A SEMIGROUP APPROACH TO THE STRONG ERGODIC THEOREM OF THE MULTISTATE STABLE POPULATION PROCESS

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In this paper we first formulate the dynamics of multistate stable population processes as a partial differential equation. Next, we rewrite this equation as an abstract differential equation in a Banach space, and solve it by using the theory of strongly continuous semigroups of bounded linear operators. Subsequently, we investigate the asymptotic behavior of this semigroup to show the strong ergodic theorem which states that there exists a stable distribution independent of the initial distribution. Finally, we introduce the dual problem in order to obtain a logical definition for the reproductive value and we discuss its applications.

KEY WORDS: semigroup, ergodicity, multistate process, differential equation, reproductive value.

CONTENTS

1. The multistate stable population process
2. Spectral properties of the population operator and its semigroup
3. Asymptotic behavior of the population semigroup: The Strong Ergodic Theorem
4. The dual problem
References
Appendix
1. THE MULTISTATE STABLE POPULATION PROCESS

We consider here a sufficiently large closed one-sex population which is divided into n "states". The states may correspond to geographical regions, social status, physiological features, or other classifications. Let \( p_i(a,t) \), 1 \( \leq \) \( i \) \( \leq \) \( n \), be the subpopulation in the age interval \( (a,a+da) \) at time \( t \) in the \( i \)th state. We define the population vector as:

\[
p(a,t) \equiv (p_1(a,t), \ldots, p_n(a,t))^T,
\]

where "\( T \)" represents the transpose of the vector. Let \( Q(a) \) be the state-transition rate matrix, in which the \( (i,j) \)th \( (i \neq j) \) element \( q_{ij}(a) \geq 0 \) is the instantaneous transition rate at age \( a \) from state \( j \) to state \( i \), and the diagonal element \( q_{ii}(a) \) is given by \( q_{ii}(a) = -\mu_i(a) - \sum_{j \neq i} q_{ji}(a) \), where \( \mu_i(a) \geq 0 \) is the instantaneous death rate at age \( a \) in state \( i \), called the force of mortality in demographic terminology. Let \( M(a) \) be the fertility matrix, in which each entry \( m_{ij}(a) \geq 0 \) is the average number of offsprings of state \( i \) per unit time produced by an individual at age \( a \) in state \( j \). Then the multistate stable population process is described by the vector-type Lotka-von Foerster system (Inaba, 1984) as:

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p(a,t) = Q(a)p(a,t), \tag{1.1a}
\]

\[
p(0,t) = \int_0^{a_{\tau}} M(a)p(a,t) \, da, \tag{1.1b}
\]

\[
p(a,0) = k(a), \tag{1.1c}
\]

where \( k(a) = (k_1(a), \ldots, k_n(a))^T \) is the initial population vector, and \( a_{\tau} \) denotes the maximum reproductive age, i.e., \( M(a) = 0 \), for \( a \geq a_{\tau} \). The equation (1.1a) is a multistate version of the McKendrick-von Foerster equation, which describes the aging process of the population (McKendrick, 1926; von Foerster, 1959). The equation (1.1b), called "renewal equation", describes the birth process of the population. The model appear in various contexts. In what follows, we shall give some examples from human demography and cell biology.

Example 1: Multiregional demographic model

Let \( p_i(a,t) \) be the resident population of the \( i \)th geographical region in the age interval \( (a,a+da) \) at time \( t \), and let \( q_{ij}(a) \) \( (i \neq j) \) be the
instantaneous migration rate from the \( j \)th region to the \( i \)th region. Furthermore, \( M(a) \) is a diagonal matrix given by

\[
M(a) \triangleq \begin{pmatrix}
    m_1(a) & 0 & \\
    0 & \ddots & 0 \\
    0 & & m_n(a)
\end{pmatrix},
\]

where \( m_i(a) \) is the fertility function of residents in the \( i \)th region. In this case, our system (1.1a)–(1.1c) describes the multiregional demographic model with continuous time proposed by Le Bras (1971) and Rogers (1975).

**Example 2: Two sex linear population models**

Let \( p(a,t) = (p_m(a,t), p_f(a,t))^T \), where \( p_m(a,t) \) is the male population and \( p_f(a,t) \) the female population. Let \( Q(a) \) be a \( 2 \times 2 \) matrix such that

\[
Q(a) \triangleq \begin{pmatrix}
    -\mu_m(a) & 0 \\
    0 & -\mu_f(a)
\end{pmatrix},
\]

where \( \mu_m(a) \) is the force of mortality of the male population and \( \mu_f(a) \) is the force of mortality of the female population. Then the system (1.1a)–(1.1c) represents a two sex linear population model proposed by Bartlett (1970). If \( m_{11}(a) = 0, m_{22}(a) = 0 \), then we obtain a very artificial two sex model given by Pollard (1948) (J. H. Pollard, 1973). Further, if \( m_{11}(a) = 0, m_{21}(a) = 0 \), then we have a female dominance model due to Goodman (1967).

**Example 3: Budding yeast population model**

Hamada, et al. (1982) proposed a very interesting model of budding yeast population. They say: "The mother of daughter cells produced by a mother of budding yeast, *Saccharomyces cerevisiae*, can be identified by counting bud scars left on the cell wall of the mother. We say that a cell is of stage \( n \) when it has \( n \) bud scars" (Hamada, et al., 1982, p. 393).

Let \( z_i(t,s)ds \) be the number of cells in stage \( i \) at time \( t \) who have spent time between \( s \) and \( s + ds \) since the beginning of the cell cycle they are in. The parameter \( s \) is called "cycle phase". Then the Hamada–Kanno–Kano model (HKK model) of the stage structured
A yeast population is described by

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) x_i(t, s) = -\lambda_i(s)x_i(t, s), \quad i \geq 0,
\]

\[
x_i(t, 0) = \int_0^\infty \lambda_{i-1}(s)x_{i-1}(t, s) \, ds, \quad i \geq 1,
\]

\[
x_0(t, 0) = \sum_{j \geq 1} x_j(t, 0),
\]

where \(\lambda_i(s)\) (\(i \geq 0\)), called the reproduction rate, is the probability per unit time of a cell of stage \(i\) producing a daughter. If we assume that cells in stage \(n\) lose their ability of reproduction, then (HKK) model is reduced to the system

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) x(t, s) = Q(s)x(t, s),
\]

\[
x(t, 0) = \int_0^\infty M(s)x(t, s) \, ds,
\]

\[
x(0, s) = k(s),
\]

where \(x(t, s) = (x_0(t, s), \ldots, x_{n-1}(t, s))^\tau\),

\[
Q(s) \overset{df}{=} \begin{pmatrix}
-\lambda_0(s) & 0 \\
\vdots & \ddots & \ddots \\
0 & -\lambda_{n-1}(s) & \vdots
\end{pmatrix},
\]

\[
M(s) \overset{df}{=} \begin{pmatrix}
\lambda_0(s) & \ldots & \lambda_{n-1}(s) \\
\lambda_0(s) & 0 & \vdots \\
0 & \lambda_{n-2}(s) & 0
\end{pmatrix},
\]

and \(k(s)\) is a given initial condition. The reader may find another formulation of the model in Gyllenberg (1986).

We now introduce some notations: Let \(C^n\) denote the \(n\)-dimensional vector space with norm

\[
|x| \overset{df}{=} \sum_{i=1}^n |x_i|, \quad x = (x_1, \ldots, x_n)^\tau \in C^n.
\]
Let $B(C^n, C^n)$ be the linear space of $n \times n$ matrices with norm
\[
|A| \overset{\text{def}}{=} \sup_{|x|=1} |Ax|, \quad A \in B(C^n, C^n).
\]
It is easy to see that if $a_{ij}$ is the $(i, j)$th element of $A \in B(C^n, C^n)$, then
\[
|A| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|.
\]
(1.2)

Let $L^1 = L^1(0, \omega; C^n)$ be the Banach space of equivalence classes of Lebesgue integrable functions from $[0, \omega]$ to $C^n$ which agree almost everywhere on $(0, \omega)$, with norm
\[
\|\phi\| \overset{\text{def}}{=} \int_0^\omega |\phi(a)| da, \quad \phi \in L^1.
\]

Let $L^1_+$ be the positive cone of $L^1$; $L^1_+ \overset{\text{def}}{=} \{\phi; \phi \in L^1, \text{ a.e. } \phi \geq 0\}$. Let $L^\infty(0, \omega; E)$ be the Banach space of essentially bounded functions from $[0, \omega]$ to $E$, where $E = C^n$, $E = R$ etc., and let $L^\infty_+(0, \omega; E)$ be the positive cone of $L^\infty$. From now on, we adopt the following assumptions:

\textbf{(H-1).} Let $a_n$ be the maximum life span of the population and let $a_r < a_m \leq +\infty$.

\textbf{(H-2).} The state space of age distribution functions $p(\cdot, t)$ is $L^1(0, \omega; C^n)$, where $\omega$ is a real number such that $0 < a_r \leq \omega < a_m$.

\textbf{(H-3).} $\mu_i(a)$, $q_{ij}(a)$ ($i \neq j$, $1 \leq i, j \leq n$) are nonnegative continuous functions on $[0, \omega]$.

