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HISASHI INABA

Graduate School of Mathematical Sciences, University of Tokyo, Japan

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ON A PANDEMIC THRESHOLD THEOREM OF THE EARLY KERMACK–McKENDRICK MODEL WITH INDIVIDUAL HETEROGENEITY

Hisashi Inaba
Graduate School of Mathematical Sciences, University of Tokyo, Japan

A pandemic threshold theorem of the Kermack–McKendrick epidemic system with individual heterogeneity is proved from the definition of $R_0$ by Diekmann, Heesterbeek, and Metz. The early Kermack–McKendrick epidemic model is extended to recognize individual heterogeneity, where the state variable indicates an epidemiological state or genetic, physiological, or behavioral characteristics such as risk of infection. With the basic reproduction number $R_0$ for the heterogeneous population, the final size equation of the limit epidemic starting from a completely susceptible steady state at $t = -\infty$ has a unique positive solution if and only if $R_0 > 1$. The main result is that the positive solution of the final size equation gives the lower bound of the intensity of any epidemic starting from a host population composed of susceptible and a few infected individuals who spread on a noncompact domain of the trait variable.

Keywords: basic reproduction number; final size equation; Kermack-McKendrick model; pandemic threshold theorem

1. INTRODUCTION

David G. Kendall (1957) proposed a spatial extension of the early Kermack–McKendrick epidemic model (Kermack and McKendrick, 1927) and stated his Pandemic Threshold Theorem for the whole space $\mathbb{R}^2$. Bailey (1975: 176) defined a pandemic as when the proportion of individuals contracting the disease, whatever the distance from the initial focus of infection is, is greater or equal to the root of the final size equation, provided that the basic reproduction number exceeds unity.

Diekmann (1978), Thieme (1977a, 1977b), and Webb (1980, 1981) extended Kendall’s pandemic threshold result to the infection-age structured Kermack–McKendrick model with continuous individual heterogeneity. However, those early studies for the spatial Kermack–McKendrick model preceded the definition of the basic reproduction number $R_0$ for heterogeneous populations (Diekmann et al., 1990). Rass and Radcliffe (2003) did not explicitly use the basic reproduction number $R_0$, but formulated their pandemic threshold from the spectral radius of the transmission matrix, which is the basic reproduction number. They assume a separable mixing transmission kernel composed of a distance function of a continuous spatial variable and finite-dimensional trait variables. The definition of final size is different from the original definition of Kermack and McKendrick.

Address correspondence to Hisashi Inaba, Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan. E-mail: inaba@ms.u-tokyo.ac.jp
I consider the pandemic threshold for the early Kermack–McKendrick model with continuous individual heterogeneity based on the modern $R_0$ theory by adopting a general transmission kernel and the original definition of final size. I shall formulate the early Kermack–McKendrick model to recognize individual heterogeneity, where the continuous trait variable reflects not only a geographical distribution but also a biological or social heterogeneity of individuals and the transmission of the infectious agent among different individuals. I shall define the basic reproduction number for the heterogeneous infected population and prove that the final size equation of the limit epidemic, starting from a completely susceptible steady state at $t = 0$, has a unique positive solution if and only if $R_0 > 1$. The main result is that the positive solution of the final size equation of the limit epidemic gives the lower bound of the intensity of any epidemic starting from a host population composed of susceptible and few infected individuals with a noncompact domain of heterogeneity.

2. THE MODEL AND $R_0$

I extend the model of Kermack–McKendrick (1927) to take into account individual heterogeneity expressed by continuous variables. Let $\xi$ be a scalar or a vector parameter with domain $\Omega \subset \mathbb{R}^n$. $\xi$ indicates any epidemiological state, or genetic, physiological, behavioral characteristics such as the risk of infection. I assume that the heterogeneity parameter of an individual does not change during the relatively short time of the epidemic. For example, age is time-dependent, but it can be taken as constant when the epidemic is brief enough. I only deal with an epidemic which ends with the disappearance of the infected because there is no supplement of susceptible.

