e^{tA}X \subset D(A), \quad \frac{d}{dt}e^{tA} = Ae^{tA} \in \mathcal{B}(X);$$
2. There exist $M > 0$ and $\beta \in \mathbb{R}$ such that for any $t > 0$
$$\|Ae^{tA}\| \le Mt^{-1}e^{\beta t}.$$

In addition, if the above 1 and 2 hold with $Me \ge 1$, then one can choose
$$\theta = \arcsin[(Me)^{-1}].$$

Remark. Let A be a generator of a C_0-semigroup on X, and assume the condition 1. Then the condition 2 holds for some $M > 0$ and $\beta \in \mathbb{R}$ if and only if there exist $\delta, K > 0$ such that for any $t \in (0, \delta]$
$$\|Ae^{tA}\| \le Kt^{-1}. \qquad (\diamondsuit)$$
In fact, the necessity is obvious, and let us show the sufficiecy. If ($\diamondsuit$) holds, the condition 2 for $t \in (0, \delta]$ is straightforward, and it suffices to discuss $t > \delta$. By Corollary 3.4 we can find $L, \gamma > 0$ such that for any $t \ge 0$
$$\|e^{tA}\| \le Le^{\gamma t}. \qquad (\clubsuit)$$
By ($\diamondsuit$) and ($\clubsuit$) it follows that for any $t > \delta$
$$\|Ae^{tA}\| = \|Ae^{\delta A}\|\|e^{(t-\delta)A}\| \le KL\delta^{-1}e^{-\gamma\delta}e^{\gamma t}.$$
Hence the condition 2 is verified also for $t > \delta$.

Proof. Necessity. Suppose that A is a generator of an analytic semigroup defined on $\mathbb{C}_\theta$ with $\theta \in (0, \pi/2]$. The condition 1 is obvious by definition of a generator, see also Proposition 3.12. To verify the condition 2 we use Theorem 3.16 to write
$$Ae^{zA} = \frac{1}{2\pi i}\int_\Gamma e^{z\zeta}\zeta(\zeta - A)^{-1}\, d\zeta.$$
Let $z = t > 0$ be small, fix any $\omega \in (0, \theta)$ and choose Γ as
$$\Gamma = \left\{t^{-1}e^{i\tau};\ |\tau| \le \pi/2 + \omega\right\} \cup \left\{se^{\pm i(\pi/2+\omega)};\ s \ge t^{-1}\right\}.$$
Then a computation similar to Step 4 of the proof of Theorem 3.16 shows that for any small $t > 0$
$$\|Ae^{tA}\| \le \frac{M_\omega}{2\pi t}\left[e(\pi + 2\omega) + 2\int_1^\infty e^{-s\sin\omega}\, ds\right],$$
where M_ω is from Theorem 3.16. This and the above Remark certainly implies the condition 2.

Sufficiency. Assume the conditions 1 and 2 of the assertion. We may let $Me \ge 1$ by retaking $M > 0$ larger if necessary.

Step 1. We first prove that e^{tA} is infinitely differentiable in norm in $\mathcal{B}(X)$ with respect to $t > 0$, and moreover that for any $n \in \mathbb{N}$ and $t > 0$
$$(e^{tA})^{(n)} = (Ae^{(t/n)A})^n, \quad \left\|(Ae^{(t/n)A})^n\right\| \le e^{\beta t}(Mn/t)^n. \qquad (\heartsuit)$$
In fact, the latter estimate of ($\heartsuit$) is clear from the condition 2. For any $\epsilon > 0$ and $t > \epsilon$, if we rewrite
$$(e^{tA})' = Ae^{tA} = e^{(t-\epsilon)A}Ae^{\epsilon A},$$
then the last formula is clearly differentiable in $t > \epsilon$. Repeating this argument, we can differentiate e^{tA} in $t > 0$ as many times as we would like. Moreover, the same argument indeed shows the former expression of ($\heartsuit$).

Step 2. Here we prove e^{tA} extends as an analytic operator-valued function $U\colon \mathbb{C}_\theta \to \mathcal{B}(X)$ for $\theta = \arcsin[(M\mathrm{e})^{-1}]$. By Step 1 and Taylor's theorem for any $t, a > 0$ and $n \in \mathbb{N}$ we can find $\tau > 0$ between a and t such that

$$\mathrm{e}^{tA} = \mathrm{e}^{aA} + \sum_{k=1}^{n-1} \frac{(t-a)^k}{k!}(A\mathrm{e}^{(a/k)A})^k + \frac{(t-a)^n}{n!}(A\mathrm{e}^{(\tau/n)A})^n.$$

The above remainder term is estimated by ($\heartsuit$) from Step 1 and Stirling's approximation as

$$\left\| \frac{(t-a)^n}{n!}(A\mathrm{e}^{(\tau/n)A})^n \right\| \le \frac{|t-a|^n}{n!}\mathrm{e}^{\beta\tau}\left(\frac{Mn}{\tau}\right)^n \le \frac{\mathrm{e}^{\beta\tau}}{(2\pi n)^{1/2}}\left(\frac{|t-a|M\mathrm{e}}{\tau}\right)^n.$$

Now let us choose $\epsilon \in (0,1)$ sufficiently small, so that

$$\frac{\epsilon M\mathrm{e}}{1-\epsilon} < 1.$$

Then for any $t, a > 0$ with $|t-a| < \epsilon a$, as $n \to \infty$,

$$\left\| \frac{(t-a)^n}{n!}(A\mathrm{e}^{(\tau/n)A})^n \right\| \le \frac{\mathrm{e}^{\beta(1+\epsilon)a}}{(2\pi n)^{1/2}}\left(\frac{\epsilon M\mathrm{e}}{1-\epsilon}\right)^n \to 0.$$

Hence e^{tA} is analytic in $t > 0$, and the analytic extension is given by the pewer series

$$U(z) = \mathrm{e}^{aA} + \sum_{k=1}^{\infty} \frac{(z-a)^k}{k!}(A\mathrm{e}^{(a/k)A})^k. \qquad (\spadesuit)$$

Computations similar to the above show that ($\spadesuit$) is convergent for $z \in \mathbb{C}$ and $a > 0$ with $|z-a| < a/(M\mathrm{e})$, and thus e^{tA} is extends analytically to a sector $\mathbb{C}_\theta$ with $\theta = \arcsin[(M\mathrm{e})^{-1}]$.

Step 3. By the identity theorem the analytic extension U of e^{tA} from Step 2 satisfies that for any $z, w \in \mathbb{C}_\theta$

$$U(z+w) = U(z)U(w).$$

Hence to verify U is an analytic semigroup it suffices to show that for any $\omega = \arcsin \epsilon \in (0,\theta)$ with $\epsilon \in (0, 1/(M\mathrm{e}))$

$$\operatorname*{s-lim}_{\mathbb{C}_\omega \ni z \to 0} U(z) = 1.$$

For that we first let $u \in D(A)$. By the expression ($\spadesuit$) from Step 2 for any $z \in \mathbb{C}$ and $a > 0$ with $|z-a| < \epsilon a$

$$U(z)u - \mathrm{e}^{aA}u = \sum_{k=1}^{\infty} \frac{(z-a)^k}{k!}(A\mathrm{e}^{(a/k)A})^{k-1}\mathrm{e}^{(a/k)A}Au.$$

Using the condition 2 and the estimate ($\clubsuit$) from the previous Remark, we can bound

$$\|U(z)u - \mathrm{e}^{aA}u\| \le \frac{aL\mathrm{e}^{(\beta+\gamma)a}\|Au\|}{M} \sum_{k=1}^{\infty} \frac{|z-a|^k}{k!}\left(\frac{Mk}{a}\right)^k.$$

Computations similar to Step 2 show the last sum is bounded uniformly in $z \in \mathbb{C}$ and $a > 0$ with $|z-a| < \epsilon a$, and thus as $z \to 0$ and $a \to +0$ with $|z-a| < \epsilon a$

$$\|U(z)u - u\| \le \|U(z)u - \mathrm{e}^{aA}u\| + \|\mathrm{e}^{aA}u - u\| \to 0.$$

To verify the same limit for general $u \in X$, due to denseness of $D(A) \subset X$, it suffices to show $U(z)$ is bounded uniformly in small $z \in \mathbb{C}_\omega$. However, this can be shown by the expression ($\spadesuit$) and computations similar to Step 2 again. We omit the detail. $\square$

Problem. 1. Show that e^{zA} extends entirely in $z \in \mathbb{C}$ if the conditions 1 and 2 from Theorem 3.17 hold with $Me < 1$. In particular, $A \in \mathcal{B}(X)$ in this case.

2. Discuss if it is possible to take

$$\theta > \arcsin[(Me)^{-1}]$$

in general in Theorem 3.17.

Hint for 2. Suppose A has an eigenvalue $z = \lambda + i\mu \in \mathbb{C}$, and then it would follow from the condition 2 that for any $t > 0$

$$|ze^{tz}| \leq Mt^{-1}e^{\beta t}, \quad \text{or} \quad |z|te^{-t(\beta-\lambda)} \leq M.$$

Since $\beta > \lambda$ would hold, we could deduce

$$|z|(\beta - \lambda)^{-1} \leq Me,$$

and therefore ... what does this mean? □

§ 3.5 Semigroups on Hilbert Space

Definition. Let $\mathcal{H}$ be a complex vector space. We call a mapping $(\cdot,\cdot)\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ an **inner product** if it satisfies

1. For any $u, v \in \mathcal{H}$ one has $(u, v) = \overline{(v, u)}$;
2. For any $a, b \in \mathbb{C}$ and $u, v, w \in \mathcal{H}$ one has $(u, av+bw) = a(u, v)+ b(u, w)$;
3. For any $u \in \mathcal{H}$ one has $(u, u) \geq 0$, and $(u, u) = 0$ if and only if $u = 0$.

We call a pair $(\mathcal{H}, (\cdot,\cdot))$ of a vector space $\mathcal{H}$ and an inner product $(\cdot,\cdot)$ on $\mathcal{H}$ an **inner product space**. We denote it simply by $\mathcal{H}$.

Remark. We follow a convention that an inner product is linear in the second variable, and conjugate-linear in the first.

Proposition 3.18. An inner product space is a normed space with respect to the **natural norm**

$$\|u\| = \sqrt{(u, u)} \quad \text{for } u \in \mathcal{H},$$

and hence has the natural metric.

Proof. The proof is omitted. □

Definition. An inner product space that is complete with respect to the natural metric is called a **Hilbert space**.

For the rest of this section we let $\mathcal{H}$ be a Hilbert space.

Definition. Let A be a linear operator on $\mathcal{H}$. We define the **numerical range** of A as

$$\nu(A) = \Big\{(u, Au) \in \mathbb{C};\ u \in D(A),\ \|u\| = 1\Big\}.$$

Remark. The numerical range $\nu(A)$ may be considered an "outer approximation" of the spectrum $\sigma(A)$.

Example. Let A be a square matrix of order d. With $(\cdot,\cdot)$ being the standard inner product on $\mathbb{C}^d$, we have

$$\nu(A) = \Big\{(u, Au) \in \mathbb{C};\ u \in \mathbb{C}^d,\ \|u\| = 1\Big\}.$$

Then it follows that

$$\sigma(A) \subset \nu(A).$$

In fact, if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then letting $u \in \mathbb{C}^d$ be the associated unit eigenvector, we obtain

$$\lambda = \lambda\|u\|^2 = (u, \lambda u) = (u, Au) \in \nu(A).$$

Problem. Show that, if A is unitarily similar to a diagonal matrix, then $\nu(A)$ coincides with the convex hull of $\sigma(A)$.

Definition. Let A be a densely defined closed operator on $\mathcal{H}$.

1. A is said to be **accretive** if

   $$\nu(A) \subset \{z \in \mathbb{C};\ \mathrm{Re}\, z \geq 0\}.$$

   A is said to be **maximal accretive**, or m-**accretive**, if in addition there does not exist a proper accretive extension.

2. A is said to be **dissipative** if

   $$\nu(A) \subset \{z \in \mathbb{C};\ \mathrm{Re}\, z \leq 0\}.$$

   A is said to be **maximal dissipative**, or m-**dissipative**, if in addition there does not exist a proper dissipative extension.