\textbf{(H-4).} $m_{ij}(a) \in L^\infty_+(0, \omega; R)$ for $1 \leq i, j \leq n$, $m_{ij}(a) = 0$ for $a \geq a_r$, $1 \leq i, j \leq n$.

The assumption (H-1) implies that the population considered here can survive beyond its finite maximum reproductive age. This assumption is appropriate for human demographic applications. Furthermore, the assumption $\omega < a_m$ is needed to make $\mu_i(a)$ ($1 \leq i \leq n$) bounded on the age space $[0, \omega]$, which simplifies the following arguments. For, $\mu_i(a)$ has a non-integrable singularity in $a = a_m$, i.e., $\int_0^{a_m} \mu_i(a) da = +\infty$, because the survival rate function $\exp(-\int_0^a \mu_i(\rho) d\rho)$ must be equal to zero in $a = a_m$. If $\omega = a_m$, then we must adopt the weighted $L^1$ space as the state space of age distributions instead of $L^1$ in order to avoid this singularity (Diekmann, et al., 1984). Under the assumption (H-1), however, the assumption
(H-2) does not reduce the value of the method of investigation. In fact, the population alive in the postreproductive age cannot affect the numbers in the reproductive age groups at a later point of time.

Here we introduce the survival rate matrix $L(a)$, which is the multi-state analog of the survival rate function in a single-state population. $L(a)$ is defined as the solution of the matrix differential equation:

$$\frac{d}{da} L(a) = Q(a) L(a), \quad L(0) = I,$$

(1.2)

where $I$ denotes the $n \times n$ identity matrix. From the theory of ordinary differential equations, we know that the survival rate matrix $L(a)$ is uniquely determined by the system (1.2), and the following holds.

$$\det L(a) = \exp \left( \int_0^a \sum_{i=1}^n q_{ii}(\zeta) d\zeta \right).$$

(1.3)

From (1.3), we know that $\det L(a) \neq 0$ for all $a \in [0, \omega]$, and the inverse matrix $L^{-1}(a)$ always exists for all $a \in [0, \omega]$. The $(i, j)$th element of $L(a)$ denotes the rate at which a person born in the $j$th state will survive and be in the $i$th state at age $a$. Furthermore, we define the transition rate matrix $L(b, a)$ as

$$L(b, a) = L(b) L^{-1}(a).$$

(1.4)

The $(i, j)$th element $l_{ij}(b, a)$ of $L(b, a)$ denotes the rate at which a person in the $j$th state at age $a$ will survive to age $b$ in the $i$th state. We can prove the following.

**Lemma 1.1** Let $L(b, a)$ be the transition rate matrix. Then the following hold:

1. $L(b, a)$ is nonnegative,
2. $|L(b, a)| \leq \exp\{-\mu(b - a)\}$, where $\mu \overset{\text{def}}{=} \inf_{i,a} \mu_i(a) \geq 0$.

**Proof** Clearly, $L(b, a)$ satisfies

$$\frac{d}{db} L(b, a) = Q(b) L(b, a), \quad L(a, a) = I.$$

Let $\eta \overset{\text{def}}{=} \sup_{i,a} |q_{ii}(a)|$. Then $Q(a) + \eta I$ is a nonnegative matrix, and we have

$$\frac{d}{db} \{L(b, a) \exp(\eta(b - a))\} = [Q(b) + \eta I] L(b, a) \exp(\eta(b - a)).$$
Accordingly, we have the following representation

\[ L(b, a) \exp(\eta(b - a)) = I + \int_a^b [Q(\rho) + \eta I] d\rho + \int_a^b [Q(\rho_1) + \eta I] \int_a^{\rho_1} [Q(\rho_2) + \eta I] d\rho_2 d\rho_1 + \cdots. \]

Since the right-hand side is nonnegative, then we know that \( L(b, a) \) is nonnegative. Next, let \( l_{ij}(b, a) \) denote the \((i, j)\)th element of \( L(b, a) \). Then

\[
\frac{d}{db} l_{ij}(b, a) = \sum_{k=1}^n q_{ik}(b) l_{kj}(b, a), \quad l_{ij}(a, a) = \delta_{ij},
\]

\[
\frac{d}{db} \sum_{i=1}^n l_{ij}(b, a) = \sum_{k=1}^n \sum_{i=1}^n q_{ik}(b) l_{kj}(b, a) = \sum_{k=1}^n (-\mu_k(b)) l_{kj}(b, a)
\]

\[ \leq (-\mu) \sum_{i=1}^n l_{ij}(b, a). \]

Thus we obtain

\[
\sum_{i=1}^n l_{ij}(b, a) \leq \exp(-\mu(b - a)).
\]

This shows that \(|L(b, a)| \leq \exp(-\mu(b - a))\). This completes our proof. ☐

2. SPECTRAL PROPERTIES OF THE POPULATION OPERATOR AND ITS SEMIGROUP

We now define the linear operator \( A \), called the population operator (Song, et al. 1985), as follows:

\[
A\phi(a) \overset{\text{def}}{=} -\frac{d}{da} \phi(a) + Q(a)\phi(a), \quad \phi \in D(A), \quad (2.1)
\]

where the domain \( D(A) \) is defined by

\[
D(A) \overset{\text{def}}{=} \left\{ \phi \in L^1(0, \omega; C^n); \phi \text{ is absolutely continuous}, \quad \phi(0) = \int_0^\omega M(a)\phi(a) da \right\}.
\]
Then we can formulate the system \((1.1)\) as the initial value problem (or Cauchy problem) related to the abstract differential equation in the space \(L^1(0,\omega; C^n)\):

\[
\frac{d}{dt} p(t) = Ap(t), \quad p(0) = k, \tag{2.2}
\]

where \(p(t) = p(\cdot, t) \in L^1(0,\omega; C^n)\). Note that only the nonnegative solution of \((2.2)\) has physical significance.

In the sequel, we shall first investigate the spectral properties of the population operator \(A\). We shall see that our assumption \(\omega < +\infty\) is essential to the spectral structure of the population operator \(A\). If \(\omega < +\infty\), in other words if the age space is finite, then \(A\) is an operator with compact resolvent and has only a discrete spectrum (i.e., the spectrum of \(A\) consists entirely of isolated eigenvalues with finite multiplicities). If the age space is infinite, \(A\) may have another kind of spectrum (Webb, 1984, 1985). However we will not treat this case here.

Secondly, we will show that the operator \(A\) is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators in \(L^1(0,\omega; C^n)\). In the next section, we will discuss the asymptotic behavior of this semigroup.

First of all, it is not difficult to verify the next proposition, although we omit the proof.

**Proposition 2.1** The population operator \(A\) is a closed linear operator in \(L^1(0,\omega; C^n)\).

Next, we shall investigate the resolvent \(R(\lambda, A) = (\lambda I - A)^{-1}, \lambda \in C,\) of the closed operator \(A\). Defining the characteristic matrix \(\Psi^* : C \to B(C^n, C^n)\) by

\[
\Psi^*(\lambda) \overset{\text{def}}{=} \int_0^\omega \Psi(a) \exp(-\lambda a) da,
\]

where \(\Psi(a) \overset{\text{def}}{=} M(a)L(a)\) is called the generalized net maternity function; \(\Psi^*(0)\) is called the net reproduction rate matrix (Willekens and Rogers, 1978). Using these definitions, we can prove the following.

**Proposition 2.2** Let \(A\) be the population operator in \(L^1\) as \((2.1)\). The following hold:

1. Let \(\sigma(A)\) be the spectrum of the operator \(A\) and let \(P_\sigma(A)\) denote the point spectrum of the operator \(A\). Then
\[ \sigma(A) = P_\sigma(A) = \{ \lambda; \lambda \in C, \; \det(I - \Psi^*(\lambda)) = 0 \}, \]
\[ = \{ \lambda; \lambda \in C, \; 1 \in \sigma(\Psi^*(\lambda)) \}, \]
\[ = \{ \lambda; \lambda \in C, \; \lambda \text{ is a pole of } 1/\det(I - \Psi^*(\lambda)) \}, \]
\[ = \{ \lambda; \lambda \in C, \; \lambda \text{ is a pole of } R(\lambda, A) \}. \] (2.3)

2. Let \( \rho(A) \) be the resolvent set of \( A \). If \( \lambda \in \rho(A) \), then \( R(\lambda, A) \) is a compact operator as
\[ R(\lambda, A)\psi(a) = \exp(-\lambda a)L(a)\left(I - \int_0^\omega \Psi(v)\exp(-\lambda v) dv\right)^{-1} \]
\[ \times \int_0^\omega \Psi(v)\exp(-\lambda u) \int_0^u \exp(\lambda v)L^{-1}(v)\psi(v) dv du \]
\[ + \exp(-\lambda a)L(a) \int_0^a \exp(\lambda v)L^{-1}(v)\psi(v) dv. \] (2.4)

3. If \( \lambda \in \sigma(A) \), then the algebraic multiplicity of \( \lambda \) is finite and the geometric eigenspace of \( A \) with respect to \( \lambda \) is given by
\[ N(\lambda I - A) = \{ \exp(-\lambda a)L(a)x; \; x \in N(I - \Psi^*(\lambda)) \}, \] (2.5)
where \( N(\cdot) \) denotes the null space.