$S(t, \xi)$ is the susceptible, $i(t, \tau, \xi)$ the infected, and $R(t, \xi)$ the recovered population density with state $\xi$ at time $t$, where $\tau$ denotes the time elapsed since infection. Then the Kermack–McKendrick model with individual heterogeneity is

$$
\frac{dS(t, \xi)}{dt} = -\lambda(t, \xi)S(t, \xi),
\frac{di(t, \tau, \xi)}{dt} + \frac{di(t, \tau, \xi)}{d\tau} = -\gamma(\tau, \xi)i(t, \tau, \xi),
i(t(0, 0, \xi)) = \lambda(t, \xi)S(t, \xi),
\frac{dR(t, \xi)}{dt} = \int_0^\infty \gamma(\tau, \xi)i(t, \tau, \xi)\,d\tau,
$$

where the force of infection $\lambda(t, \xi)$ is

$$
\lambda(t, \xi) = \int_0^\infty \int_\Omega \beta(\tau, \xi, \eta)i(t, \tau, \eta)\,d\eta\,d\tau.
$$

$\beta(\tau, \xi, \eta)$ denotes the transmission coefficient from the infective with time elapsed since infection $\tau$ in state $\eta$ to the susceptible in state $\xi$. The recovery rate in state $\eta$ and time elapsed since infection $\tau$ is denoted $\gamma(\tau, \eta)$. $S(0, \xi) = S_0(\xi)$ and $i(0, \tau, \xi) = i_0(\tau, \xi)$ are the initial data. $N(\xi)$ is the density of the total population at state $\xi$:

$$
N(\xi) := S(t, \xi) + \int_0^\infty i(t, \tau, \xi)\,d\tau + R(t, \xi),
$$
which I assume as time-independent. I assume that \( S_0 \) and \( N \) belong to \( L^1_+(\Omega) \cap L^\infty(\Omega) \), \( f_0 \in L^1_+(\mathbb{R}_+ \times \Omega) \), and \( N(\xi) \geq S_0(\xi) > 0 \) for almost all \( \xi \in \Omega \). Then there exists a disease-free steady state composed of completely susceptible individuals \((N(\xi), 0, 0)\).

The linearized equation for the infected at \((N, 0, 0)\) is given by

\[
\begin{aligned}
\frac{\partial y(t, \tau, \xi)}{\partial \tau} + \frac{\partial y(t, \tau, \xi)}{\partial \xi} &= -\gamma(\tau, \xi)y(t, \tau, \xi), \\
y(t, 0, \xi) &= N(\xi) \int_0^{\infty} \beta(\tau, \xi, \eta)y(t, \tau, \eta) \, d\eta \, d\tau,
\end{aligned}
\]

where \( y(t, \tau, \xi) \) denotes the density of the infected population in the linear invasion phase.

Integrating the McKendrick Eq. (4) along the characteristic line,

\[
y(t, \tau, \xi) = \begin{cases} 
  b(t - \tau, \xi) \Gamma(\tau, \xi), & t - \tau > 0, \\
  \frac{\Gamma(\tau, \xi)}{\Gamma(\tau - t, \xi)} y(0, t - \xi, \xi), & t - \tau > 0,
\end{cases}
\]

where \( b(t, \xi) := y(t, 0, \xi) \) is the density of newly infected individuals in the initial invasion phase and

\[
\Gamma(\tau, \xi) := \exp\left(-\int_0^\tau \gamma(x, \xi) dx\right)
\]

is the survival rate at state \( \xi \).

Inserting the expression of Eq. (5) into the boundary condition of Eq. (4), we know that the newly infected population density \( b(t, \xi) \) at the disease-free steady state without recovered individuals satisfies the renewal equation:

\[
b(t, \xi) = N(\xi)G(y(0, \cdot))(t, \xi) + N(\xi) \int_0^t \int_\Omega \Psi(\tau, \xi, \eta)b(t - \tau, \eta) \, d\eta \, d\tau,
\]

where

\[
\Psi(\tau, \xi, \eta) := \beta(\tau, \xi, \eta)\Gamma(\tau, \eta),
\]

\[
G(y(0, \cdot))(t, \xi) := \int_0^\infty \int_\Omega \beta(\tau, \xi, \eta) \frac{\Gamma(\tau, \eta)}{\Gamma(\tau - t, \eta)} y(0, t - \tau, \eta) \, d\eta \, d\tau.
\]