Remarks. 1. We will not discuss accretive operators. They are just introduced as opposed to dissipative operators.

Remarks (Continued). 2. These notions are generalized to a Banach space with some variations. In particular, 'maximal dissipative' and 'm-dissipative' are often distinguished there.

3. Obviously, A is dissipative if and only if for any $u \in D(A)$

   $$\mathrm{Re}(u, Au) \leq 0.$$

   If in addition A generates a C_0-semigroup, then this implies that for any $u \in D(A)$

   $$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{e}^{tA}u\|^2 = 2\,\mathrm{Re}(\mathrm{e}^{tA}u, A\mathrm{e}^{tA}u) \leq 0.$$

   We can interpret it as a dissipation of certain energy $\|\mathrm{e}^{tA}u\|^2$, as time passes, of the system under consideration.

Proposition 3.19. Let A be a dissipative operator on $\mathcal{H}$. Then the following conditions are equivalent to each other.

1. A is maximal dissipative.
2. For all $\mathrm{Re}\, z > 0$ one has $\mathrm{Ran}(z - A) = \mathcal{H}$.
3. For some $\mathrm{Re}\, z > 0$ one has $\mathrm{Ran}(z - A) = \mathcal{H}$.
4. For all $\mathrm{Re}\, z > 0$ one has $z \in \rho(A)$.
5. For some $\mathrm{Re}\, z > 0$ one has $z \in \rho(A)$.

Remark. The m-dissipativity on a Banach space is usually defined by employing either of the above conditions 2–5.

Proof. Let us first note that for any $\operatorname{Re} z > 0$ and $u \in D(A)$

$$(\operatorname{Re} z)\|u\|^2 \le \operatorname{Re}(u, (z - A)u), \qquad (\spadesuit)$$

hence

$$(\operatorname{Re} z)\|u\| \le \|(z - A)u\|. \qquad (\heartsuit)$$

1 ⇒ 2: Let $\operatorname{Re} z > 0$. Then, due to ($\heartsuit$) and that A is closed, the subspace $\operatorname{Ran}(z - A) \subset \mathcal{H}$ is closed. Set

$$N = (\operatorname{Ran}(z - A))^\perp,$$

and then by ($\spadesuit$) we can see $D(A) \cap N = \{0\}$. Now define an extension B of A as

$$B(u + v) = Au - \bar{z}v \quad \text{for } u + v \in D(A) + N =: D(B).$$

This operator B is dissipative. In fact, for any $u + v \in D(A) + N$

$$\begin{aligned}
&\operatorname{Re}(u + v, B(u + v)) \\
&= \operatorname{Re}(u + v, Au - \bar{z}v) \\
&= \operatorname{Re}(u, Au) + \operatorname{Re}(u, -\bar{z}v) + \operatorname{Re}(v, zu) + \operatorname{Re}(v, -\bar{z}v) \\
&= \operatorname{Re}(u, Au) - (\operatorname{Re} z)\|v\|^2 \le 0.
\end{aligned}$$

Since A is maximal dissipative, it follows that $D(A) = D(B)$, and hence $N = \{0\}$, or $\operatorname{Ran}(z - A) = \mathcal{H}$.

2 ⇒ 3: This is trivial.

3 ⇒ 1: Let $\operatorname{Re} z > 0$ satisfy $\operatorname{Ran}(z - A) = \mathcal{H}$, and take any dissipative extension B of A. Then for any $u \in D(B)$ by the assumption there exists $v \in D(A)$ such that

$$(z - B)u = (z - A)v = (z - B)v.$$

Since B also satisfies ($\heartsuit$), it follows that

$$(\operatorname{Re} z)\|u - v\| \le \|(z - B)(u - v)\| = 0, \quad \text{or} \quad u = v.$$

This says $D(B) \subset D(A)$, and thus A maximal dissipative.

2 ⇒ 4: For any $\operatorname{Re} z > 0$ the operator $z - A$ is injective due to ($\heartsuit$), hence has the inverse $(z - A)^{-1}$. Then by the closed graph theorem $(z - A)^{-1}$ is bounded, and thus $z \in \rho(A)$ follows.

4 ⇒ 2: This is trivial by the definition of resolvent.

3 ⇔ 5: We can argue similarly to the above 2 ⇔ 4. □

Theorem 3.20. Let A be a linear operator on $\mathcal{H}$. The following conditions are equivalent:

1. A is maximal dissipative.
2. A is a generator of a contraction semigroup.

Proof. We first let A be maximal dissipative. In particular, A is closed and densely defined. In addition, by Proposition 3.19 it follows that $(0, \infty) \subset \rho(A)$. Moreover, due to ($\heartsuit$) in the proof of Proposition 3.19, for any $\lambda > 0$ and $u \in D(A)$

$$\lambda\|u\| \le \|(\lambda - A)u\|,$$

which implies for any $n \in \mathbb{N}$

$$\|(\lambda - A)^{-n}\| \le \|(\lambda - A)^{-1}\|^n \le \lambda^{-n}.$$

Now the Hille–Yosida theorem verifies the condition 2.

Next, let A be a generator of a contraction semigroup. By the Hille–Yosida theorem A is closed and densely defined on $\mathcal{H}$. Moreover, $(0,\infty) \subset \rho(A)$ and for any $\lambda > 0$

$$\lambda \|(\lambda - A)^{-1}\| \leq 1.$$

Thus for any $\lambda > 0$ and $u \in D(A)$

$$\lambda\|u\| = \lambda\|(\lambda - A)^{-1}(\lambda - A)u\| \leq \|(\lambda - A)u\|,$$

so that

$$\|Au\|^2 - 2\lambda \operatorname{Re}(u, Au) = \|(\lambda - A)u\|^2 - \lambda^2\|u\|^2 \geq 0.$$

Now this implies that for any $u \in D(A)$

$$\operatorname{Re}(u, Au) \leq 0,$$

and hence A is dissipative. Since $(0,\infty) \subset \rho(A)$, it follows from Proposition 3.19 that A is maximal dissipative. □

Definition. Let A be a densely defined closed operator on $\mathcal{H}$. A is said to be **sectorial** if there exists $\theta \in (0, \pi/2)$ such that

$$\nu(A) \subset \mathbb{C} \setminus \mathbb{C}_{\pi/2+\theta}.$$

A is said to be **maximal sectorial**, or m-**sectorial**, if in addition there does not exist a proper sectorial extension of A.

Remarks. 1. A definition of sectorial operators varies according to context. It is often the case that $-A$ for the above A is defined to be sectorial.

2. A genaralization of a sectorial operator to a Banach space corresponds to an m-sectorial operator on a Hilbert space.

Proposition 3.21. Let A be a sectorial operator on $\mathcal{H}$ with $\nu(A) \subset \mathbb{C} \setminus \mathbb{C}_{\pi/2+\theta}$ for some $\theta \in (0, \pi/2)$. Then the following conditions are equivalent to each other.

1. A is maximal sectorial.
2. For all $z \in \mathbb{C}_{\pi/2+\theta}$ one has $\operatorname{Ran}(z - A) = \mathcal{H}$.
3. For some $z \in \mathbb{C}_{\pi/2+\theta}$ one has $\operatorname{Ran}(z - A) = \mathcal{H}$.
4. For all $z \in \mathbb{C}_{\pi/2+\theta}$ one has $z \in \rho(A)$.
5. For some $z \in \mathbb{C}_{\pi/2+\theta}$ one has $z \in \rho(A)$.

Proof. It suffices to repeat the proof of Proposition 3.19 for $e^{i\omega}A$ with $|\omega| \leq \theta$. The detail is omitted. □

Theorem 3.22. Let A be a linear operator on $\mathcal{H}$. The following conditions are equivalent.

1. A is maximal sectorial.
2. A is a generator of an analytic contraction semigroup.

Proof. First let A be maximal sectorial, and let $\theta \in (0, \pi/2)$ be the associated angle. Then $e^{i\omega}A$ for any $|\omega| \leq \theta$ is maximal dissipative, hence by Theorem 3.20 generates a contraction semigroup. This implies that $e^{i\omega}A$ for any $|\omega| \leq \theta$ satisfies the conditions of the Hille–Yosida theorem with $\beta = 0, M = 1$, and then by Theorem 3.14 A generates an analytic contraction semigroup.

We can go backward along the above arguments, and therefore the converse is also true. □

Section 4

Application to PDEs, I

§ 4.1 Schwartz Distributions

Let $\Omega \subset \mathbb{R}^d$ be an open subset, and we write $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$.

Definition. A linear functional $T: \mathcal{D}(\Omega) \to \mathbb{C}$ is called a **Schwartz distribution** on Ω if for any compact subset $K \subset \Omega$ there exist $C > 0$ and $k \in \mathbb{N}_0$ such that for any $\phi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \phi \subset K$

$$|\langle T, \phi \rangle| \le C \max_{x \in K,\, |\alpha| \le k} |\partial^\alpha \phi(x)|.$$

Here we have written $\langle T, \phi \rangle = T\phi = T(\phi)$. We denote the set of all the Schwartz distributions on Ω by $\mathcal{D}'(\Omega)$.

We denote the set of all the locally integrable functions on Ω by

$$L^1_{\mathrm{loc}}(\Omega) = \left\{ u\colon \Omega \to \mathbb{C};\ \forall K \Subset \Omega\ u|_K \in L^1(K) \right\}.$$

Note that for any $p \in [1, \infty]$ the inclusion $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ holds.

Proposition 4.1. For any $u \in L^1_{\mathrm{loc}}(\Omega)$ let $T_u\colon \mathcal{D}(\Omega) \to \mathbb{C}$ be a linear functional defined as

$$\langle T_u, \phi \rangle = \int_\Omega u(x)\phi(x)\,\mathrm{d}x \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Then one has $T_u \in \mathcal{D}'(\Omega)$. Moreover, the linear mapping

$$L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega), \quad u \mapsto T_u$$

is injective, i.e., if $u, v \in L^1_{\mathrm{loc}}(\Omega)$ satisfy $T_u = T_v$ as distributions, then $u = v$ a.e. on Ω.

Proof. The proof is omitted (**Problem**). □

Remark. In the following we identify $u \in L^1_{\mathrm{loc}}(\Omega)$ and $T_u \in \mathcal{D}'(\Omega)$, writing simply

$$u = T_u,$$

and regard

$$L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega).$$

In particular, $L^p(\Omega) \subset \mathcal{D}'(\Omega)$ for any $p \in [1, \infty]$.

Definition. For any $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$ define $\partial^\alpha T \in \mathcal{D}'(\Omega)$ as

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Problem. 1. Prove $\partial^\alpha T \in \mathcal{D}'(\Omega)$.

2. Prove $\partial^\alpha T_u = T_{\partial^\alpha u}$ for any $u \in C^\infty(\Omega)$.

For any $k \in \mathbb{N}_0$ define the **Sobolev space of order** k as

$$H^k(\Omega) = \left\{ u \in \mathcal{D}'(\Omega);\ |\forall \alpha| \le k\ \partial^\alpha u \in L^2(\Omega) \right\}.$$

Here ∂^α is of course understood as a distributional derivative. $H^k(\Omega)$ is a Hilbert space with respect to the inner product

$$(v, u)_{H^k} = \sum_{|\alpha| \le k} (\partial^\alpha v, \partial^\alpha u)_{L^2}.$$

In addition, define

$$H^1_0(\Omega) = \overline{\mathcal{D}(\Omega)} \ \text{ in } H^1(\Omega).$$

$H^1_0(\Omega)$ is regarded as the space of functions with the **Dirichlet boundary condition** on $\partial\Omega$.

§ 4.2 Drift-Diffusion Equation

Let $\Omega \subset \mathbb{R}^d$ be a domain, and $\mathcal{P}$ be a differential operator on Ω of the form

$$\mathcal{P} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{d} \left(b_i^{(1)}(x) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} b_i^{(2)}(x) \right) + c(x).$$

We discuss a **Cauchy problem** of the PDE

$$\frac{\partial}{\partial t} u = \mathcal{P} u \ \text{ in } (0, \infty) \times \Omega$$

for unknown function $u = u(t, x)$ with **Cauchy data**

$$u(0, \cdot) = u_0 \ \text{ on } \Omega, \quad u = 0 \ \text{ on } (0, \infty) \times \partial\Omega.$$

In addition, if Ω is unbounded, we further impose

$$\lim_{x \in \Omega,\, |x| \to \infty} u(\cdot, x) = 0 \ \text{ on } (0, \infty).$$

Remark. We can physically interpret the coefficients, for example, as follows.