4. \( \rho(A) \supset \{ \lambda; \Re \lambda > \overline{m} - \mu \} \) where \( \overline{m} \) is the essential supremum \( \sup_{0 \leq a \leq \omega} |M(a)| < +\infty \), and if \( \lambda \in R, \lambda > \overline{m} - \mu \), then
\[ R(\lambda, A)(L_+^1) \subset L_+^1. \] (2.6)

5. If \( \lambda \in \sigma(A) \), then \( \overline{\lambda} \in \sigma(A) \), where the overbar denotes complex conjugation.

6. For any \( a > -\infty \), the operator \( A \) has only finitely many eigenvalues in the strip \( \Re \lambda \geq a \).

**Proof of (1), (2) and (3)** The equation \( (\lambda I - A)\phi = \psi, \; \psi \in L^1, \phi \in D(A) \), can be solved as (2.4), if and only if \( \det(I - \Psi^*(\lambda)) \neq 0 \). Thus we obtain
\[ \sigma(A) = \{ \lambda; \lambda \in C, \; \det(I - \Psi^*(\lambda)) = 0 \} = \{ \lambda; \lambda \in C, \; 1 \in \sigma(\Psi^*(\lambda)) \}. \]

Since the first operator on the right-hand side of (2.4) has finite rank and the second operator is a Volterra-type integral operator, then \( R(\lambda, A) \) is compact for \( \lambda \in \rho(A) \) and the spectrum of \( A \) is discrete (Kato, 1976, p. 187). If \( \lambda \in \{ \lambda; \lambda \in C, \; \det(I - \Psi^*(\lambda)) = 0 \} \), then the matrix \( \Psi^*(\lambda) \) has eigenvalue one and a corresponding eigenvector.
$x \in C^n$, $x \neq 0$, such that $\Psi^*(\lambda)x = x$. Let $\psi_\lambda(a) = \exp(-\lambda a)L(a)x$. Then it follows immediately that

$$A\psi_\lambda(a) = \lambda \psi_\lambda(a), \quad \psi_\lambda \in D(A).$$

This shows that $\lambda$ is the eigenvalue of $A$, and that $\psi_\lambda(a)$ is the eigenvector of $A$ associated with $\lambda$. Since $\det(I - \Psi^*(\lambda))$ is an entire function of $\lambda$, then $R(\lambda, A)$ is meromorphic for $|\lambda| < \infty$, and has poles in $\{\lambda; \lambda \in C, \det(I - \Psi^*(\lambda)) = 0\}$. Then we have $\sigma(A) = \{\lambda; \lambda \in C, \lambda$ is a pole of $1/\det(I - \Psi^*(\lambda))\} = \{\lambda; \lambda \in C, \lambda$ is a pole of $R(\lambda, A)\}$. This completes our proof of (1), (2) and (3). $\square$

**Proof of (4)** Let $F(\lambda)$, $\lambda \in R$, denote the Frobenius root of the characteristic matrix $\Psi^*(\lambda)$, let $\psi^*_i(\lambda)$ be the $(i,j)$th element of $\Psi^*(\lambda)$, let $\hat{\Psi}^*(\lambda)$ be the nonnegative matrix, in which the $(i,j)$th element is $|\psi^*_i(\lambda)|$, and let $\hat{F}(\lambda)$ be the Frobenius root of $\hat{\Psi}^*(\lambda)$. Let $r_\sigma(A)$ denote the spectral radius of the bounded operator $A$. Then it can be shown that $r_\sigma(\Psi^*(\lambda)) \leq \hat{F}(\lambda)$ (Gantmacher, 1960, p. 57). On the other hand, from $\Psi^*(\lambda) \leq \Psi^*(\Re \lambda)$, we obtain $\hat{F}(\lambda) \leq F(\Re \lambda)$. Then we have $r_\sigma(\Psi^*(\lambda)) \leq F(\Re \lambda)$. From

$$|\Psi^*(\Re \lambda)| \leq \int_0^\omega |\Psi(v)| \exp(-\Re \lambda v) dv$$

$$\leq \int_0^\omega |M(v)||L(v)| \exp(-\Re \lambda v) dv$$

$$\leq \frac{m}{\Re \lambda + \mu} \{1 - \exp[-(\Re \lambda + \mu)\omega]\},$$

we obtain

$$F(\Re \lambda) \leq \max_j \sum_{i=1}^n \psi^*_i(\Re \lambda) = |\Psi^*(\Re \lambda)|$$

$$\leq \frac{m}{\Re \lambda + \mu} \{1 - \exp[-(\Re \lambda + \mu)\omega]\}.$$ 

Therefore, if $\Re \lambda > \overline{m} - \mu$, $r_\sigma(\Psi^*(\lambda)) \leq F(\Re \lambda) < 1$. Thus $\det(I - \Psi^*(\lambda)) \neq 0$, for $\Re \lambda > \overline{m} - \mu$. This shows that $\rho(A) \supset \{\lambda; \lambda \in C, \Re \lambda > \overline{m} - \mu\}$. Further, if $\lambda \in R, \lambda > \overline{m} - \mu$, then $(I - \Psi^*(\lambda))^{-1} > 0$ because $(I - \Psi^*(\lambda))$ is nonnegatively invertible (Nikaido, p. 102, p.
107). Since the transition rate matrix is nonnegative, then we have
\[
\int_0^\omega \Psi(u) \exp(-\lambda u) \int_0^u \exp(\lambda v)L^{-1}(v)\psi(v) dv \, du \\
= \int_0^\omega M(u) \exp(-\lambda u) \int_0^u \exp(\lambda v)L(u,v)\psi(v) dv \, du \geq 0,
\]
\[
\exp(-\lambda a)L(a) \int_0^a \exp(\lambda v)L^{-1}(v)\psi(v) dv \\
= \exp(-\lambda a) \int_0^a \exp(\lambda v)L(a,v)\psi(v) dv \geq 0,
\]
for \( \psi \in L^1_+ \). Accordingly,
\[
R(\lambda, A)\psi \geq 0, \quad \text{for} \quad \psi \in L^1_+.
\]
This completes the proof. \( \square \)

**Proof of (5)** This proposition is evident from (2.3) and the fact that
\[
\det(I - \Psi^*(\lambda)) = \det(I - \Psi^*(\bar{\lambda})). \quad \square
\]

**Proof of (6)** Since \( R(\lambda, A) \) is a compact resolvent, the spectrum \( \sigma(A) \) has no accumulation point different from \( \infty \). Then if we assume that there exists an infinite number of eigenvalues \( \lambda_n \) \( (n = 1, 2, \ldots) \) of \( A \) such that \( \lambda_n = \alpha_n + i\beta_n, a \leq a_n \leq m - \mu \), we can choose a subsequence, \( \lambda_{n(k)} = \alpha_{n(k)} + i\beta_{n(k)}, k = 1, 2, \ldots \), which has the property \( \alpha_{n(k)} \to \alpha^*, \beta_{n(k)} \to \infty \) when \( k \to \infty \). It follows that
\[
\left| \psi_{ij}^*(\lambda_{n(k)}) - \int_0^\omega \exp(-\alpha^*\zeta)\exp(-i\beta_{n(k)}\zeta)\psi_{ij}(\zeta) d\zeta \right| \\
\leq \int_0^\omega |\exp(-\alpha^*\zeta) - \exp(-\alpha_{n(k)}\zeta)||\psi_{ij}(\zeta)| d\zeta \to 0 \quad (k \to \infty).
\]
From the Riemann–Lebesgue lemma, we obtain
\[
\int_0^\omega \exp(-\alpha^*\zeta)\exp(-i\beta_{n(k)}\zeta)\psi_{ij}(\zeta) d\zeta \to 0 \quad (k \to \infty).
\]
Then we have \( \psi_{ij}^*(\lambda_{n(k)}) \to 0 \) \( (k \to \infty) \). This is a contradiction, because \( \det(I - \Psi^*(\lambda_{n(k)})) = 0 \), for \( k = 1, 2, \ldots \). This completes our proof. \( \square \)

**Proposition 2.3** \( D(A) \) is dense in \( L^1(0, \omega; C^n) \).
PROOF If \( \lambda > \overline{m} - \mu \), we can define \( \phi_{\lambda} = \lambda (\lambda I - A)^{-1} \psi \) for all \( \psi \in L^1(0, \omega; C^n) \). Since \( \phi_{\lambda} \in D(A) \), it is sufficient to show that \( \phi_{\lambda} \to \psi \) (\( \lambda \to \infty \)) in \( L^1(0, \omega; C^n) \). From (2.4), we have

\[
\phi_{\lambda}(a) = \lambda \exp(-\lambda a) L(a) (I - \Psi^*(\lambda))^{-1} \\
\times \int_0^\omega \Psi(u) \exp(-\lambda u) \int_0^u \exp(\lambda v) L^{-1}(v) \psi(v) dv du \\
+ \lambda \exp(-\lambda a) L(a) \int_0^a \exp(\lambda v) L^{-1}(v) \psi(v) dv. \tag{2.7}
\]