Diekmann et al. (1990, 2013) showed that the basic reproduction number \( R_0 \) of the renewal Eq. (7) is given by the spectral radius \( r(K) \) of the linear positive operator, called the next generation operator, \( K \) on \( L^1(\Omega) \) defined by

\[
(Ku)(\xi) := N(\xi) \int_0^\infty \int_\Omega \Psi(\tau, \xi, \eta)u(\eta) \, d\eta \, d\tau, \quad u \in L^1(\Omega).
\]

I introduce a technical assumption for the parameters:

**Assumption 2.1.**

1. \( \beta \) and \( \gamma \) are uniformly bounded nonnegative measurable functions, and \( \inf_{(\tau, \eta) \in \mathbb{R}_+ \times \Omega} \gamma(\tau, \eta) > 0 \).
2. \(\beta\) is comparable with a positive separable mixing function, that is, there exist functions \(\beta_1 \in L_+^\infty(\Omega), \beta_2 \in L_+^\infty(\mathbb{R}^+ \times \Omega)\), and a number \(\alpha > 1\) such that

\[
\beta_1(\xi)\beta_2(\tau, \eta) \leq \beta(\tau, \xi, \eta) \leq \alpha \beta_1(\xi)\beta_2(\tau, \eta), \tag{11}
\]

where \(\inf_{\xi \in \Omega} \beta_1(\xi) > 0\) and \(\beta_2(\tau, \eta) > 0\) for almost all \((\tau, \eta) \in \mathbb{R}^+ \times \Omega\).

3. The following holds uniformly for \((\tau, \eta) \in \mathbb{R}^+ \times \Omega,\)

\[
\lim_{h \to 0} \int_{\Omega} |\beta(\tau, \xi + h, \eta) - \beta(\tau, \xi, \eta)| \, d\xi = 0. \tag{12}
\]

The assumption in Eq. (11) looks restrictive but not unrealistic. For example, if \(\beta_1\) reflects the susceptibility (for example, the immunity level based on the past infection experience) and \(\beta_2\) is the infectivity of host individuals in the evolutionary epidemic (Inaba, 2001, 2002), Eq. (11) is reasonable because there is no clear correlation between those two factors. For example, it is known that the infectivity of HIV can be affected by the genetic character of host individuals. But it is not appropriate if \(\xi\) and \(\eta\) denote spatial positions because the spatial correlation may be strong.

From Assumption 2.1, \(G\) is a positive linear operator from \(L_+^1(\mathbb{R}^+ \times \Omega)\) into itself.

**Proposition 2.2.** Under Assumption 2.1, \(K\) is a nonsupporting and compact operator.

**Proof.** The definition of nonsupporting operator is given in the Appendix (Sawashima, 1964; Marek, 1970; Inaba, 1990). Let \(Y = KB\), where \(B\) is a unit ball of \(L^1(\Omega)\). If \(f \in Y\), there exists a \(u \in L^1(\Omega)\) such that \(f = Ku\). Then

\[
\int_{\Omega} |f(\xi + h) - f(\xi)| \, d\xi \leq \int_0^\infty d\tau \int_{\Omega} d\eta \Gamma(\tau, \eta) |u(\eta)|
\]

\[
\times \int_{\Omega} |N(\xi + h)\beta(\tau, \xi + h, \eta) - N(\xi)\beta(\tau, \xi, \eta)| \, d\xi, \tag{13}
\]

where

\[
\int_0^\infty d\tau \int_{\Omega} d\eta \Gamma(\tau, \eta) |u(\eta)| \leq \frac{1}{\inf_{\eta} \gamma} \tag{14}
\]

because \(\Gamma(\tau, \eta) \leq e^{-(\inf_{\eta})\tau}\) and \(u \in B\). Besides,

\[
\int_{\Omega} |N(\xi + h)\beta(\tau, \xi + h, \eta) - N(\xi)\beta(\tau, \xi, \eta)| \, d\xi
\]

\[
\leq \int_{\Omega} N(\xi + h)|\beta(\tau, \xi + h, \eta) - \beta(\tau, \xi, \eta)| \, d\xi
\]