- $(a_{ij})_{i,j}$ represents the diffusivity depending on directions.
- $(b_i^{(1)})_i$ and $(b_j^{(2)})$ provide a velocity field of the media.
- c represents a rate of self-creation or self-annihilation.

In order to discuss the unique solvability of the given Cauchy problem we need to fix a "mathematical framework" to deal with it. Here we are going to reformulate it in terms of the functional analysis with the following assumptions on the coefficients.

Assumption 4.2. 1. For any $i, j = 1, \dots, d$ and $k = 1, 2$

$$a_{ij}, b_i^{(k)}, c \in L^\infty(\Omega) = L^\infty(\Omega; \mathbb{C}).$$

2. There exists $\epsilon > 0$ such that for any $x \in \Omega$

$$\mathrm{Re}(a_{ij}(x))_{i,j} := \frac{1}{2} \left(a_{ij}(x) + \overline{a_{ji}(x)} \right)_{i,j} \ge \epsilon$$

as a quadratic form on $\mathbb{C}^d$, i.e., for any $(x, \xi) \in \Omega \times \mathbb{C}^d$

$$\mathrm{Re} \sum_{i,j=1}^{d} a_{ij}(x) \bar{\xi}_i \xi_j \ge \epsilon |\xi|^2.$$

Now we define a realization P of $\mathcal{P}$ on $L^2(\Omega)$ as

$$D(P) = \left\{ u \in H^1_0(\Omega);\ \mathcal{P} u \in L^2(\Omega) \right\}, \quad P = \mathcal{P}|_{D(P)}.$$

Remarks. 1. We use the notation $\mathcal{P}$ for a general distributional derivative, and distinguish it from its restriction P, an operator on $L^2(\Omega)$.

2. To be sure, let us discuss how we should interpret

$$\mathcal{P}u = \sum_{i,j=1}^{d} \partial_i a_{ij} \partial_j u + \sum_{i=1}^{d} \Big(b_i^{(1)} \partial_i u + \partial_i b_i^{(2)} u \Big) + cu \in \mathcal{D}'(\Omega)$$

for $u \in H_0^1(\Omega)$. We understand the term $\partial_i a_{ij} \partial_j u$ as a distributional derivative of $a_{ij}\partial_j u \in L^2(\Omega)$ which is a product of $a_{ij} \in L^\infty(\Omega)$ and $\partial_j u \in L^2(\Omega)$. If one first considered $\partial_j u \in \mathcal{D}'(\Omega)$, then we could not take a product $a_{ij}\partial_j u$ even in $\mathcal{D}'(\Omega)$. The term $\partial_i b_i^{(2)} u$ is understood similarly. On the other hand, the remaining terms are are naturally in $L^2(\Omega)$ as products of $b_i^{(1)}, c \in L^\infty(\Omega)$ and $\partial_i u, u \in L^2(\Omega)$, respectively.

Theorem 4.3. Under Assumption 4.2 there exists $\gamma \in \mathbb{R}$ such that $P - \gamma$ is maximal sectorial on $L^2(\Omega)$. In particular, the operator P generates an analytic semigroup on $L^2(\Omega)$.

Corollary 4.4. For any $u_0 \in L^2(\Omega)$ an evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Pu(t) \quad \text{for } t > 0, \quad u(0) = u_0 \qquad (\clubsuit)$$

has a unique solution in

$$\Big\{ u \in C\big([0,\infty); L^2(\Omega)\big) \cap C^1\big((0,\infty); L^2(\Omega)\big);\ \forall t > 0\ u(t) \in D(P) \Big\},$$

which is given by

$$u(t) = \mathrm{e}^{tP} u_0 \quad \text{for } t \geq 0.$$

Proof. The assertion follows from Theorems 4.3 and Proposition 3.12. □

Proof of Theorem 4.3. Step 1. Let $\gamma \in \mathbb{R}$, and define a quadratic form q on $\mathcal{D}(\Omega)$ as, for $u, v \in \mathcal{D}(\Omega)$,

$$\begin{aligned} q(v,u) &= -(v, (\mathcal{P} - \gamma)u)_{L^2} \\ &= \sum_{i,j=1}^{d} (\partial_i v, a_{ij}\partial_j u)_{L^2} - \sum_{i=1}^{d} \Big((v, b_i^{(1)} \partial_i u)_{L^2} - (\partial_i v, b_i^{(2)} u)_{L^2} \Big) \\ &\quad - (v, cu)_{L^2} + \gamma (v,u)_{L^2}. \end{aligned}$$

Here we claim that, if we fix sufficiently large $\gamma \in \mathbb{R}$, then there exist $c_1, C_1 > 0$ such that for any $u, v \in \mathcal{D}(\Omega)$

$$\operatorname{Re} q(u,u) \geq c_1 \|u\|_{H^1}^2, \quad |q(v,u)| \leq C_1 \|v\|_{H^1} \|u\|_{H^1}. \qquad (\heartsuit)$$

In particular, q extends uniquely to a bounded quadratic form defined on $H_0^1(\Omega)$.

Let us show ($\heartsuit$). The latter inequality from ($\heartsuit$) is clear by Assumption 4.2 and the Cauchy–Schwarz inequality. As for the former, by Assumption 4.2 and the Cauchy–Schwarz inequality again there exists $C_2 > 0$ such that for any $u \in \mathcal{D}(\Omega)$

$$\begin{aligned} \operatorname{Re} q(u,u) \geq \epsilon \sum_{i=1}^{d} \|\partial_i u\|_{L^2}^2 - C_2 \sum_{i=1}^{d} \|u\|_{L^2} \|\partial_i u\|_{L^2} - C_2 \|u\|_{L^2}^2 \\ + \gamma \|u\|_{L^2}^2. \end{aligned}$$

We further apply the Cauchy–Schwarz inequality, to obtain

$$\operatorname{Re} q(u,u) \geq \frac{\epsilon}{2} \sum_{i=1}^{d} \|\partial_i u\|_{L^2}^2 + \Big(\gamma - C_2 - \frac{dC_2}{2\epsilon} \Big) \|u\|_{L^2}^2.$$

Then the assertion follows for sufficiently large fixed $\gamma \in \mathbb{R}$.

In the following we consider q as defined on $H_0^1(\Omega)$. Of course, ($\heartsuit$) holds true for any $u, v \in H_0^1(\Omega)$ for this extended q.

Step 2. We next prove that there exists an isomorphism between Hilbert spaces:

$$J\colon H_0^1(\Omega) \to H_0^1(\Omega)$$

such that for any $u, v \in H_0^1(\Omega)$

$$q(v,u) = (v, Ju)_{H^1}.$$

(This is essentially the **Lax–Milgram theorem**.)

Let $u \in H_0^1(\Omega)$. By (♡) and the Riesz representation theorem there uniquely exists $Ju \in H_0^1(\Omega)$ such that for any $v \in H_0^1(\Omega)$

$$q(v,u) = (v, Ju)_{H^1}.$$

By the uniqueness this correspondence $J\colon H_0^1(\Omega) \to H_0^1(\Omega)$ is clearly linear. Furthermore, for any $u \in H_0^1(\Omega)$

$$c_1 \|u\|_{H^1} \le \|Ju\|_{H^1} \le C_1 \|u\|_{H^1}. \qquad (\diamondsuit)$$

In fact, it follows from (♡) that

$$c_1 \|u\|_{H^1}^2 \le \operatorname{Re} q(u,u) = \operatorname{Re}(u, Ju)_{H^1} \le \|u\|_{H^1} \|Ju\|_{H^1},$$

and that

$$\|Ju\|_{H^1} = \sup_{\|v\|_{H^1}=1} |(v, Ju)_{H^1}| = \sup_{\|v\|_{H^1}=1} |q(v,u)| \le C_1 \|u\|_{H^1}.$$

Hence J is bounded and injective.

Due to (◇) it suffices to show that J is surjective. By (◇) the subspace $\operatorname{Ran} J \subset H_0^1(\Omega)$ is closed. If $u \in (\operatorname{Ran} J)^\perp$, then by (♡)

$$c_1 \|u\|_{H^1}^2 \le \operatorname{Re} q(u,u) = \operatorname{Re}(u, Ju)_{H^1} = 0,$$

so that $u = 0$. Thus $\operatorname{Ran} J = H_0^1(\Omega)$, and the claim is verified.

Step 3. Here we prove that $D(P)$ is dense in $L^2(\Omega)$. For that let us show

$$J^{-1}\big(H_0^1(\Omega) \cap H^2(\Omega)\big) \subset D(P). \qquad (\spadesuit)$$

In fact, let $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Then for any $v \in \mathcal{D}(\Omega)$

$$\begin{aligned}(v, (-\Delta + 1)u)_{L^2} &= (v,u)_{H^1}\\ &= q(v, J^{-1}u)\\ &= -(v, \mathcal{P} J^{-1} u)_{L^2} + \gamma (v, J^{-1}u)_{L^2},\end{aligned}$$

which implies

$$J^{-1}u \in D(P), \quad PJ^{-1}u = \mathcal{P} J^{-1} u = \Delta u - u + \gamma J^{-1} u,$$

hence (♠). By Step 2 and the general theory subspaces

$$J^{-1}\big(H_0^1(\Omega) \cap H^2(\Omega)\big) \subset H_0^1(\Omega), \quad H_0^1(\Omega) \subset L^2(\Omega)$$

are dense in each topology. This and (♠) imply the claim.

Step 4. Here we prove that P is closed. Let $u_1, u_2, \ldots \in D(P)$ satisfy as $n \to \infty$

$$u_n \to u \ \text{ in } L^2(\Omega), \quad Pu_n \to w \ \text{ in } L^2(\Omega).$$

We first claim that we then actually have

$$u_n \to u \ \text{ in } H_0^1(\Omega).$$

In fact, we have for any $u \in D(P)$ and $v \in \mathcal{D}(\Omega)$

$$q(v,u) = -(v, Pu)_{L^2} + \gamma (v,u)_{L^2},$$

and, if we let $v \to u$ in $H_0^1(\Omega)$, it follows that

$$q(u,u) = -(u, Pu)_{L^2} + \gamma (u,u)_{L^2}. \qquad (\clubsuit)$$

By (♡) and (♣) we obtain

$$c_1\|u_n-u_m\|_{H^1}^2 \le -\operatorname{Re}(u_n-u_m, Pu_n-Pu_m)_{L^2}+\gamma\|u_n-u_m\|_{L^2}^2,$$

hence the claim. Now by definition for any $v\in\mathcal{D}(\Omega)$

$$(v,\mathcal{P}u_n)_{L^2}=(v,Pu_n)_{L^2},$$

and here we take the limit $n\to\infty$. Due to the above claim it follows that

$$(v,\mathcal{P}u)_{L^2}=(v,w)_{L^2},$$

so that

$$u\in D(P),\quad Pu=w.$$

Thus P is closed.

Step 5. We prove that $P-\gamma$ is sectorial, but it is rather straightforward. In fact, (♣) and (♡) implies that for any $u\in D(P)$

$$\begin{aligned}\left|\operatorname{Im}(u,(P-\gamma)u)\right| &= |\operatorname{Im} q(u,u)| \le |q(u,u)| \le C_1\|u\|_{H^1}^2\\ &\le \frac{C_1}{c_1}\operatorname{Re} q(u,u) = -\frac{C_1}{c_1}\operatorname{Re}(u,(P-\gamma)u).\end{aligned}$$

Step 6. Now we prove that $P-\gamma$ is maximal sectorial. Due to Proposition 3.21 it suffices to show $0\in\rho(P-\gamma)$, since then a neighborhood of 0 is contained in $\rho(P-\gamma)$. By (♡) and (♣) for any $u\in D(P)$

$$c_1\|u\|_{H^1}^2 \le \operatorname{Re} q(u,u) = -\operatorname{Re}(u,(P-\gamma)u).$$

Thus $P-\gamma$ is injective, and $(P-\gamma)^{-1}$ exists.