Then we can write \( \| \phi_{\lambda} - \Psi \| \leq J_1 + J_2 \), where

\[
J_1 = \lambda \int_0^\omega \int_0^u \Psi(u) \exp(-\lambda u) \int_0^u \exp(\lambda v) L^{-1}(v) \psi(v) dv du \| da, \\
J_2 = \int_0^\omega \lambda \exp(-\lambda a) L(a) \int_0^a \exp(\lambda v) L^{-1}(v) \psi(v) dv - \psi(a) \| da.
\]

It is easy to verify the following inequality:

\[
J_1 \leq \frac{1 - \exp[-(\lambda + \mu)\omega]}{\lambda - (\overline{m} - \mu)} \cdot \frac{\overline{m}}{\lambda + \mu} \cdot \| \psi \|.
\]

Then we obtain \( \lim_{\lambda \to \infty} J_1 = 0 \). On the other hand,

\[
J_2 \leq \int_0^\omega |L(a)| \lambda \exp(-\lambda a) \int_0^a \exp(\lambda v) L^{-1}(v) \\
\times \psi(v) dv - L^{-1}(a) \psi(a) \| da \\
\leq \int_0^\omega \lambda \exp(-\lambda a) \int_0^a \exp(\lambda v) L^{-1}(v) \psi(v) dv - L^{-1}(a) \psi(a) \| da.
\]

It can be shown that the right-hand side of the above inequality tends to 0 as \( \lambda \to \infty \) (Webb, 1985, p. 92). Then \( \lim_{\lambda \to \infty} J_2 = 0 \). Therefore, we conclude that \( \lim_{\lambda \to \infty} \| \phi_{\lambda} - \psi \| = 0 \). This shows that \( D(A) \) is dense in \( L^1 \). This is the end of our proof. \( \square \)

**Proposition 2.4** The population operator \( A \) is an infinitesimal generator of a strongly continuous semigroup \( T(t) \) which satisfies the following

1. \( \| T(t) \| \leq \exp((\overline{m} - \mu) t) \). \( \tag{2.8} \)
2. \( T(t)(L^1_+) \subset L^1_+ \) for all \( t \geq 0 \). \( \tag{2.9} \)
PROOF We first show the following estimates;

\[ \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda - (m - \mu)}, \quad \text{for} \quad \lambda > m - \mu. \]

From (2.7), we can write

\[ \|(\lambda I - A)^{-1}\psi\| \leq \frac{1}{\lambda} J_1 + \int_0^\omega \exp(-\lambda a) \int_0^a \exp(\lambda v)|L(a, v)||\psi(v)|\, dv\, da, \]

where

\[ J_1 \leq \frac{\lambda m}{\lambda + \mu} \cdot \frac{1}{\lambda - (m - \mu)} \cdot \|\psi\|, \]

\[ \int_0^\omega \exp(-\lambda a) \int_0^a \exp(\lambda v)|L(a, v)||\psi(v)|\, dv\, da \leq \frac{1}{\lambda + \mu} \|\psi\|. \]

Thus we have

\[ \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda + \mu} \left[ \frac{m}{\lambda - (m - \mu)} + 1 \right] = \frac{1}{\lambda - (m - \mu)}. \]

By using the Hille-Yosida Theorem, we know that the closed, densely defined operator \(A\) is the infinitesimal generator of a strongly continuous semigroup \(T(t) = \exp(At)\) such that

\[ \|T(t)\| \leq \exp[(m - \mu)t]. \]

Next, by the formula of Hille, we have

\[ T(t) = s \cdot \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} = s \cdot \lim_{n \to \infty} \left( \frac{n}{t} \right)^n R \left( \frac{n}{t}, A \right)^n, \quad (t \geq 0), \]

where \(s \cdot \lim\) denotes strong convergence. From (2.6), we have

\[ R \left( \frac{n}{t}, A \right) (L^1_+) \subset L^1_+ \quad \text{for} \quad \frac{n}{t} \geq m - \mu, \]

then we know that \(T(t)(L^1_+) \subset L^1_+\). \(\square\)

In the following, we refer to the semigroup \(T(t)\) generated by the population operator \(A\) as the population semigroup. From the theory of strongly continuous semigroups, it follows that for \(\phi \in D(A)\), \(T(t)\phi = \exp(At)\phi\) is continuously differentiable and \(T(t)\phi \in D(A)\), and

\[ \frac{d}{dt} T(t)\phi = AT(t)\phi = T(t)A\phi, \quad (2.10) \]
It can be shown that $T(t)\phi$, $\phi \in D(A)$ is the unique strict solution of the Cauchy problem (2.2) (Belloni-Morante, 1979, p. 163). Therefore, the population vector can be represented by the population semigroup as $p(a,t) = (T(t)k)(a)$, $k(a) = p(a,0)$.

The next proposition shows the compactness property of the population semigroup $T(t)$, which plays an essential role in the study of its ultimate behavior as time evolves.

**Proposition 2.5** Let $T(t)$ be the population semigroup. Then $T(t)$ is compact for $t \geq \omega$.

**Proof** It can be shown that the population semigroup $T(t)$, $0 \leq t \leq \omega$, admits the direct representation (Prüss, 1981) as

$$
(T(t)k)(a) = \begin{cases} 
L(a, a-t)k(a-t) & \text{for a.e. } a \in (t, \omega), \\
L(a)b_t(t-a; k) & \text{for a.e. } a \in (0, t),
\end{cases}
$$

(2.11)

where $b_t(x; k)$ denotes the solution of the integral equation

$$
b_t(x; k) = (G_t \cdot k)(x) + \int_0^x \Psi(a)b_t(x - a; k) \, da, \quad x \in [0,t],
$$

(2.12)

and $G_t$ is an operator such that

$$
G_t : L^1(0, \omega; C^n) \to L^1(0, t; C^n),
$$

$$(G_t \cdot k)(x) = \int_x^\omega \Psi(a)L^{-1}(a-x)k(a-x) \, da.
$$

Let $V_t$ be the integral operator defined by

$$
V_t : L^1(0, t; C^n) \to L^1(0, t; C^n), \quad (V_t \cdot f)(x) = \int_0^x \Psi(a)f(x - a) \, da.
$$

Then the integral equation (2.12) can be solved as

$$
b_t(x; k) = ((I - V_t)^{-1}G_t \cdot k)(x).
$$

From (2.11), we have the representation

$$
(T(\omega)k)(a) = L(a)((I - V_\omega)^{-1}G_\omega \cdot k)(\omega - a).
$$

Since $G_\omega$ is compact and $V_\omega$ is bounded with spectral radius zero, we know that $T(\omega)$ is a compact linear operator. From $T(t) = T(t-\omega)T(\omega)$, $t \geq \omega$, and $T(t-\omega)$ is bounded, $T(t)$ is compact for $t \geq \omega$. This completes our proof. $\square$

A strongly continuous semigroup $T(t)$, $t \geq 0$, is called eventually compact, if there exists $t_0 > 0$ such that $T(t_0)$ is compact and hence
$T(t)$ is compact for all $t \geq t_0$ (Nagel, 1986, p. 40). From the above proposition, we know that the population semigroup is eventually compact.

From the direct representation (2.11), we observe that if the generalized net maternity function $f(a)$ and the initial data $k(a)$ are differentiable, the semigroup solution $(T(t)k)(a)$ of (2.2) is differentiable almost everywhere for $(a,t) \in [0,\omega] \times [0,\infty)$ and then almost everywhere agrees with the classical solution of (1.1).

3. ASYMPTOTIC BEHAVIOR OF THE POPULATION SEMIGROUP: THE STRONG ERGODIC THEOREM

In this section we shall formulate a necessary and sufficient condition for existence of a stable distribution of the Cauchy problem (2.2). Here a stable distribution with intrinsic growth rate $\lambda \in R$ is defined as an element $\psi \in L^1$ such that for $k \in L^1$ the following holds: there is a constant $c$ such that

$$\exp(-\lambda t)T(t)k \rightarrow c\psi \quad \text{as} \quad t \rightarrow \infty,$$

where $\lambda$ is independent of the initial data $k \in L^1$ (Heijmans, 1985). For most applications, $c\psi$ must be nonnegative in order to make physical interpretations possible.

In demographic terminology, the theorems which state that the age distribution of a closed population is asymptotically independent of this initial distribution are called ergodic theorems (Cohen, 1979). The strong ergodic theorem assumes time-independent age-specific birth and death rates. From the above definitions, it is apparent that if the population semigroup $T(t)$ of the problem (2.2) has a stable distribution, then the strong ergodic theorem holds for this population process. Historically speaking, the strong ergodic theorem of a single-state population was first proposed by Sharpe and Lotka (1911), and was first provided with a rigorous proof by Feller (1941) using Laplace transform techniques. Recently, Prüss (1981, 1983a, 1983b), Diekmann, et al. (1984), Webb (1984, 1985) and Song, et al. (1985) independently gave a semigroup proof of the Sharpe-Lotka theorem. The main aim of this section is to establish the strong ergodic theorem for the multistate stable population process.