\[
+ \int_{\Omega} \beta(\tau, \xi, \eta)|N(\xi + h) - N(\xi)| \, d\xi, \tag{15}
\]
where the right hand side goes to zero as $h \to 0$ uniformly for $\tau$ and $\eta$, because $\beta$ is upper bounded and $N \in L^1(\Omega) \cap L^\infty(\Omega)$. For a large positive number $A$ and $C_A: = \{x=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n: |x_j| \leq A, j=1, \ldots, n\},$

$$\int_\Omega |f(\xi)| \, d\xi \leq x\|N\beta_1\|_{L^1} \frac{\|N\beta_2\|_{L^\infty}}{\inf \gamma}. \quad (16)$$

Then $KB$ is bounded and

$$\lim_{A \to \infty} \int_{\Omega \setminus C_A} |f(\xi)| \, d\xi = 0 \quad (17)$$

holds uniformly for $f \in Y$. The compactness of $K$ follows from the Fréchet–Kolmogolov criterion (Dunford and Schwartz, 1958: 301, theorem 21). From Eq. (11),

$$(K\phi)(\xi) \geq N(\xi)\beta_1(\xi)\langle x^*, \phi \rangle, \quad (18)$$

where $N\beta_1 \in L^1_+(\Omega)$ and $x^*$ is a strictly positive functional defined by

$$\langle x^*, \phi \rangle = \int_\Omega \int_0^\infty \beta_2(\tau, \eta)\Gamma(\tau, \eta)\phi(\eta) \, d\tau \, d\eta, \quad \phi \in L^1_+(\Omega) \quad (19)$$

and $\langle x^*, \phi \rangle$ is the value of the functional $x^*$ at $\phi \in L^1$. Then

$$K^n\phi \geq \langle x^*, \phi \rangle \langle x^*, N\beta_1 \rangle^{n-1} N\beta_1, \quad (20)$$

which means that for every $n \geq 1$, $K^n\phi$ is a quasi-interior point (also called nonsupporting point) for any $\phi \in L^1_+(\Omega) \setminus \{0\}$. Then $K$ is a nonsupporting operator. □

From Proposition 2.2, Assumption 2.1 implies that the next generation operator is nonsupporting and compact, although $\Omega$ may not be a compact set. From the theory of nonsupporting operators (Appendix or Theorem 2.2, and theorem 2.3 in Marek (1970)), the spectral radius $r(K)$ is the dominant positive eigenvalue of $K$ associated with a positive eigenfunction. From the renewal theorem, the Malthusian parameter

$$\lambda_0 := \lim_{t \to \infty} \frac{\ln \|b(t, \cdot)\|_{L^1}}{t} \quad (21)$$

exists and $\text{sgn}(\lambda_0) = \text{sgn}(R_0 - 1)$ (Inaba, 2012). Then the epidemic outbreak occurs if $R_0 > 1$, while it does not if $R_0 < 1$.

3. THE INITIAL VALUE PROBLEM AND A RESULT FOR A COMPACT DOMAIN

Here I introduce the final size of the epidemic and explain an existing result related to my motivation. Consider an epidemic with given initial data at $t=0$. 


The integration of the McKendrick Eq. (1) along the characteristic line yields:

\[ i(t, \tau, \xi) = \begin{cases} B(t - \tau, \xi) \Gamma(\tau, \xi), & t - \tau > 0, \\ \frac{\Gamma(t, \xi)}{\Gamma(\tau, \xi)} i(0, \tau - t, \xi), & t - \tau > 0, \end{cases} \tag{22} \]

where \( B(t, \xi) := i(t, 0, \xi) = -\frac{\partial S(t, \xi)}{\partial t} \) is the density of newly infected individuals. Then I obtain that

\[
\frac{1}{S(t, \xi)} \frac{\partial S(t, \xi)}{\partial t} = -\lambda(t, \xi) = \int_0^t \int_\Omega \Psi(\tau, \xi, \eta) \frac{\partial S}{\partial t}(t - \tau, \eta) \, d\eta \, d\tau - G(i_0)(t, \xi), \tag{23} \]

where \( i_0 := i(0, \tau, \xi) \).