By the closed graph theorem it suffices to show that $P-\gamma$ is surjective. For that let $u\in L^2(\Omega)$. Then, for any $v\in H_0^1(\Omega)$

$$|(v,-u)_{L^2}| \le \|v\|_{L^2}\|u\|_{L^2} \le \|v\|_{H^1}\|u\|_{L^2},$$

and therefore by the Riesz representation theorem there exists $w\in H_0^1(\Omega)$ such that for any $v\in\mathcal{D}(\Omega)$

$$\begin{aligned}(v,-u)_{L^2} &= (v,w)_{H^1} = q(v,J^{-1}w)\\ &= -(v,\mathcal{P}J^{-1}w)_{L^2}+\gamma(v,J^{-1}w)_{L^2}.\end{aligned}$$

Now it follows that

$$J^{-1}w\in D(P),\quad (P-\gamma)J^{-1}w=u$$

and hence $P-\gamma$ is surjective.

Step 7. Finally we prove that P generates an analytic semigroup. By Step 6 and Theorem 3.22 the operator $P-\gamma$ generates an analytic semigroup defined on $\mathbb{C}_\theta$ for some $\theta\in(0,\pi/2)$. Set

$$U(z)=\mathrm{e}^{\gamma z}\mathrm{e}^{z(P-\gamma)}\quad\text{for } z\in\mathbb{C}_\theta,$$

and then U is obviously an analytic semigroup, and its generator coincides with P. Hence we are done. □

Remark. As for Step 7, we may also use Theorems 3.14 or 3.16, instead.

Corollary 4.5. Define $\|\cdot\|_q\colon H_0^1(\Omega)\to\mathbb{R}$ as, for $u\in H_0^1(\Omega)$,

$$\|u\|_q=(\operatorname{Re} q(u,u))^{1/2},$$

where q is from Step 1 of the proof of Theorem 4.3. Then $\|\cdot\|_q$ is a norm on $H_0^1(\Omega)$, and is equivalent to $\|\cdot\|_{H^1}$. Moreover, define $(\cdot,\cdot)_q\colon H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$ as, for $u,v\in H_0^1(\Omega)$,

$$(u,v)_q=\frac{1}{4}\Big(\|u+v\|_q^2-\|u-v\|_q^2+\mathrm{i}\|u+\mathrm{i}v\|_q^2-\mathrm{i}\|u-\mathrm{i}v\|_q^2\Big).$$

Then $(\cdot,\cdot)_q$ is an inner product on $H_0^1(\Omega)$ compatible with $\|\cdot\|_q$.

Proof. To prove the former assertion, due to Step 1 of the proof Theorem 4.3, it suffices to verify the triangle inequality for $\|\cdot\|_q$. By a direct computaion it further reduces to verify that for any $u,v\in H_0^1(\Omega)$

$$|\operatorname{Re}(q(u,v)+q(v,u))|\le 2\|u\|_q\|v\|_q.$$

However, this easily follows by taking the discriminant of

$$t^2\|u\|_q^2+t\operatorname{Re}(q(u,v)+q(v,u))+\|v\|_q^2=\|tu+v\|_q^2\ge 0.$$

As for the latter assertion, it suffices to verify the **parallelogram law**: For any $u,v\in H_0^1(\Omega)$

$$\|u+v\|_q^2+\|u-v\|_q^2=2\left(\|u\|_q^2+\|v\|_q^2\right).$$

This immediately follows by a direct computation employing that q is a quadratic form on $H_0^1(\Omega)$. □

§ 4.3 Wave Equation with Certain Damping

Similarly to the previous section, let $\Omega\subset\mathbb{R}^d$ be a domain, and

$$\mathcal{P}=\sum_{i,j=1}^d\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}+\sum_{i=1}^d\left(b_i^{(1)}(x)\frac{\partial}{\partial x_i}+\frac{\partial}{\partial x_i}b_i^{(2)}(x)\right)+c(x).$$

We discuss a Cauchy problem of the PDE

$$\frac{\partial^2}{\partial t^2}u=\mathcal{P}u\quad\text{in }(0,\infty)\times\Omega$$

for unknown function $u=u(t,x)$ with Cauchy data

$$u(0,\cdot)=u_0,\quad\frac{\partial u}{\partial t}(0,\cdot)=u_1\quad\text{on }\Omega,\qquad u=0\quad\text{on }(0,\infty)\times\partial\Omega.$$

If Ω is unbounded, we of course impose

$$\lim_{x\in\Omega,\,|x|\to\infty}u(\cdot,x)=0\quad\text{on }(0,\infty).$$

Remark. We can physically interpret the coefficients, for example, as follows.

- $(a_{ij})_{i,j}$ provides squares of the wave propagation speeds depending on directions.
- $(b_i^{(1)})_i$ and $(b_i^{(2)})_i$ represent a certain damping or amplifying effect.
- c represents a certain external field.

Assumption 4.6. 1. For any $i,j = 1,\dots,d$ and $k = 1,2$

$$a_{ij}, b_i^{(k)}, c \in L^\infty(\Omega) = L^\infty(\Omega;\mathbb{C}).$$

2. For each $x \in \Omega$ the matrix $(a_{ij}(x))_{i,j}$ is Hermitian. Moreover, there exists $\epsilon > 0$ such that for any $x \in \Omega$

$$(a_{ij}(x))_{i,j} \geq \epsilon$$

as a quadratic form on $\mathbb{C}^d$.

Under Assumption 4.6 define a realization P of $\mathcal{P}$ on $L^2(\Omega)$ as in the previous section, and we set

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega), \quad A = \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix}, \quad D(A) = D(P) \times H_0^1(\Omega).$$

The inner product $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{H}$ is defined as, for $(u,v),(f,g) \in \mathcal{H}$,

$$((f,g),(u,v))_{\mathcal{H}} = (f,u)_{q_0} + (g,v)_{L^2}.$$

Here $(\cdot,\cdot)_{q_0}$ is an inner product on $H_0^1(\Omega)$ from Corollary 4.5 associated with a differential operator

$$\mathcal{P}_0 := \sum_{i,j=1}^d \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^d \left(-\overline{b_i^{(2)}(x)} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} b_i^{(2)}(x) \right).$$

Theorem 4.7. Under Assumption 4.6 there exists $\gamma \in \mathbb{R}$ such that $A - \gamma$ is maximal dissipative on $\mathcal{H}$. In particular, A generates a C_0-semigroup on $\mathcal{H}$.

Corollary 4.8. For any $(u_0,u_1) \in D(P) \times H_0^1(\Omega)$ an evolution equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) = Pu(t) \quad \text{for } t > 0, \quad u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1 \qquad (\clubsuit)$$

has a unique solution in

$$\Big\{ u \in C\big([0,\infty); H_0^1(\Omega)\big) \cap C^1\big((0,\infty); H_0^1(\Omega)\big) \cap C^1\big([0,\infty); L^2(\Omega)\big) \cap C^2\big((0,\infty); L^2(\Omega)\big);\ \forall t > 0\ u(t) \in D(P) \Big\},$$

which is given by the first component of $\mathrm{e}^{tA}(u_0,u_1)$.

Proof of Theorem 4.7. By Theorem 4.3 P is closed and densely defined on $L^2(\Omega)$. Then clearly A is also closed and densely defined on $\mathcal{H}$. In addition, for any $(u,v) \in D(A)$

$$\mathrm{Re}\big((u,v),(A-\gamma)(u,v)\big)_{\mathcal{H}} = \mathrm{Re}(u,v)_{q_0} + \mathrm{Re}(v,Pu)_{L^2} - \gamma\|u\|_{q_0}^2 - \gamma\|v\|_{L^2}^2.$$

Here by Corollary 4.5 and definition of $(\cdot,\cdot)_{q_0}$, fixing a constant $\gamma_0 \in \mathbb{R}$ that defines q_0, we can write

$$\begin{aligned}\mathrm{Re}(u,v)_{q_0} &= -\frac{1}{2}\Big(\mathrm{Re}(u,(P_0-\gamma_0)v)_{L^2} + \mathrm{Re}(v,(P_0-\gamma_0)u)_{L^2}\Big) \\ &= -\frac{1}{2}\mathrm{Re}(v,(P_0+P_0^*)u)_{L^2} + \gamma_0\,\mathrm{Re}(v,u)_{L^2}.\end{aligned}$$

Thus, if we note

$$P - \frac{1}{2}(P_0 + P_0^*) = \sum_{i=1}^d \left(b_i^{(1)}(x) + \overline{b_i^{(2)}(x)} \right) \frac{\partial}{\partial x_i} + c(x),$$

we can bound by the Cauchy–Schwarz inequality as

$$\mathrm{Re}\Big((u,v),(A-\gamma)(u,v)\Big)_{\mathcal{H}} \le -\gamma\|u\|_{q_0}^2 + C_1\|u\|_{H^1}^2 - (\gamma - C_1)\|v\|_{L^2}^2.$$

Therefore, by Corollary 4.5, letting $\gamma \in \mathbb{R}$ be sufficiently large, we can verify that $A-\gamma$ is dissipative.

To prove $A-\gamma$ is maximal dissipative, let $(f,g) \in \mathcal{H}$, and we solve

$$(A-\gamma)(u,v) = (f,g) \quad \text{or} \quad v - \gamma u = f,\ Pu - \gamma v = g,$$

for $(u,v) \in D(A)$. Eliminating v, we have

$$(P-\gamma^2)u = \gamma f + g.$$

Since $P-\gamma^2$ is maximal sectorial for sufficiently large γ, we can find a solution $u \in D(P)$. Then it suffices to take

$$v = \gamma u + f \in H_0^1(\Omega).$$

The last assertion follows similarly to Theorem 4.3. □

Proof of Corollary 4.8. Let $(u_0,u_1) \in D(P) \times H_0^1(\Omega) = D(A)$, and set

$$(u,v) = \mathrm{e}^{tA}(u_0,u_1).$$

Then it follows that

$$u \in C^1\Big([0,\infty); H_0^1(\Omega)\Big), \quad v \in C^1\Big([0,\infty); L^2(\Omega)\Big),$$

and, moreover, that for any $t > 0$

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = v(t), \quad \frac{\mathrm{d}v}{\mathrm{d}t}(t) = Pu(t), \quad u(t) \in D(P).$$

Hence u is a solution to the Cauchy problem (♣) belonging to the asserted function space, or in fact a slightly better space.

Conversely, let u be a solution to the Cauchy problem (♣) belonging to the asserted function space. Define

$$w \in C\Big([0,\infty);\mathcal{H}\Big) \cap C^1\Big((0,\infty);\mathcal{H}\Big)$$

as, for $t \ge 0$,

$$w(t) = \left(u(t), \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right).$$

It obviously satisfies

$$w(t) \in D(A) \quad \text{for any } t > 0,$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t) = Aw(t) \quad \text{for } t > 0, \qquad w(0) = (u_0,u_1).$$

Then by the uniqueness from Corollary 3.11 we obtain

$$w = \mathrm{e}^{tA}(u_0,u_1).$$

Hence we are done. □

Section 5
Application to PDEs, II

§ 5.1 Growth of Generalized Eigenfunction

○ **Settings**

Let $\Omega \subset \mathbb{R}^d$ be a domain. In this section we discuss **generalized eigenfunctions** for the **free Schrödinger operator**

$$H = H_0 = \frac{1}{2}p^2 = -\frac{1}{2}\Delta.$$

Here $p_i = -\mathrm{i}\partial_i$, $i = 1, \dots, d$, denote the **momentum operators**. In the following we shall often work in the Hilbert space

$$\mathcal{H} = L^2(\Omega),$$

and its inner product is denoted by $\langle \cdot, \cdot \rangle$, which is conjugate-linear in the first variable. The associated norm is denoted by $\| \cdot \|$.

Throughout the section we assume the following.

Assumption 5.1. There exists an **escape function** $f \in C^\infty(\Omega)$ such that:

1. The image $f(\Omega)$ coincides with $[1, \infty)$;
2. For any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \ge 1$ the derivative $\partial^\alpha f$ is bounded.
3. There exists $r_0 \ge 1$ such that for any $x \in \Omega$ with $f(x) \ge r_0$
$$f(x) = r(x) = |x|;$$
4. The gradient vector field $\nabla f \in \mathfrak{X}(\Omega)$ is **forward complete**, i.e., the integral curve for ∇f exists for any initial point $x \in \Omega$ and any non-negative time parameter $t \ge 0$.