In the recent paper of Webb (1986), the following result is essentially proved, although we need to adjust his result to our situation. A proof of the proposition is given in the appendix.
Proposition 3.1 Let $T(t)$, $t \geq 0$ be a strongly continuous semigroup of bounded linear operators in the Banach space $X$ and let $A$ be the infinitesimal generator of $T(t)$. $T(t)$ has a stable distribution $\psi_0$ with intrinsic growth rate $\lambda_0 \in \mathbb{R}$ if and only if $\lambda_0$ is a strictly dominant, simple eigenvalue of $A$ and $\omega_1(A) < \lambda_0$, where $\omega_1(A)$ denotes the essential growth bound of $T(t)$. Further, $\psi_0$ is an eigenvector of $A$ associated with $\lambda_0$ and the following holds:

$$\lim_{t \to \infty} \exp(-\lambda_0 t)T(t) = P_0,$$

where $P_0$ is the eigenprojection with rank one corresponding to $\lambda_0$ defined by

$$P_0 \phi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} \phi d\lambda,$$

where $\Gamma$ is a positively oriented closed curve in the complex plane enclosing $\lambda_0$ but no other point of $\sigma(A)$.

If $T(t)$ is eventually compact, then it is easy to verify that $\omega_1(A) = -\infty$ (Appendix, Lemma A and B). Thus the population semigroup $T(t)$ has a stable distribution if and only if the population operator $A$ has a strictly dominant and simple eigenvalue. The next proposition provides a sufficient condition to prove there exists a strictly dominant, simple eigenvalue of the population operator $A$.

Proposition 3.2 Assume that the net reproduction rate matrix $\Psi^*(0)$ is indecomposable. Then the population operator $A$ has a real, strictly dominant eigenvalue $r_0$ which is algebraically simple. In addition,

1. if $F(0) = 1$, then $r_0 = 0$ (critical),
2. if $F(0) > 1$, then $r_0 > 0$ (supercritical),
3. if $F(0) < 1$, then $r_0 < 0$ (subcritical),

where $F(0)$ is the Frobenius root of $\Psi^*(0)$. Further, the eigenprojection $P_0$ associated with $r_0$ is a nonnegative operator such that

$$P_0 \psi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} \psi d\lambda$$

$$= \frac{\exp(-r_0 \alpha) L(a)}{d}{\mid}_{\lambda=r_0} \det(I - \Psi^*(\lambda)) \psi_{0}$$

$$= \exp(-r_0 u) \Psi(u) \int_{0}^{u} \exp(r_0 v) L^{-1}(v) \psi(v) dv du,$$
where $\Gamma$ is a positively oriented closed curve in the complex plane enclosing $r_0$, but no other point of $\sigma(A)$, and adj$(I - \Psi^*(r_0))$ denotes the adjugate matrix of the matrix $(I - \Psi^*(r_0))$.

**Remark**  A nonnegative $n \times n$ matrix $C = (c_{ij})$ is called decomposable if there exist two subsets $K$ and $H$ of integers $L = \{1, 2, \ldots, n\}$ such that

$$K \cap H = \emptyset, \quad K \cup H = L, \quad c_{ij} = 0 \quad \text{for} \quad i \in K, \quad j \in H.$$ 

$C = (c_{ij})$ is called indecomposable, if it is not decomposable and is not the zero matrix of order one (Nikaido, 1968, p. 105).

**Proof**  Under the assumption, $\Psi^*(r)$ is indecomposable for all $r \in R$ and has the Frobenius root $F(r) > 0$, which is a strictly monotone decreasing continuous function of $r \in R$. Since $F(r)$ satisfies the inequality (Nikaido, 1968, p. 108):

$$0 < \min_j \sum_{i=1}^n \psi^*_{ij}(r) \leq F(r) \leq \max_j \sum_{i=1}^n \psi^*_{ij}(r),$$

then we obtain

$$\lim_{r \to -\infty} F(r) = +\infty, \quad \lim_{r \to +\infty} F(r) = 0.$$ 

Accordingly, the characteristic equation $F(r) = 1$ has a unique real root $r_0$. And then $F(0) = 1, r_0 = 0; F(0) > 1, r_0 > 0; F(0) < 1, r_0 < 0$. From $1 \in \sigma(\Psi^*(r_0))$, we know that $r_0 \in \sigma(A)$. Next, we show that $r_0$ is greater than the real part of any other eigenvalue of $A$. Let $1 \in \sigma(\Psi^*(\lambda_1)), \lambda_1 \neq r_0$. It is easy to see that if $\text{Im} \lambda \neq 0$, then $\Psi^*(\lambda) \leq \Psi^*(\text{Re} \lambda)$, where $x \leq y$ means $x \geq y, x \neq y$. Since $\Psi^*(\text{Re} \lambda)$ is indecomposable, then we have $\hat{F}(\lambda) < F(\text{Re} \lambda)$. Hence if $\lambda_1 \neq 0$, then

$$F(r_0) = 1 \leq r_\sigma(\Psi^*(\lambda_1)) \leq \hat{F}(\lambda_1) < F(\text{Re} \lambda_1).$$

Since $F(r)$ is a strictly monotone decreasing function, we know that $\text{Re} \lambda_1 < r_0$. If $\lambda_1$ is real, then $F(\lambda_1) \neq 1$ because $r_0(\neq \lambda_1)$ is a unique real root of the equation $F(r) = 1$. From $1 \in \sigma(\Psi^*(\lambda_1))$, we obtain $F(r_0) = 1 \leq r_\sigma(\Psi^*(\lambda_1)) \leq F(\lambda_1)$. Since $1 \neq F(\lambda_1)$, then $1 = F(r_0) < F(\lambda_1)$. This shows that $\lambda_1 < r_0$. Therefore, we obtain $\text{Re} \lambda < r_0$ for any $\lambda \in \sigma(A) \setminus \{r_0\}$. Thus $r_0$ is a strictly dominant, real eigenvalue of $A$. Next, we prove that $r_0$ is algebraically simple. Since $\Psi^*(\lambda)$ is an
analytic function of $\lambda$, then $\Psi^*(\lambda)$ can be expanded in a Taylor series:

$$\Psi^*(\lambda) = \sum_{n \geq 0} (\lambda - r_0)^n K_n. \quad (3.5)$$

Since $\Psi^*(r_0) = K_0$ is a nonnegative, indecomposable matrix which has the Frobenius root one, then the right and left eigenvectors, denoted by $\psi_0(0), \nu_0(0)$, respectively, associated with the Frobenius root one of $K_0$ are positive vectors. From

$$K_1 = \frac{d}{d\lambda} \Psi^*(\lambda) \bigg|_{\lambda = r_0} = - \int_0^\omega \zeta \exp(-r_0\zeta) \Psi(\zeta) d\zeta, \quad (3.6)$$

we obtain $\langle \psi_0(0), -K_1\psi_0(0) \rangle \neq 0$, where $\langle \ , \ \rangle$ denotes the inner product of $C^n$. From the operator residue theorem of Schumitzky and Wenska (1975), we know that $r_0$ is a simple pole of $(I - \Psi^*(\lambda))^{-1}$. From (2.4), the resolvent $R(\lambda, A)$ has a simple pole in $\lambda = r_0$. Thus we have $R(P_0) = N(r_0 I - A)$ (Yosida, 1980, p. 229). From Proposition 2.2-(3), we obtain

$$\dim R(P_0) = \dim N(r_0 I - A) = \dim N(I - \Psi^*(r_0)) = 1.$$

This shows that $r_0$ is algebraically simple. Finally, since $r_0$ is a pole of order one of $(\lambda I - A)^{-1}$, we have

$$P_0 \psi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} \psi d\lambda = \lim_{\lambda \to r_0} (\lambda - r_0) R(\lambda, A) \psi$$

$$= \frac{\exp(-r_0a) L(a)}{\det(I - \Psi^*(\lambda))} \text{adj}(I - \Psi^*(r_0))$$

$$\times \int_0^\omega \exp(-r_0\zeta) \Psi(\zeta) \int_0^\zeta \exp(r_0\eta) L^{-1}(\eta) \psi(\eta) d\eta d\zeta.$$

From Proposition 2.4 and Proposition 3.1, we know that $T(t) = \exp(tA)$ is nonnegative, and $P_0 \phi = \lim_{t \to -\infty} \exp(-r_0 t) T(t) \phi \geq 0$ for $\phi \in L^1$. Then $P_0$ is nonnegative. \(\square\)

From Proposition 2.2, 3.1 and 3.2, we obtain the following proposition immediately.

**Proposition 3.3 (The Strong Ergodic Theorem)** Let the net reproduction rate matrix $\Psi^*(0)$ be indecomposable. Then the population semigroup $T(t)$ has the stable distribution $\psi_0$ with intrinsic growth rate $r_0$. Here $r_0$ is the strictly dominant and simple eigenvalue of
the population operator \( A \) and \( \psi_0 \) is the eigenvector of \( A \) associated with \( r_0 \) given by \( \psi_0(a) = \exp(-r_0 t)L(a)\psi_0(0) \), where \( \psi_0(0) \) is a right eigenvector associated with the Frobenius root one of \( \Psi^*(r_0) \).