Define the cumulative force of infection as

\[
\Lambda(t, \xi) := \int_0^t \int_\Omega \lambda(x, \xi) \, dx = -\ln \frac{S(t, \xi)}{S_0(\xi)}, \tag{24} \]

where \( S_0(\xi) = S(0, \xi) > 0 \). Integrating both sides of Eq. (23) with respect to time from 0 to \( t \) yields the nonlinear renewal equation:

\[
\Lambda(t, \xi) = g(t, \xi) + \int_0^t \int_\Omega \Psi(\tau, \xi, \eta) S_0(\eta) f(\Lambda(t - \tau, \eta)) \, d\eta \, d\tau, \tag{25} \]

where

\[
f(x) := 1 - e^{-x}, \quad g(t, \xi) := \int_0^t G(i_0)(\sigma, \xi) \, d\sigma. \tag{26}\]

A convenient framework for studying Eq. (25) is the Banach space \( C_T = C([0, T]; BC(\Omega)) \) of continuous functions on \( [0, T] \) with values in \( BC(\Omega) \), which is the set of bounded continuous functions, equipped with the norm \( \| \Lambda \|_{C_T} = \sup_{0 \leq t \leq T} \| \Lambda(t, \cdot) \|_{BC(\Omega)} \) (Diekmann, 1978).

From Assumption 2.1,

\[
\sup_{\xi \in \Omega} \int_0^\infty \int_0^\infty \Psi(\tau, \xi, \eta) N(\eta) \, d\tau \, d\eta < \infty. \tag{27} \]

Then \( \Lambda(t, \xi) \) is uniformly bounded, continuous, and monotone increasing with respect to time \( t \), so \( \Lambda(\infty, \xi) := \lim_{t \to \infty} \Lambda(t, \xi) \) exists in the sense of uniform convergence in compact sets of \( \Omega \). It becomes the solution of the limiting equation:

\[
\Lambda(\infty, \xi) = g(\infty, \xi) + \int_\Omega \int_0^\infty \Psi(\tau, \xi, \eta) S_0(\eta) f(\Lambda(\infty, \eta)) \, d\tau \, d\eta. \tag{28} \]
The intensity of the epidemic or the final size of the epidemic at $\xi$ (Kermack and McKendrick, 1927; Bailey, 1975) is defined by
\[
p(\xi) := 1 - \frac{S(\infty, \xi)}{N(\xi)} = 1 - \frac{S_0(\xi)}{N(\xi)} e^{-\Lambda(\infty, \xi)},
\]
(29)

The final size is the proportion of the total number of host individuals who contract the disease provided that the initial population is composed of susceptible and infected. Rass and Radcliffe (2003) modeled epidemics initiated from outside and their final size was the proportion of the total number of the initial susceptible who contract the disease, so Rass and Radcliffe assumed that $S_0(\xi) = N(\xi)$ and the final size was $1 - S(\infty, \xi)/S_0(\xi)$.

Based on Eq. (28), Diekmann (1978) extended Kendall’s pandemic threshold theorem to the Kermack–McKendrick model with time elapsed since infection. Define
\[
s_0 := \inf_{\xi \in \Omega} S_0(\xi), \quad \psi_0 := \inf_{\xi \in \Omega} \int_0^\infty \int_0^\infty \Psi(\tau, \xi, \eta) d\tau d\eta.
\]
(30)

Diekmann (1978, theorem 4.1) proved that if $\Omega$ is compact and connected, $s_0\psi_0(x) > x$ for $0 < x < q$, and if for each $\xi \in \Omega$ there exists $\delta = \delta(\xi) > 0$ such that the set $\{x: |x - \xi| \leq \delta\} \cap \Omega$ is contained in the support of $\int_0^\infty \Psi(\xi, \zeta, \eta) d\zeta$, then $\Lambda(\infty, \xi) = q$ for all $\xi \in \Omega$, where $q$ is the largest nonnegative root of $R_0(1 - e^{-x}) = x$ with $R_0 := s_0\psi_0$. As
\[
p(\xi) \geq 1 - e^{-\Lambda(\infty, \xi)} \geq 1 - e^{-q} = \frac{q}{R_0}
\]
(31)

if $R_0 > 1$, a positive lower bound of $p(\xi)$ is given by the unique positive root $p = q/R_0$ of the intensity equation $1 - x = e^{-R_0 x}$.