Remarks. 1. This is an assumption imposed on the domain Ω. We have assumed almost nothing on the set

$$\{x \in \Omega;\ f(x) < r_0\},$$

which could possibly be unbounded.

2. The arguments of the section directly generalize to a manifold with asymptotically Euclidean and/or hyperbolic funnel ends. In the present setting each component of the set
$$\{x \in \Omega;\ f(x) > r_0\}$$
may be considered an **end** of Ω.
3. We can also include appropriate potential and metric perturbations, but we omit them for simplicity.

○ **Dirichlet realization**

Let

$$\mathcal{H}^1 = H_0^1(\Omega), \quad \mathcal{H}^{-1} = (\mathcal{H}^1)'.$$

Note that we may embed and regard $\mathcal{H}^{\pm 1} \subset \mathcal{D}'(\Omega)$.

Lemma 5.2. H is bounded as an operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$.

Proof. Let $\psi \in \mathcal{H}^1$. Then for any $\phi \in \mathcal{D}(\Omega)$

$$|\langle \phi, H\psi \rangle| = \frac{1}{2}|\langle p\phi, p\psi \rangle| \le \frac{1}{2}\|\phi\|_{\mathcal{H}^1}\|\psi\|_{\mathcal{H}^1}.$$

This implies that $H\psi \in \mathcal{H}^{-1}$, and moreover that H is bounded as $\mathcal{H}^1 \to \mathcal{H}^{-1}$. □

We realize the associated operator H on $\mathcal{H}$ by restricting its domain to

$$D(H) = \left\{\psi \in \mathcal{H}^1;\ H\psi \in \mathcal{H}\right\}.$$

It coincides with the Dirichlet realization discussed in Section 4.

Remark. In this section we shall NOT really notationally distinguish distributional derivatives and the associated operators, e.g., on $\mathcal{H}$. Hence the meaning of the notation H changes according to the context.

Problem. Show that the operator H on $\mathcal{H}$ is self-adjoint, i.e., $H = H^*$.

Solution. In this proof we always regard H as an operator on $\mathcal{H}$ with the given domain $D(H)$.

Step 1. We first show that H is symmetric. It suffices to verify that for any $\psi, \phi \in D(H)$

$$\langle \phi, H\psi\rangle = \frac{1}{2}\langle p\phi, p\psi\rangle = \langle H\phi, \psi\rangle. \qquad (\spadesuit)$$

Choose $\phi_j \in \mathcal{D}(\Omega)$ such that $\phi_j \to \phi$ in $\mathcal{H}^1$, and then

$$\langle \phi, H\psi\rangle = \frac{1}{2}\lim_{j\to\infty}\langle \phi_j, p^2\psi\rangle = \frac{1}{2}\lim_{j\to\infty}\langle p\phi_j, p\psi\rangle = \frac{1}{2}\langle p\phi, p\psi\rangle.$$

The latter identity of ($\spadesuit$) follows similarly.

Step 2. We next show H is closed essentially by repeating Step 4 of the proof of Theorem 4.3. Let $\psi_j \in D(H)$ satisfy as $j \to \infty$

$$\psi_j \to \psi \ \text{ in } \mathcal{H}, \quad H\psi_j \to \phi \ \text{ in } \mathcal{H}.$$

Then, due to ($\spadesuit$)

$$\|\psi_j - \psi_k\|^2_{\mathcal{H}^1} = \|\psi_j - \psi_k\|^2 + 2\langle \psi_j - \psi_k, H\psi_j - H\psi_k\rangle,$$

and this implies that

$$\psi_j \to \psi \ \text{ in } \mathcal{H}^1, \ \text{ in particular } \psi \in \mathcal{H}^1.$$

In addition, by the assymption for any $\eta \in \mathcal{D}(\Omega)$

$$\langle \eta, H\psi\rangle = \langle H\eta, \psi\rangle = \lim_{j\to\infty}\langle H\eta, \psi_j\rangle = \lim_{j\to\infty}\langle \eta, H\psi_j\rangle = \langle \eta, \phi\rangle,$$

so that

$$H\psi = \phi \in \mathcal{H}.$$

Hence H is certainly closed.

Step 3. Here we show $-1/2 \in \rho(H)$ essentially by repeating Step 6 of the proof of Theorem 4.3. In fact, by ($\spadesuit$) for any $\psi \in D(H)$

$$\|\psi\|^2 \le \|p\psi\|^2 + \|\psi\|^2 = \langle \psi, 2H\psi\rangle + \langle \psi, \psi\rangle \le \|\psi\|\|(2H+1)\psi\|,$$

and this implies that $2H + 1$ is injective.

On the other hand, let $\phi \in \mathcal{H}$. Since $\langle \phi, \cdot\rangle$ provides a bounded linear functional on $\mathcal{H}^1$, there exists $\psi \in \mathcal{H}^1$ such that for any $\eta \in \mathcal{D}(\Omega)$

$$\langle \phi, \eta\rangle = \langle \psi, \eta\rangle_{\mathcal{H}^1} = \langle 2H\psi, \eta\rangle + \langle \psi, \eta\rangle.$$

Then it follows that

$$\psi \in D(H), \quad (2H+1)\psi = \phi,$$

and hence $2H + 1$ is surjective.

By Step 2 and the closed graph theorem we obtain $-1/2 \in \rho(H)$.

Step 4. Finally we show that H is self-adjoint. By Step 1 it suffices to show $D(H^*) \subset D(H)$. Let $\psi \in D(H^*)$. Then due to Step 3 there exists $\phi \in D(H)$ such that

$$(H + 1/2)\phi = (H^* + 1/2)\psi.$$

Then by (♠) from Step 1 for any $\eta \in D(H)$

$$\begin{aligned}\langle (2H+1)\eta, \psi\rangle &= \langle \eta, (2H^*+1)\psi\rangle \\ &= \langle \eta, (2H+1)\phi\rangle = \langle (2H+1)\eta, \phi\rangle.\end{aligned}$$

It follows that

$$\psi = \phi \in D(H),$$

and thus H is self-adjoint. □

○ **Function spaces**

We introduce for $s \in \mathbb{R}$

$$\mathcal{H}_s = f^{-s}\mathcal{H}, \quad \mathcal{H}_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\Omega).$$

We also intruduce the **Agmon–Hörmander spaces** defined as

$$\mathcal{B}^* = \Big\{\psi \in \mathcal{H}_{\mathrm{loc}};\ \|\psi\|_{\mathcal{B}^*} := \sup_{\nu\in\mathbb{N}_0} 2^{-\nu/2}\|F_\nu\psi\|_{\mathcal{H}} < \infty\Big\},$$
$$\mathcal{B}^*_0 = \Big\{\psi \in \mathcal{B}^*;\ \lim_{\nu\to\infty} 2^{-\nu/2}\|F_\nu\psi\|_{\mathcal{H}} = 0\Big\}.$$

Here we have set for each $\nu \in \mathbb{N}_0$

$$F_\nu = F\left(\left\{x \in \Omega;\ 2^\nu \le f(x) < 2^{\nu+1}\right\}\right),$$

where $F(\omega)$ is the characteristic function for a subset $\omega \subset \Omega$.

Problem. Show the following inclusions hold for any $s > 1/2$:

$$\mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq \mathcal{B}^*_0 \subsetneq \mathcal{B}^* \subsetneq \mathcal{H}_{-s}.$$

In addition, show $\mathcal{B}^*_0$ coincides with the closure of $\mathcal{D}(\Omega)$ in $\mathcal{B}^*$.

Furthermore, choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \le 1, \\ 0 & \text{for } t \ge 2, \end{cases} \qquad \chi' \le 0,$$

and define $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(\Omega)$ for $n, m \in \mathbb{N}_0$ as

$$\chi_n = \chi(f/2^n), \quad \bar{\chi}_n = 1 - \chi_n, \quad \chi_{m,n} = \bar{\chi}_m\chi_n.$$

Then we introduce

$$\mathcal{N} = \left\{\psi \in \mathcal{H}_{\mathrm{loc}};\ \forall n \in \mathbb{N}_0\ \chi_n\psi \in \mathcal{H}^1\right\}.$$

This is a space of functions on Ω satisfying the Dirichlet boundary condition on $\partial\Omega$, possibly with infinite $\mathcal{H}^1$-norms.

○ **Main theorem: Rellich's theorem**

Theorem 5.3 (Rellich). If $\phi \in \mathcal{H}_{\mathrm{loc}}$ and $\lambda > 0$ satisfy

1. $(H - \lambda)\phi = 0$ in the distributional sense,
2. there exists $l \in \mathbb{N}_0$ such that $\bar{\chi}_l\phi \in \mathcal{B}^*_0 \cap \mathcal{N}$,

then $\phi \equiv 0$ in Ω. In particular, the operator H on $\mathcal{H}$ has no positive eigenvalues, i.e., $\sigma_{\mathrm{p}}(H) \cap (0, \infty) = \emptyset$.

Remarks. 1. For each $\lambda > 0$ we can show

$$\mathcal{E}_\lambda := \{\phi \in \mathcal{B}^* \cap \mathcal{N};\ (H - \lambda)\phi = 0\} \neq \{0\},$$

and therefore the space $\mathcal{B}_0^*$ in the assertion is optimal with respect to a configuration weight. Physically, the growth rate of $\mathcal{B}^*$ conforms with that of a stationary wave with minimal source and sink only at infinity.

2. We can drop the space $\mathcal{N}$ in the assertion if the obstacle Ω^c is bounded.

3. We shall prove Theorem 5.3 by a commutator method according to I.–Skibsted '20. We will realize and investigate a commutator with the help of some C_0-semigroup on $\mathcal{H}$. See also a book by Amrein–Boutet de Monvel–Georgescu.

§ 5.2 Commutator Realization

○ **Semigroup of radial translations**

Let $y\colon \mathcal{M} \to \Omega$ with $\mathcal{M} \subset \mathbb{R} \times \Omega$ be the **maximal flow** generated by ∇f. By definition it satisfies

$$\partial_t y_i(t,x) = (\partial_i f)(y(t,x)) \quad \text{for } i = 1, \dots, d, \qquad y(0,x) = x.$$

Note that by Assumption 5.1

$$[0,\infty) \times \Omega \subset \mathcal{M}.$$

Define the associated **radial translation** of a function $\psi \in \mathcal{H}$ as

$$(T(t)\psi)(x) = \begin{cases} J(t,x)^{1/2}\psi(y(t,x)) & \text{if } (t,x) \in \mathcal{M}, \\ 0 & \text{otherwise,} \end{cases}$$

where $J(t,\cdot)$ denotes the Jacobian for $y(t,\cdot)\colon \Omega \to \Omega$.

Proposition 5.4. 1. For each $t \geq 0$, $T(t)$ provides a surjective partial isometry on $\mathcal{H}$ with initial subspace $L^2(y(t,\Omega))$. Moreover, $T(t)$ with $t \geq 0$ form a C_0-semigroup on $\mathcal{H}$.

2. For each $t \geq 0$, $T(-t)$ provides an isometry on $\mathcal{H}$ with final subspace $L^2(y(t,\Omega))$. Moreover, $T(-t)$ with $t \geq 0$ form a C_0-semigroup on $\mathcal{H}$.

3. For any $t \in \mathbb{R}$

$$T(t)^* = T(-t).$$

4. For any $\psi \in \mathcal{H}$ and $(t,x) \in \mathcal{M}$

$$(T(t)\psi)(x) = \exp\left(\frac{1}{2}\int_0^t (\Delta f)(y(s,x))\,\mathrm{d}s\right)\psi(y(t,x)).$$

Proof. 1. Let $t \geq 0$. By change of variables for any $\psi \in \mathcal{H}$

$$\|T(t)\psi\|^2 = \int_{y(t,\Omega)} |\psi(x)|^2\,\mathrm{d}x,$$

and hence $T(t)$ is a partial isometry on $\mathcal{H}$ with initial subspace $L^2(y(t,\Omega))$. Obviously, $T(t)$ with $t \geq 0$ form a one-parameter semigroup by the corresponding properties of the flow y and the Jacobian J. Note that these properties also guarantee that $T(t)$ is surjective for each $t \geq 0$. In fact, for any $\psi \in \mathcal{H}$

$$\psi = T(t)T(-t)\psi, \quad T(-t)\psi \in \mathcal{H}.$$

Finally to see the strong continuity of $T(t)$ in $t \geq 0$ it suffices to verify it on a dense subspace $\mathcal{D}(\Omega) \subset \mathcal{H}$. This is straightforward due to smoothness of y and J.