Let \( N(t) = \|p(\cdot, t)\| = \int_0^\alpha |p(a, t)| da \). Then \( N(t) \) is the total population at time \( t \) of the multistate population considered here and \( p(a, t)/N(t) \equiv C(a, t) \) is a vector, in which the \( i \)th element denotes the share of the population in the \( i \)th state at age \( a \) and time \( t \). Let \( \psi_0(a) \) be the eigenvector associated with the strictly dominant and simple eigenvalue \( r_0 \) of the population operator \( A \). By virtue of the strong ergodic theorem, there exist numbers \( \gamma \geq 0, \eta > 0 \) such that

\[
p(a, t) = (T(t)k)(a) = \gamma \cdot \psi_0(a) \exp(r_0 t) + O(\exp\{(r_0 - \eta)t\})
\]

where \( O(\exp\{(r_0 - \eta)t\}) \cdot \exp\{-\eta |t|\} \) is bounded as \( t \to \infty \). Hence

\[
N(t) = \|T(t)k\| = \gamma \cdot \exp(r_0 t)\|\psi_0\| + O(\exp\{(r_0 - \eta)t\})
\]

Therefore, it is easy to see that

\[
C(a, t) = \frac{p(a, t)}{N(t)} = \frac{\psi_0(a)}{\|\psi_0\|} + O(\exp(-\eta t))
\]

This shows that the age-by-state distribution of the multistate stable population converges to a constant distribution \( \psi_0(a)/\|\psi_0\| \), which is independent of the initial condition, and uniquely determined by mortality rates, inter-states transition rates and fertility rates. In a word, the strong ergodicity holds for this population process.

The strong ergodic theorem implies that the zero equilibrium solution of (2.2) is globally asymptotically stable if \( r_0 < 0 \). On the other hand, Proposition 3.2 shows that \( r_0 < 0 \) if and only if \( F(0) < 1 \), where \( F(0) \) is the Frobenius root of the net reproduction rate matrix \( \Psi^*(0) \).

It is well known in mathematical economics that one of the following conditions are sufficient to show \( F(0) < 1 \) (Nikaido, 1968, p. 90, p. 94).

1. (Brauer-Solow's Condition)

\[
\max_{1 \leq i \leq n} \sum_{j=1}^n \psi_{ij}^*(0) < 1, \quad \text{or} \quad \max_{1 \leq i \leq n} \sum_{j=1}^n \psi_{ij}^*(0) < 1,
\]
(2) (Hawkins-Simon’s Condition) \((I - \Psi^*(0))\) has the \(n\) positive upper left-hand corner principal minors, i.e.,

\[
\begin{vmatrix}
1 - \psi^*_{11}(0) & \cdots & -\psi^*_{1k}(0) \\
\vdots & \ddots & \vdots \\
-\psi^*_{k1}(0) & \cdots & 1 - \psi^*_{kk}(0)
\end{vmatrix} > 0 \quad (k = 1, \ldots, n).
\]

Therefore, either one of (1) and (2) is sufficient for globally asymptotic stability of the zero equilibrium solution of (2.2).

Moreover, if there exist complex eigenvalues \(\lambda_j, j = 1, 2, \ldots, n\), of the population operator \(A\), then we can order \(\lambda_1, \lambda_2, \ldots\) such that \(\text{Re}\lambda_1 \geq \text{Re}\lambda_2 \geq \ldots\) according to Proposition 2.2. Let \(r\) be a number such that

\[
\min\{\text{Re}\lambda_i; 1 \leq i \leq n\} > r > \max\{\text{Re}\lambda; \lambda \in \sigma(A) \setminus \Lambda\},
\]

where \(\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\). In this case, the population semigroup \(T(t)\) admits the asymptotic expansion as follows (Belleni-Morante, 1979; Jörgens, 1958; Ukai, 1976):

\[
T(t)k = \sum_{i=1}^{n} T(t)P_i k + O(\exp(rt)),
\]

\[
= \sum_{i=1}^{n} \exp(\lambda_i t) \left( P_i + \sum_{j=1}^{m_i-1} \frac{t^j}{j!} D_i^j \right) k + O(\exp(rt)), \quad (3.7)
\]

where \(P_i\) is the eigenprojection associated with \(\lambda_i\), \(D_i\) is the eigen-nilpotent associated with \(\lambda_i\) and \(m_i\) is the algebraic multiplicity of \(\lambda_i\). Let \(\text{Im}\lambda_i \neq 0\) and let \(\overline{\lambda}_i = \lambda_j\). Then it follows immediately that

\[
P_j = \overline{P}_i, \quad D_j = \overline{D}_i. \quad (3.8)
\]

From (3.7) and (3.8), we know that the oscillatory behavior is possible in the transient phase before the stable distribution is attained if there exists a complex eigenvalue of \(A\). In fact, if \(\text{Im}\lambda_i \neq 0\), there exists an oscillatory term on the right-hand side of (3.7) such that

\[
\exp(\lambda_i t)(P_i k)(a) + \exp(\lambda_j t)(P_j k)(a) = 2 \exp(\text{Re}\lambda_i t)(P_i k)(a) \cos(\text{Im}\lambda_i t + \theta_i(a)),
\]

where \(\theta_i(a)\) is the phase angle.
where

\[ |(P_k)(a)| = \sqrt{(\text{Re}(P_k)(a))^2 + (\text{Im}(P_k)(a))^2}, \]

\[ \theta_i(a) = \tan^{-1} \left( \frac{\text{Im}(P_k)(a)}{\text{Re}(P_k)(a)} \right). \]

The oscillatory behavior of the structured population has been discussed by several authors (Coale, 1972; Feller, 1941; Hamada and Nakamura, 1982).

Indecomposability of the net reproduction rate matrix \( \Psi^*(0) \) has a simple demographic meaning. If \( \Psi^*(0) \) is decomposable, there exist two subsets \( K \) and \( H \) of integers \( L = \{1, 2, \ldots, n\} \) such that

\[ K \cap H = \phi, \quad K \cup H = L, \quad \psi_{ij}^*(0) = 0 \quad \text{for} \quad i \in K, \quad j \in H. \quad (3.9) \]

Hence,

\[ \sum_{k=1}^{n} m_{ik}(a)l_{kj}(a) = 0 \quad \text{for} \quad i \in K, \quad j \in H, \quad \text{a.e.} \quad a \in [0, \omega]. \quad (3.10) \]

This means that individuals born in state \( j \in H \) cannot have babies of state \( i \in K \). We divide individuals into two subclasses \( K \) and \( H \) by their birthstages, that is, the subclass \( K \) is the set of individuals born in states belonging to \( K \) and the subclass \( H \) is the set of individuals born in states belonging to \( H \). Then the subclass \( K \) is "closed" with respect to its self-reproduction in the sense that individuals of the subclass \( K \) can be reproduced only by individuals of the subclass \( K \). Indecomposability of \( \Psi^*(0) \) guarantees that there is no subclass which is closed for its reproduction.

4. THE DUAL PROBLEM

In this section we shall state some results concerning the dual operator \( A^* \) of the population operator \( A \). The dual operator \( A^* \) of \( A \) is given by

\[ A^*v = \frac{d}{da}v(a) + Q(a)^T v(a) + M(a)^T v(0) \quad \text{for} \quad v \in D(A^*), \quad (4.1) \]

where \( D(A^*) = \{ v \in L^\infty(0, \omega; C^n); \ A^*v \in L^\infty(0, \omega; C^n), \ v(\omega) = 0 \} \), and \( \tau \) denotes the transpose of the matrix. In fact, from the definition of \( A^* \), we have \( \langle A^*x, y \rangle = \langle x, Ay \rangle \) for all \( x \in D(A^*) \) and all \( y \in D(A) \).
where \((x, y)\) is defined by
\[
(x, y) = \int_0^\omega x(a)^r y(a^r) da \quad \text{for} \quad x \in L^\infty, \quad y \in L^1.
\]

It is easy to verify the following
\[
\int_0^\omega (A^* x(a))^r y(a^r) da = \int_0^\omega x(a)^r (Ay(a))^r da \\
= \int_0^\omega \left[ x(0)^r M(a) + \frac{d}{da} x(a)^r + x(a)^r Q(a) \right] y(a^r) da.
\]

Thus we have (4.1). A proof of the following theorem is straightforward and shall be omitted (Kato, 1976, p. 183).