This threshold result means that if $R_0 > 1$, the epidemic outbreak ultimately occurs everywhere in $\Omega$, no matter how small the initial infected population (the hair-trigger effect). Diekmann (1978) shows that the same kind of threshold result holds for the noncompact domain $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}^2$ when the transmission coefficient $\beta$ is given by a separable function $\beta(\tau, \xi, \eta) = f_1(\tau) f_2(\xi - \eta)$, but this kind of separable function is not consistent with Assumption 2.1.

Diekmann’s pandemic threshold result is not based on the basic reproduction number $R_0$ defined in the previous section. From the definition of the next generation operator $K$, we have $KS_0 \geq R_0 N$. Then we have
\[
\langle f^*, KS_0 \rangle = \langle K^* f^*, S_0 \rangle = R_0 \langle f^*, S_0 \rangle \geq R_0 \langle f^*, N \rangle,
\]
(32)

where $f^*$ is the adjoint positive eigenfunctional of $K$ associated with the eigenvalue $R_0 = r(K)$. Therefore,
\[
R_0 \geq \frac{\langle f^*, N \rangle}{\langle f^*, S_0 \rangle} R_0 \geq R_0,
\]
(33)
which means that the condition \( R_0 > 1 \) implies the invasion condition \( R_0 > 1 \). In order to calculate \( R_0 \), the initial density of susceptible \( S_0 \) is necessary but usually difficult to know.

Therefore, I investigate the existence of a positive lower bound to the intensity of the epidemic as a positive solution of the final size equation, without knowledge of the initial data, in the case \( R_0 > 1 \).

4. THE FINAL SIZE OF THE LIMIT EPIDEMIC

I consider the limit epidemic starting from a susceptible steady state at \( t = -\infty \); the population size of initially infected is infinitesimally small. In a real situation, an epidemic in a large scale population can start from a few cases (Metz and Diekmann, 1986, section 4.1; Diekmann et al., 2013, section 8.4), so the limit epidemic is not far from reality.

Integrating the McKendrick Eq. (1) along the characteristic line, we have

\[
i(t, \tau, \zeta) = B(t - \tau, \zeta) \Gamma(\tau, \zeta),
\]

where

\[
B(t, \zeta) := i(t, 0, \zeta) = \lambda(t, \zeta) S(t, \zeta) = - \frac{\partial S(t, \zeta)}{\partial t}
\]

is the density of newly infected individuals. Then

\[
\frac{1}{S(t, \zeta)} \frac{\partial S(t, \zeta)}{\partial t} = -\lambda(t, \zeta) = \int_0^\infty \int_\Omega \Psi(\tau, \zeta, \eta) S(t - \tau, \eta) d\eta d\tau.
\]

Define the cumulative force of infection as

\[
\Lambda(t, \zeta) := \int_{-\infty}^t \lambda(x, \zeta) dx = - \ln \frac{S(t, \zeta)}{N(\zeta)},
\]

where \( S(-\infty, \zeta) = N(\zeta) > 0 \) for all \( \zeta \in \Omega \).

Integrating Eq. (36) with respect to \( t \) from \( -\infty \) to \( t \) yields the nonlinear renewal equation:

\[
\Lambda(t, \zeta) = \int_0^\infty \int_\Omega \Psi(\tau, \zeta, \eta) N(\eta) f(\Lambda(t - \tau, \eta)) d\eta d\tau.
\]

Again \( \Lambda(t, \zeta) \) is uniformly bounded, continuous, and monotone increasing with respect to time \( t \), so that \( \Lambda(\infty, \zeta) := \lim_{t \to \infty} \Lambda(t, \zeta) \) exists and is the solution of the limiting equation:

\[
\Lambda(\infty, \zeta) = \int_\Omega \int_0^\infty \Psi(\tau, \zeta, \eta) N(\eta) f(\Lambda(\infty, \eta)) d\tau d\eta.
\]
The intensity of the epidemic at state $\xi$ for the limit epidemic is

$$p_{\infty}(\xi) := 1 - \frac{S(\infty, \xi)}{N(\xi)} = 1 - \frac{S(\infty, \xi)}{S(-\infty, \xi)}.$$  \hfill (40)