2. Note that for any $t \geq 0$ and $\psi \in \mathcal{H}$ by change of variables

$$\|T(-t)\psi\|^2 = \int_\Omega |\psi(x)|^2 \, dx.$$

Then we can argue more or less similarly to the assertion 1. We only note that the final subspace of $T(-t)$ for $t \geq 0$ is determined by the identity

$$\psi = T(-t)T(t)\psi \quad \text{for any } \psi \in L^2(y(t,\Omega)).$$

We omit the rest of the aruguments.

3. This is a direct consequence of change of variables and the (semi)group properties of y and J. We omit the detail.

4. Obviously, it suffices to show that for any $(t,x) \in \mathcal{M}$

$$\partial_t J(t,x) = (\Delta f)(y(t,x))J(t,x).$$

Differentiating the definition of determinant, we can write

$$\partial_t J(t,x) = \sum_{i=1}^{d} \det(\mathcal{J}^{(i)}(t,x)),$$

where matrix-valued functions $\mathcal{J}^{(i)}$ are given by

$$\mathcal{J}^{(i)}_{jk} = \begin{cases} \partial_k y_j & \text{for } j \neq i, \\ \partial_t \partial_k y_i & \text{for } j = i. \end{cases}$$

However, we can compute

$$\partial_t \partial_k y_i(t,x) = \partial_k[(\partial_i f)(y(t,x))] = (\partial_l \partial_i f)(y(t,x))\partial_k y_l(t,x).$$

Here the Einstein convention is adopted without tensorial superscripts. Then, since determinant is alternating and multilinear,

$$\det(\mathcal{J}^{(i)}(t,x)) = (\partial_i^2 f)(y(t,x))J(t,x).$$

Thus the assertion is verified. We are done. □

◦ **Generator**

Define a differential operator A as

$$A = \mathrm{Re}\, p_f = \frac{1}{2}(p_f + p_f^*) = p_f - \frac{\mathrm{i}}{2}(\Delta f) = p_f^* + \frac{\mathrm{i}}{2}(\Delta f)$$

with

$$p_f = -\mathrm{i}\partial_f, \quad \partial_f = (\partial_i f)\partial_i.$$

We let $A_\pm$ be the corresponding operators on $\mathcal{H}$ defined as

$$D(A_+) = \{\psi \in \mathcal{H};\ A\psi \in \mathcal{H}\}, \quad A_+ = A|_{D(A_+)},$$

and

$$A_- = \overline{A|_{\mathcal{D}(\Omega)}}.$$

Problem. Show that $A|_{\mathcal{D}(\Omega)}$ is closable as an operator on $\mathcal{H}$.

Proposition 5.5. The operators $\pm\mathrm{i}A_\pm$ generate C_0-semigroups formed by $T(\pm t)$ with $t \geq 0$:

$$T(\pm t) = \mathrm{e}^{\pm \mathrm{i}tA_\pm},$$

respectively. Moreover, they satisfy

$$A_- \subset A_+, \quad A_\pm^* = A_\mp,$$

respectively, and in particular

$$\mathcal{D}(\Omega) \subset D(H) \subset \mathcal{H}^1 \subset D(A_-) \subset D(A_+).$$

Remark. After the proof we will write simply $A = A_\pm$, and distinguish $\mathrm{e}^{\pm \mathrm{i}tA} = \mathrm{e}^{\pm \mathrm{i}tA_\pm}$, respectively, only by their signs.

Proof. By definitions of $A_\pm$ it is not difficult to verify that

$$A_- \subset A_+, \quad A_-^* = A_+.$$

In particular, we have the asserted inclusions. By taking the adjoint we also obtain

$$A_+^* = A_-^{**} = \overline{A_-} = A_-.$$

Now it remains to show that the generators of $T(\pm t)$, denoted for the moment by $\pm \mathrm{i}B_\pm$, coincide with $\pm\mathrm{i}A_\pm$, respectively. Let us start with the lower sign. First note that by Proposition 5.4 and the Hille–Yosida theorem

$$A_- \subset B_-, \quad T(-t)\mathcal{D}(\Omega) \subset \mathcal{D}(\Omega) \ \text{ for any } t \geq 0. \qquad (\heartsuit)$$

Since $\mathrm{i} \in \rho(B_-)$ by the Hille–Yosida theorem again, it follows by ($\heartsuit$) that $A_- - \mathrm{i}$ is injective, and that

$$(A_- - \mathrm{i})^{-1} \subset (B_- - \mathrm{i})^{-1}.$$

Assume $\psi \in (\mathrm{Ran}(A_- - \mathrm{i}))^\perp$. Then by ($\heartsuit$) for any $\phi \in \mathcal{D}(\Omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi, T(-t)\phi\rangle = -\mathrm{i}\langle \psi, A_- T(-t)\phi\rangle = \langle \psi, T(-t)\phi\rangle,$$

so that

$$\langle \psi, T(-t)\phi\rangle = \mathrm{e}^t \langle \psi, \phi\rangle.$$

Letting $t \to \infty$, we can deduce $\psi = 0$, and hence

$$(A_- - \mathrm{i})^{-1} = (B_- - \mathrm{i})^{-1}.$$

This implies $A_- = B_-$.

We next show $A_+ = B_+$. Let $\psi \in D(B_+)$. Then for any $\phi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \langle \phi, B_+\psi\rangle &= \lim_{t\to+0} \langle \phi, (\mathrm{i}t)^{-1}(T(t)-1)\psi\rangle \\ &= \lim_{t\to+0} \langle (-\mathrm{i}t)^{-1}(T(-t)-1)\phi, \psi\rangle = \langle A\phi, \psi\rangle, \end{aligned}$$

and hence $\psi \in D(A_+)$ and $B_+\psi = A_+\psi$, i.e.,

$$B_+ \subset A_+.$$

Conversely, let $\psi \in D(A_+)$. Then for any $\phi \in \mathcal{D}(\Omega)$ and $t > 0$

$$\begin{aligned} \langle \phi, (\mathrm{i}t)^{-1}(T(t)-1)\psi\rangle &= (\mathrm{i}t)^{-1}\langle (T(-t)-1)\phi, \psi\rangle \\ &= t^{-1}\left\langle \int_0^t AT(-s)\phi\,\mathrm{d}s, \psi\right\rangle \\ &= \left\langle \phi, t^{-1}\int_0^t T(s)A_+\psi\,\mathrm{d}s\right\rangle, \end{aligned}$$

so that

$$(\mathrm{i}t)^{-1}(T(t)-1)\psi = t^{-1}\int_0^t T(s)A_+\psi\,\mathrm{d}s.$$

Letting $t \to +0$, we conclude $D(A_+) \subset D(B_+)$. □

○ Radial and spherical decomposition

We introduce a differential operator

$$L = p_i \ell_{ij} p_j \ \text{ with } \ell_{ij} = \delta_{ij} - (\partial_i f)(\partial_j f),$$

which may be considered the spherical part of $-\Delta$ on the set $\{x \in \Omega;\ f(x) \geq r_0\}$. (Let us note here again that the Einstein convention is always assumed.)

Lemma 5.6. One has a decomposition

$$H = \frac{1}{2}A^2 + \frac{1}{2}L + q \quad \text{with } q = \frac{1}{8}(\Delta f)^2 + \frac{1}{4}(\partial_f \Delta f).$$

Proof. We can compute, e.g., as

$$H = \frac{1}{2}p_f^* p_f + \frac{1}{2}L = \frac{1}{2}\Big(A - \frac{\mathrm{i}}{2}(\Delta f)\Big)\Big(A + \frac{\mathrm{i}}{2}(\Delta f)\Big) + \frac{1}{2}L.$$

We omit the rest of the computations. □

Problem. Show that on the set $\{x \in \Omega;\ f(x) \geq r_0\}$

$$0 \leq \ell \leq 1, \quad \ell_{ij}(\partial_j f) = 0, \quad Lf = 0, \quad q = \frac{(d-1)(d-3)}{8f^2}.$$

Lemma 5.7. For each $t \geq 0$, $\mathrm{e}^{-\mathrm{i}tA}$ is bounded as $\mathcal{H}^1 \to \mathcal{H}^1$, and

$$\sup_{t\in[0,1]} \|\mathrm{e}^{-\mathrm{i}tA}\|_{\mathcal{B}(\mathcal{H}^1)} < \infty. \tag{$\diamondsuit$}$$

Moreover, $\mathrm{e}^{-\mathrm{i}tA}$ is strongly continuous in $t \geq 0$ in $\mathcal{B}(\mathcal{H}^1)$.

Proof. By Proposition 5.4 and the chain rule we can write for any $\psi \in \mathcal{D}(\Omega)$, $(-t, x) \in \mathcal{M}$ and $i = 1, \dots, d$

$$\begin{aligned} p_i(\mathrm{e}^{-\mathrm{i}tA}\psi)(x) = {} & \frac{1}{2\mathrm{i}}(\mathrm{e}^{-\mathrm{i}tA}\psi)(x)\int_0^{-t}(\partial_i y_\alpha(s,x))(\partial_\alpha \Delta f)(y(s,x))\,\mathrm{d}s \\ & + (\mathrm{e}^{-\mathrm{i}tA}p_\alpha\psi)(x)(\partial_i y_\alpha(-t,x)). \end{aligned}$$

Since derivatives of y and f are bounded, we can see from the above expression that $\mathrm{e}^{-\mathrm{i}tA}$ for each $t \geq 0$ is bounded as $\mathcal{H}^1 \to \mathcal{H}^1$. We can also see that for each $\psi \in \mathcal{D}(\Omega)$ the $\mathcal{H}^1$-valued function $\mathrm{e}^{-\mathrm{i}tA}\psi$ is continuous in $t \geq 0$.

Now it remains to show ($\diamondsuit$), since then we can also deduce the strong continuity of $\mathrm{e}^{-\mathrm{i}tA} \in \mathcal{B}(\mathcal{H}^1)$ in $t \geq 0$ by density argument. Let us show that there exists $C_1 > 0$ such that for any $\psi \in \mathcal{D}(\Omega)$ and $t \in [0,1]$

$$f(t) := \langle \mathrm{e}^{-\mathrm{i}tA}\psi, (H+1)\mathrm{e}^{-\mathrm{i}tA}\psi\rangle \leq C_1\|\psi\|^2_{\mathcal{H}^1}.$$

In fact, noting that

$$[H, \mathrm{i}A] = 2p_i(\partial_i\partial_j f)p_j + (\partial_f q) + \frac{1}{2}(L\Delta f), \tag{$\clubsuit$}$$

we have

$$f'(t) = -\langle \mathrm{e}^{-\mathrm{i}tA}\psi, [H, \mathrm{i}A]\mathrm{e}^{-\mathrm{i}tA}\psi\rangle \leq C_2\|\mathrm{e}^{-\mathrm{i}tA}\psi\|^2_{\mathcal{H}^1} \leq C_3 f(t).$$

This leads to $f(t) \leq f(0)\mathrm{e}^{C_3 t}$, hence to ($\diamondsuit$). □

Problem. Verify the identity ($\clubsuit$).

◦ Commutator and C_0-semigroup

Here we formulate a weighted commutator

$$[H, \mathrm{i}A]_\Theta := \mathrm{i}(H\Theta A - A\Theta H)$$

first as a (quadratic) form on $\mathcal{D}(\Omega)$, and then extend it as a bounded form on $\mathcal{H}^1$, see Proposition 5.9.

A weight Θ will be given explicitly when applied in Section 5.3, but for simplicity we for the moment assume only the following.

Assumption 5.8. A weight $\Theta = \Theta(f)$ is a smooth function only of f, and satisfies

$$f \geq r_0 \text{ on } \operatorname{supp}\Theta, \quad \Theta \geq 0, \quad |\Theta^{(k)}| \leq C_k \text{ for any } k \in \mathbb{N}_0,$$

where $\Theta^{(k)}$ denotes the k-th derivative of Θ in f.