**Proposition 4.1** Let \(A^*\) be the dual operator of the population operator \(A\). Then the following hold:

1. \(\sigma(A) = \sigma(A^*)\), \(\rho(A) = \rho(A^*)\) and \(R(\lambda, A)^* = R(\lambda, A^*)\) for \(\lambda \in \rho(A)\).
2. The dual operator \(R(\lambda, A)^*\) of \(R(\lambda, A)\) is expressed as:
\[
R(\lambda, A)^* u = \exp(\lambda a) L^{-1}(a)^r \int_a^\omega \exp(-\lambda \zeta) L(\zeta)^r u(\zeta) d\zeta \\
+ \exp(\lambda a) L^{-1}(a)^r \int_a^\omega \exp(-\lambda \zeta) \Psi(\zeta) d\zeta \\
\times (I - \Psi(\lambda)^r)^{-1} \int_0^\omega \exp(-\lambda \zeta) L(\zeta)^r u(\zeta) d\zeta. \quad (4.2)
\]
3. If \(\lambda \in \sigma(A^*)\), the corresponding eigenvector is given by
\[
v_\lambda(a) = \exp(\lambda a) L^{-1}(a)^r \int_a^\omega \exp(-\lambda \zeta) \Psi(\zeta)^r d\zeta \cdot v_\lambda(0), \quad (4.3)
\]
where \(v_\lambda(0)\) is the right-hand side eigenvector associated with the eigenvalue one of the matrix \(\Psi^*(\lambda)^r = \int_0^\omega \Psi(\zeta)^r \exp(-\lambda \zeta) d\zeta\).
4. If \(\lambda \in \sigma(A)\) is an isolated eigenvalue of \(A\) with finite multiplicity \(m\), then \(\overline{\lambda}\) is an eigenvalue of \(A^*\) with finite multiplicity \(m\) and the geometric multiplicity of \(\overline{\lambda}\) for \(A^*\) is equal to that of \(\lambda\) for \(A\).
5. If \(\lambda_i \neq \overline{\lambda}_j\), then \(\langle v_{\lambda_i}, \psi_{\lambda_j} \rangle = 0\), where \(\psi_{\lambda_j}\) is an eigenvector corresponding to the eigenvalue \(\lambda_j\) of \(A\) and \(v_{\lambda_i}\) is an eigenvector corresponding to the eigenvalue \(\lambda_i\) of \(A^*\). And if \(\lambda_i = \overline{\lambda}_j\), then
\[
\langle v_{\lambda_i}, \psi_{\lambda_j} \rangle = \langle v_{\lambda_i}, \psi_{\lambda_j} \rangle = v_{\lambda_i}(0)^r \left( \int_0^\omega \zeta \exp(-\lambda_i \zeta) \Psi(\zeta) d\zeta \right) \overline{\psi_{\lambda_i}(0)}. \quad (4.4)
\]
If there exists a real dominant eigenvalue $r_0$ of $A$ with multiplicity one, we can define the reproductive value vector of the system (1.1) as:

$$v_0(a) = \exp(r_0 a) L^{-1}(a)^r \int_a^\omega \exp(-r_0 \zeta) \Psi(\zeta)^r d\zeta \cdot v_0(0),$$

where $v_0(0)$ is the right-hand side eigenvector associated with the eigenvalue one of $\Psi^*(r_0)^r$, which is uniquely determined except for scalar multipliers. The reproductive value was first defined by R. A. Fisher (1958) in order to measure the extent to which an individual of a given age will, on the average, contribute to the births of future generations (Keyfitz, 1985, p. 142; Crow and Kimura, 1970, p. 20). The reader may find demographic applications of the reproductive value in Keyfitz (1985). The multistate version of the reproductive value was first introduced by Rogers and Willekens (1978) in multi-regional demography. Furthermore, we define the total reproductive value of the population $p(a,t)$ as:

$$V(t) = \langle v_0, p(a,t) \rangle = \int_0^\omega v_0(\zeta)^r p(\zeta,t) d\zeta.$$

Observe that

$$\frac{d}{dt} V(t) = \int_0^\omega v_0(\zeta)^r \frac{\partial}{\partial t} p(\zeta,t) d\zeta = \int_0^\omega v_0(\zeta)^r A p(\zeta,t) d\zeta = \langle v_0, Ap \rangle$$

$$= \langle A^* v_0, p \rangle = r_0 V(t).$$

Then the following holds:

$$V(t) = \exp(r_0 t) V(0). \quad (4.5)$$

This result is called Fisher's principle (Crow and Kimura, 1970, p. 20). The use of the total reproductive value may be found in Charlesworth (1980). Here we use the reproductive value for calculating the coefficient of the stable distribution.

**Proposition 4.2** Let the net reproduction rate matrix $\Psi^*(0)$ be indecomposable. Then the population semigroup $T(t)$ has the property:

$$\lim_{t \to -\infty} \exp(-r_0 t) (T(t)k)(a) = \frac{\langle v_0, k \rangle}{\langle v_0, \psi_0 \rangle} \psi_0(a), \quad (4.6)$$

$$= \frac{V(0)}{v_0(0)^r (-K_1) \psi_0(0)} \psi_0(a), \quad (4.7)$$
where $\psi_0(a)$ is the eigenvector of $A$ corresponding to $r_0$, $v_0(a)$ is the reproductive value vector and the matrix $K_1$ is given by (3.6).

PROOF Since $r_0$ is a simple pole of the resolvent $(\lambda I - A)^{-1}$, then we have (Yosida, 1980, p. 229, Theorem 3)

$$R(P_0) = N(r_0 I - A), \quad R(I - P_0) = R(r_0 I - A),$$

$$L^1 = N(r_0 I - A) \oplus R(r_0 I - A),$$

where $P_0$ denotes the eigenprojection associated with $r_0$. From $R(I - P_0) = R(r_0 I - A)$, there exists $\phi \in D(A)$ for $k \in L^1$ such that

$$(I - P_0)k = (r_0 I - A)\phi.$$ 

From Proposition 3.1, there exists a constant $\gamma$ such that $P_0 k = \gamma \cdot \psi_0$. Observe that

$$\langle v_0, (I - P_0)k \rangle = \langle v_0, (r_0 I - A)\phi \rangle = r_0 \langle v_0, \phi \rangle - \langle v_0, A\phi \rangle = r_0 \langle v_0, \phi \rangle - \langle A^* v_0, \phi \rangle = 0,$$

$$\langle v_0, k \rangle = \langle v_0, P_0 k + (I - P_0)k \rangle = \gamma \langle v_0, \psi_0 \rangle.$$

Since

$$\langle v_0, \psi_0 \rangle = v_0(0)^{\tau} \left( \int_0^\omega \zeta \exp(-r_0 \zeta) \Psi(\zeta) d\zeta \right) \psi_0(0) = v_0(0)^{\tau} (-K_1) \psi_0(0),$$

then we obtain $\langle v_0, \psi_0 \rangle > 0$ and

$$P_0 k = \frac{\langle v_0, k \rangle}{\langle v_0, \psi_0 \rangle} \psi_0 = \frac{V(0)}{v_0(0)^{\tau} (-K_1) \psi_0(0)} \psi_0.$$

From Proposition 3.1, we have (4.6) and (4.7) immediately. □

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REFERENCES


APPENDIX

We shall prove Proposition 3.1. First of all, we state some definitions and results, which are showed in Webb (1985, 1986). Let $A$ be a closed linear operator in the Banach space $X$. An eigenvalue $\lambda_0 \in \mathbb{R}$ is strictly dominant when $\lambda_0 > \text{Re} \lambda$ for any $\lambda \in \sigma(A) - \{\lambda_0\}$ and simple when $\dim R(P_0) = 1$, where $P_0$ denotes the eigenprojection associated with $\lambda_0$ and $R(\cdot)$ denotes the range of the operator. The generalized eigenspace $N_\lambda(A)$ for $\lambda \in \sigma(A)$ is the smallest subspace of $X$ containing $\bigcup_{k=1}^\infty N((\lambda I - A)^k)$. The essential spectrum $E\sigma(A)$ is $\{\lambda \in \sigma(A); \text{either } R(\lambda I - A) \text{ is not closed, } \lambda \text{ is a limit point of } \sigma(A), \text{ or } N_\lambda(A) \text{ is infinite dimensional}\}$. If $A$ is bounded, then the spectral radius $r_\sigma(A)$ is $\sup\{|\lambda|; \lambda \in \sigma(A)\}$, the essential spectral radius $r_{E\sigma}(A)$ is $\sup\{|\lambda|; \lambda \in E\sigma(A)\}$. If $M$ is a bounded subset of $X$, the measure of noncompactness of $M$, denoted by $\alpha[M]$, is the infimum of $\varepsilon > 0$ such that $M$ can be covered by a finite number of subsets of $X$ each with diameter no larger than $\varepsilon$. If $A$ is a bounded linear operator in $X$, the measure of noncompactness of $A$, denoted by $\alpha[A]$, is the infimum of $\varepsilon > 0$ such that $\alpha[A(M)] \leq \varepsilon \alpha[M]$ for all bounded sets $M$ in $X$.

**Lemma A** Let $A$ and $B$ be bounded linear operators in $X$. Then $\alpha[A] = 0$ if and only if $A$ is compact, $\alpha[A] \leq \|A\|$, $\alpha[A + B] \leq \alpha[A] + \alpha[B]$ and $\alpha[AB] \leq \alpha[A] \alpha[B]$.