$p_{\infty}(\xi)$ gives the ultimate proportion of recovered individuals with trait $\xi$, which is the final size of the limit epidemic. Using $p_{\infty}(\xi)$ and the fact that $S(\infty, \xi) = N(\xi)e^{-A(\infty, \xi)}$, we have

$$p_{\infty}(\xi) = 1 - e^{-A(\infty, \xi)},$$  \hfill (41)

then Eq. (39) becomes

$$- \ln(1 - p_{\infty}(\xi)) = \int_0^\infty \int_0^\infty \Psi(\tau, \xi, \eta) N(\eta) p_{\infty}(\eta) d\tau d\eta.$$  \hfill (42)

I seek a positive solution of Eq. (42) in $L^\infty$. The $L^\infty$-solution is a solution in $BC(\Omega)$ if the initial data and $\beta(\tau, \xi, \eta)$ are continuous with respect to $\xi$.

Eq. (42) is rewritten as the final size operator equation:

$$1 - \phi(\xi) = \exp(-(U^{-1}K\phi)(\xi)), \quad \phi \in L^\infty_+(\Omega),$$  \hfill (43)

where $U: L^\infty \to L^1$ is the multiplication operator:

$$(U\phi)(\xi) := N(\xi)\phi(\xi).$$  \hfill (44)

If the final size operator Eq. (43) has a positive solution, it gives the final size distribution of the limit epidemic satisfying Eq. (42).

The positive solution of Eq. (43) corresponds to the fixed point $x = F(x)$ on $L^1(\Omega)$, where $x := U\phi$ and

$$F(x) := U(1 - \exp(-(U^{-1}Kx))).$$  \hfill (45)

$F$ is a positive operator from $L^1_+(\Omega)$ into the convex bounded set $D := \{N\psi \in L^1_+(\Omega) : 0 \leq \psi \leq 1, \psi \in L^\infty(\Omega)\}$, and its Fréchet derivative $F(0)$ at the origin is the next generation operator $K$. If $F$ has a positive fixed point $x \in D$, $U^{-1}x$ gives a positive solution of the final size Eq. (43) in $L^\infty_+(\Omega)$.

**Lemma 4.1.** Under Assumption 2.1, the positive operator $F$ does not have two distinct non-zero fixed points in the cone.

**Proof.** From Proposition 7.6 in the Appendix, it is sufficient to show that under Assumption 2.1, the positive operator $F$ is monotone and concave in $E_+ = L^1_+(\Omega)$ and there exists a $u_0 \in E_+ \setminus \{0\}$ such that for any $x \in E_+$ and any $0 < t < 1$, there exists a number $\eta = \eta(x, t) > 0$ satisfying

$$F(tx) \geq tF(x) + \eta u_0.$$  \hfill (46)
Because $F$ is monotone, I show that $F(x), x \in E_+ \setminus \{0\}$ is comparable with $N$. $F(x) \leq N$ for any $x \in E_+$, and

$$(U^{-1}Kx)(\xi) \geq \langle z^*, x \rangle,$$  \hspace{1cm} (47)

where $z^* := (\inf_{\xi \in \Omega} \beta_1(\xi))x^*$ is a strictly positive functional on $L^1_+(\Omega)$, and $x^*$ is a positive functional given by Eq. (19). Therefore, for any $x \in L^1_+(\Omega) \setminus \{0\}$

$$0 < (1 - e^{-(z^*, x)})N \leq F(x) \leq N.$$  \hspace{1cm} (48)

Next

$$F(tx) - tF(x) = U(1 - e^{-tU^{-1}Kx} - t(1 - e^{-U^{-1}Kx})), \hspace{1cm} (49)$$

where

$$1 - e^{-tU^{-1}Kx} - t(1 - e^{-U^{-1}Kx}) > 0$$  \hspace{1cm} (50)

if $U^{-1}Kx > 0$ for $t \in (0, 1)$. As $U^{-1}Kx \geq \langle z^*, x \rangle > 0$ for $x \in E_+ \setminus \{0\}$, Eq. (46) holds when we choose $u_0 = N$ and a number

$$\eta(x, t) := 1 - e^{-t(z^*, x)} - t(1 - e^{-(z^*, x)})$$  \hspace{1cm} (51)

because $x \rightarrow 1 - e^{-tx} - t(1 - e^{-x})$ is a monotone function of $x$. Therefore $F$ is a concave operator and satisfies inequality (46).