Proposition 5.9. Under Assumption 5.8, as a form on $\mathcal{D}(\Omega)$,

$$[H, \mathrm{i}A]_\Theta = A\Theta' A + f^{-1}\Theta L - \frac{1}{4}\Theta''' - (\partial_f q)\Theta - \mathrm{Re}(\Theta' H).$$

Therefore $[H, \mathrm{i}A]_\Theta$ extends as a bounded form on $\mathcal{H}^1$, or equivalently as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$.

Remarks. 1. As for the second term, note by Assumption 5.8

$$f^{-1}\Theta L = L f^{-1}\Theta = p_i f^{-1}\Theta \ell_{ij} p_j.$$

2. In the **Mourre theory** the **conjugate operator** A is usually chosen as the generator of dilations.

Proof. By Lemma 5.6

$$\begin{aligned}[H, \mathrm{i}A]_\Theta &= \frac{1}{2}[A^2, \mathrm{i}A]_\Theta + \frac{1}{2}[L, \mathrm{i}A]_\Theta + [q, \mathrm{i}A]_\Theta \\ &= \frac{1}{2}A\Theta' A + \frac{1}{2}[L, \mathrm{i}A]_\Theta - (\partial_f q)\Theta - q\Theta'.\end{aligned}$$

Let us compute the second term on the last line as

$$\frac{1}{2}[L, \mathrm{i}A]_\Theta = \cdots = f^{-1}\Theta L - \frac{1}{2}\Theta' L,$$

so that

$$\begin{aligned}[H, \mathrm{i}A]_\Theta &= \frac{1}{2}A\Theta' A + f^{-1}\Theta L - \frac{1}{2}\Theta' L - (\partial_f q)\Theta - q\Theta' \\ &= \cdots = A\Theta' A + f^{-1}\Theta L - \frac{1}{4}\Theta''' - (\partial_f q)\Theta - \mathrm{Re}(\Theta' H).\end{aligned}$$

Hence the assertion is verified. □

Problem. Complete missing details of the above computations.

Next, we present an alternative expression for $[H, \mathrm{i}A]_\Theta$ employing the C_0-semigroup $\mathrm{e}^{-\mathrm{i}tA}$. We introduce an auxiliary operator

$$H_\Theta = \frac{1}{2} p_i \Theta p_i.$$

Lemma 5.10. Under Assumption 5.8 one has

$$[H, \mathrm{i}A]_\Theta = [H_\Theta, \mathrm{i}A] + A\Theta' A + \frac{1}{4}(\partial_f \Delta f)\Theta' + \frac{1}{4}(\Delta f)\Theta'', \qquad (\spadesuit)$$

and for any $\psi \in \mathcal{H}^1$

$$\langle \psi, [H_\Theta, \mathrm{i}A]\psi\rangle = \lim_{t\to+0} t^{-1}\left\langle \psi, \left(H_\Theta - \mathrm{e}^{\mathrm{i}tA} H_\Theta \mathrm{e}^{-\mathrm{i}tA}\right)\psi\right\rangle. \qquad (\heartsuit)$$

Proof. Step 1. The identity ($\spadesuit$) is due to a direct computation. The proof is omitted.

Step 2. To prove ($\heartsuit$) we claim that there exists $C_1 > 0$ such that for any $t \in [0, 1]$ and $\psi, \phi \in \mathcal{H}^1$

$$\left|\left\langle \phi, \left(H_\Theta - \mathrm{e}^{\mathrm{i}tA} H_\Theta \mathrm{e}^{-\mathrm{i}tA}\right)\psi\right\rangle\right| \le C_1 t \|\phi\|_{\mathcal{H}^1} \|\psi\|_{\mathcal{H}^1}.$$

In fact, by Proposition 5.5 we can write

$$H_\Theta - \mathrm{e}^{\mathrm{i}tA} H_\Theta \mathrm{e}^{-\mathrm{i}tA} = \int_0^t \mathrm{e}^{\mathrm{i}sA}[H_\Theta, \mathrm{i}A]\mathrm{e}^{-\mathrm{i}sA}\,\mathrm{d}s$$

as a form on $\mathcal{D}(\Omega)$. It is easy to see that H_Θ and $[H_\Theta, \mathrm{i}A]$ extend as bounded forms on $\mathcal{H}^1$ from $\mathcal{D}(\Omega)$. By using Lemma 5.7 the claim is verified first on $\mathcal{D}(\Omega)$ and then on $\mathcal{H}^1$ by continuity.

Step 3. It suffices to show (♡) for $\psi \in \mathcal{D}(\Omega)$ due to density argument employing Step 2 and continuity of the form $[H_\ominus, \mathrm{i}A]$ on $\mathcal{H}^1$. For any $\psi \in \mathcal{D}(\Omega)$ write

$$t^{-1}\left\langle \psi, \left(H_\ominus - \mathrm{e}^{\mathrm{i}tA}H_\ominus \mathrm{e}^{-\mathrm{i}tA}\right)\psi\right\rangle - \langle \psi, [H_\ominus, \mathrm{i}A]\psi\rangle$$
$$= t^{-1}\int_0^t \left\{\left\langle \mathrm{e}^{-\mathrm{i}sA}\psi, [H_\ominus, \mathrm{i}A]\mathrm{e}^{-\mathrm{i}sA}\psi\right\rangle - \langle \psi, [H_\ominus, \mathrm{i}A]\psi\rangle\right\} \mathrm{d}s.$$

Then we obtain the assertion by Lemma 5.7. □

○ **Undoing commutator**

In the following we use the notation

$$\mathrm{Im}(A\Theta H) = \frac{1}{2\mathrm{i}}(A\Theta H - H\Theta A)$$

exclusively as a form on $D(H)$, i.e., for any $\psi \in D(H)$

$$\langle \psi, \mathrm{Im}(A\Theta H)\psi\rangle = \frac{1}{2\mathrm{i}}\Big(\langle A\psi, \Theta H\psi\rangle - \langle H\psi, \Theta A\psi\rangle\Big).$$

Proposition 5.11. Under Assumption 5.8, as forms on $D(H)$,

$$[H, \mathrm{i}A]_\ominus \le 2\,\mathrm{Im}(A\Theta H).$$

Remark. The above forms coincide on $\mathcal{D}(\Omega)$, but not in general on $D(H)$ due to a contribution from boundary. Fortunately, here the contribution has a sign. We also note it vanishes if ∇f is both forward and backward complete.

Proof. Similarly to Lemma 5.10, we can write

$$2\,\mathrm{Im}(A\Theta H) = 2\,\mathrm{Im}(AH_\ominus) + A\Theta' A + \frac{1}{4}(\partial_f \Delta f)\Theta' + \frac{1}{4}(\Delta f)\Theta''$$

as a form on $D(H)$. Then by Lemma 5.10 it suffices to show

$$[H_\ominus, \mathrm{i}A] \le 2\,\mathrm{Im}(AH_\ominus)$$

as forms on $D(H)$.

Let us write, as a form on $\mathcal{H}^1$,

$$H_\ominus - \mathrm{e}^{\mathrm{i}tA}H_\ominus \mathrm{e}^{-\mathrm{i}tA} = H_\ominus\left(1 - \mathrm{e}^{-\mathrm{i}tA}\right) + \left(1 - \mathrm{e}^{\mathrm{i}tA}\right)H_\ominus$$
$$- \left(1 - \mathrm{e}^{\mathrm{i}tA}\right)H_\ominus\left(1 - \mathrm{e}^{-\mathrm{i}tA}\right).$$

Then by Lemma 5.10 and Proposition 5.5 for any $\psi \in D(H)$

$$\langle \psi, [H_\ominus, \mathrm{i}A]\psi\rangle \le \lim_{t\to+0} t^{-1}\Big\{\left\langle H_\ominus\psi, \left(1 - \mathrm{e}^{-\mathrm{i}tA}\right)\psi\right\rangle$$
$$+ \left\langle \left(1 - \mathrm{e}^{-\mathrm{i}tA}\right)\psi, H_\ominus\psi\right\rangle\Big\}$$
$$= \langle H_\ominus\psi, \mathrm{i}A\psi\rangle + \langle \mathrm{i}A\psi, H_\ominus\psi\rangle$$
$$= \langle \psi, 2\,\mathrm{Im}(AH_\ominus)\psi\rangle.$$

Hence we are done. □

§ 5.3 Proof of Main Theorem

∘ **A priori super-exponential decay estimates**

Proposition 5.12. If $\phi \in \mathcal{H}_{\mathrm{loc}}$ and $\lambda > 0$ satisfy

1. $(H-\lambda)\phi = 0$ in the distributional sense,
2. there exists $l \in \mathbb{N}_0$ such that $\bar{\chi}_l \phi \in \mathcal{B}_0^* \cap \mathcal{N}$,

then $\bar{\chi}_l \mathrm{e}^{\alpha f}\phi \in \mathcal{B}_0^*$ for any $\alpha \geq 0$.

Now we introduce an explicit weight with parameters $\alpha, \beta, R \geq 0$ and $m, n \in \mathbb{N}_0$:

$$\Theta = \Theta_{m,n,R}^{\alpha,\beta} = \chi_{m,n}\mathrm{e}^{\theta}.$$

Here the exponent θ is given by

$$\theta = \theta_R^{\alpha,\beta} = 2\alpha f + 2\beta f(1+f/R)^{-1}.$$

cf. Yosida approximation. Set for notational simplicity

$$\theta_0 = 1 + f/R,$$

and then, for example,

$$\theta' = 2\alpha + 2\beta\theta_0^{-2}, \quad \theta'' = -4\beta R^{-1}\theta_0^{-3}, \quad \ldots.$$

In particular, noting $R^{-1}\theta_0^{-1} \leq f^{-1}$, we have

$$|\theta^{(k)}| \leq C_k \beta f^{1-k}\theta_0^{-2} \quad \text{for } k = 2, 3, \ldots.$$

Lemma 5.13. Let $\lambda > 0$, and fix any $\alpha_0 \geq 0$. Then there exist $\beta, c, C, R_0 > 0$ and $n_0 \in \mathbb{N}_0$ such that for any $\alpha \in [0, \alpha_0]$, $n > m \geq n_0$ and $R \geq R_0$,

$$\begin{aligned}\mathrm{Im}(A\Theta(H-\lambda)) \geq{}& cf^{-1}\theta_0^{-1}\Theta - C\big(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2\big)f^{-1}\mathrm{e}^{\theta}\\ &+ \mathrm{Re}(\gamma(H-\lambda))\end{aligned}$$

as forms on $D(H)$, where $\gamma = \gamma_{m,n,R}$ is a function satisfying

$$\mathrm{supp}\,\gamma \subset \mathrm{supp}\,\chi_{m,n}, \quad |\gamma| \leq C_{m,n}\mathrm{e}^{\theta}.$$

Proof. Let $\lambda > 0$. To be rigorous all the estimates in Step 1 below are uniform in $\alpha \geq 0$, $\beta \in [0,1]$, $n > m \geq 0$ and $R \geq 0$ with constants $c_*, C_* > 0$ being independent of them. Finally in Step 2 we restrict the parameter ranges to verify the assertion.