**Lemma B** Let $T(t)$, $t \geq 0$ be a strongly continuous semigroup of bounded linear operators with infinitesimal generator $A$ in the Banach space $X$. Then the following hold:

1. The growth bound of $T(t);
   \begin{align*}
   \omega_0(A) \equiv & \lim_{t \to \infty} \frac{\log(\|T(t)\|)}{t} \text{ exists.}
   \end{align*}

2. The essential growth bound of $T(t);
   \begin{align*}
   \omega_1(A) \equiv & \lim_{t \to \infty} \frac{\log(\alpha[T(t)])}{t} \text{ exists.}
   \end{align*}

3. If $\gamma > \omega_0(A)$, then there exists $M(\gamma) \geq 1$ such that
   \begin{align*}
   \|T(t)\| \leq M(\gamma) \exp(\gamma t), \quad t \geq 0.
   \end{align*}

4. $\sup_{\lambda \in \sigma(A)} \text{Re} \lambda \leq \omega_0(A)$ and $\sup_{\lambda \in E\sigma(A)} \text{Re} \lambda \leq \omega_1(A)$.

5. $r_\sigma(T(t)) = \exp\{\omega_0(A)t\}$ and $r_{E\sigma}(T(t)) = \exp\{\omega_1(A)t\}$.

6. $\omega_0(A) = \max\{\omega_1(A), \sup_{\lambda \in \sigma(A) - E\sigma(A)} \text{Re} \lambda\}$. 
7. If \( \lambda_0 > \gamma > \omega_1(A) \), \( \lambda_0 \in \sigma(A) \), and \( \sup_{\lambda \in \sigma(A) - E\sigma(A), \lambda \neq \lambda_0} \text{Re} \lambda < \gamma \), then there exists a direct sum decomposition \( X = X_0 \oplus X_1 \) and associated projection \( P_i, P_i X = X_i, \ i = 0, 1 \), such that \( P_0 = (1/2\pi i) \times \int_\Gamma (\lambda I - A)^{-1} d\lambda \) (where \( \Gamma \) is a positively oriented closed curve in the complex plane enclosing \( \lambda_0 \) but no other point of \( \sigma(A) \)), \( P_1 = I - P_0 \), \( X_0 = N_{\lambda_0}(A) \), \( T(t)P_0 = \exp(tA_0)P_0, \ t \geq 0 \) (where \( A_0 \) is the restriction of \( A \) to \( N_{\lambda_0}(A) \)), and for some constant \( M_1 \geq 1 \), \( \|T(t)P_1\| \leq M_1 \exp(\gamma t)\|P_1\|, \ t \geq 0 \).

**Lemma C** Let \( A \) be a closed linear operator in the Banach space \( X \). If \( \lambda \in \sigma(A) - E\sigma(A) \), then \( \lambda \) is a pole of \( (\lambda I - A)^{-1} \) and \( \lambda \) is an eigenvalue of \( A \).

**Proof of Proposition 3.1 (sufficiency)** Assume that \( \lambda_0 \) is a strictly dominant and simple eigenvalue of the infinitesimal generator \( A \) and \( \lambda_0 > \omega_1(A) \). Since \( \lambda_0 > \omega_1(A) \geq \sup \{\text{Re} \lambda; \lambda \in E\sigma(A)\} \) (Lemma B-4), then \( \lambda_0 \in \sigma(A) \setminus E\sigma(A) \). Thus \( \lambda_0 \) is a pole of the resolvent \( (\lambda I - A)^{-1} \) and an eigenvalue of \( A \) (Lemma C). Suppose there exists a sequence \( \{\lambda_n\}, n = 1, 2, \ldots \) such that \( \lambda_n \in \sigma(A), \text{Re} \lambda_{n+1} > \text{Re} \lambda_n, \text{Re} \lambda_n \to \lambda_0 \) as \( n \to \infty \). Let \( t > 0 \) be fixed. From the spectral mapping theorem, we conclude that \( \exp(\lambda_n t) \in \sigma(T(t)) \), \( |\exp(\lambda_n t)| \geq \exp(\lambda_0 t) \) and \( |\exp(\lambda_n t)| \to \exp(\lambda_0 t) \) as \( t \to \infty \). Thus \( \sigma(T(t)) \) contains a limit point on the circle \( |z| = \exp(\lambda_0 t) \). Thus

\[
\text{re}_{E\sigma}(T(t)) = \exp(\omega_1(A)t) \geq \exp(\lambda_0 t),
\]

that is, \( \omega_1(A) \geq \lambda_0 \). This contradicts our assumption. Then there exists an \( \eta > 0 \) such that

\[
\lambda_0 - \eta \geq \omega_1(A) \quad \text{and} \quad \text{Re} \lambda \leq \lambda_0 - \eta \quad \text{if} \quad \lambda \in \sigma(A) \setminus \{\lambda_0\}.
\]

Let \( P_0 \) be the eigenprojection associated with \( \lambda_0 \) and let \( P_1 = I - P_0 \). Since \( \lambda_0 \) is a simple eigenvalue, it follows from Lemma B-(7) that \( X = R(P_0) \oplus R(P_1) = N(\lambda_0 I - A) \oplus R(\lambda_0 I - A) \), \( N(\lambda_0 I - A) = \{ \gamma \cdot \psi_0; \gamma \in C \} \) where \( \psi_0 \) is an eigenvector of \( A \) associated with \( \lambda_0 \), \( T(t)P_0 = \exp(\lambda_0 t)P_0 \) and there exists a constant \( M \geq 1 \) such that

\[
\|T(t)P_1\| \leq M \cdot \exp((\lambda_0 - \varepsilon)t)\|P_1\| \quad \text{for all} \quad \varepsilon \in (0, \eta).
\]

Therefore, we obtain

\[
\lim_{t \to \infty} \|\exp(-\lambda_0 t)T(t) - P_0\| = \lim_{t \to \infty} \|\exp(-\lambda_0 t)T(t)P_1\| \\
\leq M \cdot \exp(-\varepsilon t)\|P_1\| \to 0.
\]
Since there exists a $\gamma$ such that $P_0 \phi = \gamma \cdot \psi_0$ for all $\phi \in X$, $T(t)$ has a stable distribution $\psi_0$ with intrinsic growth rate $\lambda_0$.

(necessity) Suppose that $T(t)$ has a stable distribution $\psi_0$ with intrinsic growth rate $\lambda_0$. Let $Q\phi = \lim_{t \to \infty} \exp(-\lambda_0 t)T(t)\phi$ for $\phi \in X$. Then it is easily seen that $Q$ is a projection with rank one, $T(t)Q = QT(t) = \exp(\lambda_0 t)Q$, $t \geq 0$ and $AQ\phi = \lim_{t \to 0} (T(t)Q\phi - Q\phi)/t = \lambda_0 Q\phi$, $\phi \in X$. Thus $\lambda_0$ is an eigenvalue of $A$ and $\psi_0$ is an eigenvector associated with $\lambda_0$. Since $Q$ is a projection, there exists a direct sum decomposition of $X$ as $X = QX \oplus (I - Q)X$. Let $T_1(t)$ be the restriction of $T(t)$ to $(I - Q)X$ and let $A_1$ be the infinitesimal generator of $T_1(t)$. From
\[
\lim_{t \to \infty} \|\exp(-\lambda_0 t)T_1(t)\| = \lim_{t \to \infty} \|\exp(-\lambda_0 t)T(t)(I - Q)\| = \lim_{t \to \infty} \|\exp(-\lambda_0 t)T(t) - Q\| + \|\exp(-\lambda_0 t)T(t) - Q\|Q\| = 0,
\]
it follows that $\omega_0(A_1) < \lambda_0$, because $\exp\{\omega_0(A_1)t\} = r_\sigma(T_1(t)) \leq \|T_1(t)\|$. From Lemma B-(3), there exist numbers $\epsilon > 0$, $M(\epsilon) \geq 1$ such that
\[
\|T_1(t)\| \leq M(\epsilon)\exp\{(\lambda_0 - \epsilon)t\}, \quad t \geq 0.
\]
Since $T(t)Q$ is rank one, then
\[
\alpha[T(t)] \leq \alpha[T(t)Q] + \alpha[T(t)(I - Q)] = \alpha[T(t)(I - Q)] \leq \|T_1(t)\| \leq M(\epsilon)\exp\{(\lambda_0 - \epsilon)t\},
\]
which shows that $\omega_1(A) \leq \lambda_0 - \epsilon < \lambda_0$. Next, suppose that there exists $\lambda_1 \in \sigma(A)$ such that $\Re \lambda_1 \geq \gamma$. From $\lambda_0 > \omega_1(A)$, $\lambda_1$ is an eigenvalue of $A$. Then there exists $\phi_1 \in X$ such that $T(t)\phi_1 = \exp(\lambda_1 t)\phi_1$. Since $\lim_{t \to \infty} \exp(-\lambda_0 t)T(t)\phi_1 = \lim_{t \to \infty} \exp\{(\lambda_1 - \lambda_0)t\}\phi_1$ converges, it must follow that $\lambda_1 = \lambda_0$. Then $\lambda_0$ is a strictly dominant eigenvalue. It can be shown that $\lambda_0$ is a simple pole of $(\lambda I - A)^{-1}$ (Webb, 1986). From the same argument as the proof of sufficiency, we conclude that $\lim_{t \to \infty} \exp(-\lambda_0 t)T(t)\phi = P_0\phi$, where $P_0$ is the eigenprojection associated with $\lambda_0$. Thus $P_0 = Q$ and $\dim R(P_0) = 1$, which shows that $\lambda_0$ is simple. This completes our proof.