Step 1. By Lemmas 5.11 and 5.9 and the Cauchy–Schwarz inequality we can estimate

$$\begin{aligned}&\mathrm{Im}(A\Theta(H-\lambda))\\ &\geq \tfrac{1}{2}A\theta'\Theta A + \tfrac{1}{2}f^{-1}\Theta L\\ &\quad - \tfrac{1}{8}\theta'^3\Theta - \tfrac{3}{8}\theta'\theta''\Theta - \tfrac{1}{2}\mathrm{Re}(\Theta'(H-\lambda)) - C_1 Q\\ &\geq \tfrac{1}{2}c_1 A f^{-1}\theta_0^{-1}\Theta A + \tfrac{1}{2}c_1 f^{-1}\theta_0^{-1}\Theta L\\ &\quad + \tfrac{1}{2}A\big(\theta' - c_1 f^{-1}\theta_0^{-1}\big)\Theta A + \tfrac{1}{4}f^{-1}\Theta\big(1 - 2c_1\theta_0^{-1}\big)L\\ &\quad - \tfrac{1}{8}\theta'^3\Theta - \tfrac{3}{8}\theta'\theta''\Theta - \tfrac{1}{2}\mathrm{Re}(\Theta'(H-\lambda)) - C_2 Q.\end{aligned} \qquad (\diamondsuit)$$

Here $c_1 > 0$ is chosen small enough that the fourth term on the right-hand side of ($\diamondsuit$) is non-negative. We have also absorbed 'admissible error terms' into

$$Q = \Big[(1+\alpha^2)f^{-2}\chi_{m,n} + (1+\alpha^2)|\chi'_{m,n}| + (1+\alpha)|\chi''_{m,n}| + |\chi'''_{m,n}|\Big]\mathrm{e}^\theta + p_i\Big(f^{-2}\chi_{m,n} + |\chi'_{m,n}|\Big)\mathrm{e}^\theta p_i,$$

which will be bounded later. Let us further compute and bound terms on the right-hand side of ($\diamondsuit$). By Lemma 5.6 the first and second terms of ($\diamondsuit$) get to be

$$\begin{aligned}&\tfrac{1}{2}Af^{-1}\theta_0^{-1}\Theta A + \tfrac{1}{2}f^{-1}\theta_0^{-1}\Theta L\\ &\geq \tfrac{1}{2}\mathrm{Im}\Big(f^{-1}\theta_0^{-1}\theta'\Theta A\Big) + \tfrac{1}{2}\mathrm{Re}\Big(f^{-1}\theta_0^{-1}\Theta(A^2+L)\Big) - C_3Q\\ &\geq (\lambda - q)f^{-1}\theta_0^{-1}\Theta + \tfrac{1}{4}f^{-1}\theta_0^{-1}\theta'^2\Theta + \mathrm{Re}\Big(f^{-1}\theta_0^{-1}\Theta(H-\lambda)\Big)\\ &\quad - C_4Q.\end{aligned}$$

We combine the third, fifth and sixth terms of ($\diamondsuit$) as

$$\begin{aligned}&\tfrac{1}{2}A\Big(\theta' - c_1f^{-1}\theta_0^{-1}\Big)\Theta A - \tfrac{1}{8}\theta'^3\Theta - \tfrac{3}{8}\theta'\theta''\Theta\\ &\geq \tfrac{1}{2}\Big(A + \tfrac{\mathrm{i}}{2}\theta'\Big)\Big(\theta' - c_1f^{-1}\theta_0^{-1}\Big)\Theta\Big(A - \tfrac{\mathrm{i}}{2}\theta'\Big)\\ &\quad - \tfrac{1}{8}c_1f^{-1}\theta_0^{-1}\theta'^2\Theta + \tfrac{1}{8}\theta'\theta''\Theta - C_5Q.\end{aligned}$$

Substitute these bounds into ($\diamondsuit$), and we deduce

$$\begin{aligned}&\mathrm{Im}\Big(A\Theta(H-\lambda)\Big)\\ &\geq c_1(\lambda-q)f^{-1}\theta_0^{-1}\Theta + \tfrac{1}{8}c_1f^{-1}\theta_0^{-1}\theta'^2\Theta + \tfrac{1}{8}\theta'\theta''\Theta\\ &\quad + \tfrac{1}{2}\Big(A + \tfrac{\mathrm{i}}{2}\theta'\Big)\Big(\theta' - c_1f^{-1}\theta_0^{-1}\Big)\Theta\Big(A - \tfrac{\mathrm{i}}{2}\theta'\Big)\\ &\quad + \tfrac{1}{2}\mathrm{Re}\Big[\Big(2c_1f^{-1}\theta_0^{-1}\Theta - \Theta'\Big)(H-\lambda)\Big] - C_6Q.\end{aligned}$$

Now we bound Q as

$$\begin{aligned}Q &\leq C_7(1+\alpha^2)f^{-2}\Theta + C_7(1+\alpha^2)\Big(\chi^2_{m-1,m+1} + \chi^2_{n-1,n+1}\Big)f^{-1}\mathrm{e}^\theta\\ &\quad + 2\,\mathrm{Re}\Big[\Big(f^{-2}\chi_{m,n} + |\chi'_{m,n}|\Big)\mathrm{e}^\theta(H-\lambda)\Big].\end{aligned}$$

Then we finally obtain

$$\begin{aligned}&\mathrm{Im}(A\Theta(H-\lambda))\\ &\geq \Big[c_1(\lambda-q)f^{-1}\theta_0^{-1} + \tfrac{1}{8}c_1f^{-1}\theta_0^{-1}\theta'^2 + \tfrac{1}{8}\theta'\theta''\\ &\quad - C_8(1+\alpha^2)f^{-2}\Big]\Theta\\ &\quad + \tfrac{1}{2}\Big(A + \tfrac{\mathrm{i}}{2}\theta'\Big)\Big(\theta' - c_1f^{-1}\theta_0^{-1}\Big)\Theta\Big(A - \tfrac{\mathrm{i}}{2}\theta'\Big)\\ &\quad - C_8(1+\alpha^2)\Big(\chi^2_{m-1,m+1} + \chi^2_{n-1,n+1}\Big)f^{-1}\mathrm{e}^\theta\\ &\quad + \mathrm{Re}(\gamma(H-\lambda))\end{aligned} \qquad (\clubsuit)$$

with

$$\gamma = c_1f^{-1}\theta_0^{-1}\Theta - \tfrac{1}{2}\Theta' - 2C_6f^{-2}\Theta - 2C_6|\chi'_{m,n}|\mathrm{e}^\theta.$$

Step 3. Fix any $\alpha_0 \geq 0$. Choose $\beta \in (0,1]$ small and $n_0 \in \mathbb{N}_0$ large. Then the first term of ($\clubsuit$) is bounded below uniformly in $\alpha \in [0,\alpha_0]$, $n > m \geq n_0$ and $R \geq 0$ as

$$\begin{aligned}&\Big[c_1(\lambda-q)f^{-1}\theta_0^{-1} + \tfrac{1}{8}c_1f^{-1}\theta_0^{-1}\theta'^2 + \tfrac{1}{8}\theta'\theta'' - C_8(1+\alpha^2)f^{-2}\Big]\Theta\\ &\geq \Big[c_2f^{-1}\theta_0^{-1} - C_9\beta f^{-1}\theta_0^{-2} - C_9f^{-2}\Big]\Theta\\ &\geq c_3f^{-1}\theta_0^{-1}\Theta.\end{aligned}$$

Next, since

$$\theta' - c_1f^{-1}\theta_0^{-1} \geq 2\beta\theta_0^{-2} - C_{10}f^{-1}\theta_0^{-1},$$

by taking $R_0 > 0$ large enough the second term of (♣) is non-negative for any $\alpha \in [0, \alpha_0]$, $n > m \geq n_0$ and $R \geq R_0$. Hence the desired bound is obtained. □

Proof of Proposition 5.12. Let $\lambda > 0$, $\phi \in \mathcal{H}_{\rm loc}$ and $l \in \mathbb{N}_0$ be as in the assertion, and set

$$\alpha_0 = \sup\{\alpha \geq 0;\ \bar\chi_l e^{\alpha f}\phi \in \mathcal{B}_0^*\}.$$

Assume $\alpha_0 < \infty$, and we choose $\beta, R_0 > 0$ and $n_0 \geq 0$ as in Lemma 5.13. Note that we may assume $n_0 \geq l + 3$, so that for all $n > m \geq n_0$

$$\chi_{m-2,n+2}\phi \in D(H).$$

We let $\alpha \in \{0\} \cup [0, \alpha_0)$ such that $\alpha + \beta > \alpha_0$.

With these parameters fixed evaluate the form inequality from Lemma 5.13 on the state $\chi_{m-2,n+2}\phi \in D(H)$. Then for any $n > m \geq n_0$ and $R \geq R_0$

$$\begin{aligned}\left\|(f^{-1}\theta_0^{-1}\Theta)^{1/2}\phi\right\|^2 &\leq C_m\|\chi_{m-1,m+1}\phi\|^2 \\ &\quad + C_R 2^{-n}\|\chi_{n-1,n+1}e^{\alpha f}\phi\|^2.\end{aligned}$$

The above second term vanishes as $n \to \infty$, and consequently by Lebesgue's monotone convergence theorem

$$\left\|(\bar\chi_m f^{-1}\theta_0^{-1}e^{\theta})^{1/2}\phi\right\|^2 \leq C_m\|\chi_{m-1,m+1}\phi\|^2.$$

Next we let $R \to \infty$. Again by Lebesgue's monotone convergence theorem it follows that

$$\bar\chi_m^{1/2} f^{-1/2} e^{(\alpha+\beta)f}\phi \in \mathcal{H}.$$

Thus $\bar\chi_m^{1/2} e^{\kappa r}\phi \in \mathcal{B}_0^*$ for any $\kappa \in (0, \alpha + \beta)$, but this is a contradiction, since $\alpha + \beta > \alpha_0$. We are done. □

◦ **Absence of super-exponentially decaying eigenstates**

Proposition 5.14. If $\phi \in \mathcal{H}_{\rm loc}$ and $\lambda > 0$ satisfy

1. $(H - \lambda)\phi = 0$ in the distributional sense,
2. there exists $l \in \mathbb{N}_0$ such that $\bar\chi_l e^{\alpha f}\phi \in \mathcal{B}_0^* \cap \mathcal{N}$ for any $\alpha \geq 0$,

then $\phi \equiv 0$ in Ω.

The proof is very similar to that of Proposition 5.12. Here we choose

$$\Theta = \Theta_{m,n}^{\alpha} = \chi_{m,n}e^{2\alpha f},$$

formally letting $\beta = 0$ and $R \to \infty$ in the previous $\Theta = \Theta_{m,n,R}^{\alpha,\beta}$.

Lemma 5.15. Let $\lambda > 0$ and $\alpha_0 > 0$. Then there exist $c, C > 0$ and $n_0 \geq 0$ such that for any $\alpha > \alpha_0$ and $n > m \geq n_0$,

$$\begin{aligned}\mathrm{Im}(A\Theta(H-\lambda)) \geq\ & c\alpha^2 f^{-1}\Theta \\ & - C\alpha^2\big(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2\big)f^{-1}e^{2\alpha f} \\ & + \mathrm{Re}(\gamma(H-\lambda))\end{aligned}$$

as forms on $D(H)$, where $\gamma = \gamma_{m,n}$ is a function satisfying

$$\mathrm{supp}\,\gamma \subset \mathrm{supp}\,\chi_{m,n}, \quad |\gamma| \leq C_{m,n}\alpha e^{2\alpha f}.$$

Proof. We can prove it similarly to Lemma 5.13, and in fact it is slightly easier. We omit the proof. □

Proof of Proposition 5.14. Let $\lambda > 0$, $\phi \in \mathcal{H}_{\text{loc}}$ and $l \in \mathbb{N}_0$ be as in the assertion. Fix any $\alpha_0 > 0$, and choose $n_0 \geq 0$ as in Lemma 5.15. We may assume that $n_0 \geq l + 3$, so that for all $n > m \geq n_0$

$$\chi_{m-2,n+2}\phi \in D(H).$$

Evaluate the form inequality from Lemma 5.15 on the state $\chi_{m-2,n+2}\phi \in D(H)$, and then for any $\alpha > \alpha_0$ and $n > m \geq n_0$

$$\|f^{-1/2}\Theta^{1/2}\phi\|^2 \leq C_1\|\chi_{m-1,m+1}\mathrm{e}^{\alpha f}\phi\|^2 + C_1 2^{-n}\|\chi_{n-1,n+1}\mathrm{e}^{\alpha f}\phi\|^2.$$

The above second term to the right vanishes as $n \to \infty$, and hence by Lebesgue's monotone convergence theorem

$$\left\|\bar{\chi}_m^{1/2} f^{-1/2}\mathrm{e}^{\alpha(f-2^{m+2})}\phi\right\|^2 \leq C_1\|\chi_{m-1,m+1}\phi\|^2.$$

Now assume $\bar{\chi}_{m+2}\phi \not\equiv 0$. The left-hand side grows exponentially as $\alpha \to \infty$ whereas the right-hand side remains bounded. This is a contradiction. Thus

$$\bar{\chi}_{m+2}\phi \equiv 0.$$

Now by the unique continuation property for the second order elliptic operator H we conclude that $\phi \equiv 0$ globally on Ω. □