**Proposition 4.2.** Under Assumption 2.1, the final size operator Eq. (43) has a unique positive solution if $R_0 > 1$, while it has no positive solution if $R_0 \leq 1$.

**Proof.** From Proposition 7.7 in the Appendix, $F$ has at least one positive fixed point in $D$ if $R_0 = r(K) = r(F(0)) > 1$, and, from Lemma 4.1, it is a unique positive solution. The inequality $U^{-1}Kx > 1 - \exp(-U^{-1}Kx)$ holds for all $x \in L^1_+(\Omega) \setminus \{0\}$, which implies that $F(0)x = Kx > F(x)$, where, according to the convention of the positive operator theory, $x > y$ means that $x - y \in L^1_+(\Omega) \setminus \{0\}$. If there exists a positive fixed point $x = F(x)$, then $Kx > x$. Let $x^*$ be a strictly positive eigenfunctional of $K^*$ associated with $r(K) = R_0$. Then

$$\langle x^*, Kx \rangle = \langle K^*x^*, x \rangle = r(K)\langle x^*, x \rangle > \langle x^*, x \rangle,$$  \hspace{1cm} (52)

which implies that $r(K) > 1$, because $\langle x^*, x \rangle > 0$. Then there is no positive fixed point if $R_0 = r(K) \leq 1$. \hfill $\square$

## 5. A PANDEMIC THRESHOLD THEOREM

Consider the initial value problem of Eq. (1) that an epidemic starts at $t = 0$ in a host population composed of susceptible and infected individuals. $S(0, \xi) = S_0(\xi) \in$
$L^1_+(\Omega)$ and $i(0, \tau, \xi) = i_0(\tau, \xi) \in L^1(\mathbb{R}_+ \times \Omega)$ are the initial data, such that

$$N(\xi) = S_0(\xi) + \int_0^\infty i_0(\tau, \xi) \, d\tau.$$  \hspace{1cm} (53)

The initial infective population size is

$$\epsilon := \int_0^\infty \int_\Omega i_0(\tau, \xi) \, d\xi \, d\tau$$  \hspace{1cm} (54)

and $u_0(\tau, \xi)$ is the normalized initial distribution given by $u_0(\tau, \xi) = e^{-1}i_0(\tau, \xi)$. Then

$$g(t, \xi) = \int_0^t G(i_0)(\sigma, \xi) \, d\sigma = \epsilon g_0(t, \xi),$$  \hspace{1cm} (55)

where

$$g_0(t, \xi) := \int_0^t G(u_0)(\sigma, \xi) \, d\sigma.$$  \hspace{1cm} (56)

From Assumption 2.1, $g_0(\infty, \xi) < \infty$. In the following, $N$ and $u_0$ are fixed functions, although $\epsilon$ and $S_0$ can change.

$\Lambda(t, \xi; \epsilon)$ is the solution of the renewal equation as

$$\Lambda(t, \xi; \epsilon) = \epsilon g_0(t, \xi) + \int_\Omega \int_0^\infty \Psi(\tau, \xi, \eta)S_0(\eta) f(\Lambda(t, \eta; \epsilon)) \, d\tau \, d\eta.$$  \hspace{1cm} (57)

Then $\Lambda(\infty, \xi; \epsilon) = \lim_{t \to \infty} \Lambda(t, \xi; \epsilon)$ is a positive root of the limiting equation:

$$\Lambda(\infty, \xi; \epsilon) = g_0(\infty, \xi) + \int_\Omega \int_0^\infty \Psi(\tau, \xi, \eta)S_0(\eta) f(\Lambda(\infty, \eta; \epsilon)) \, d\tau \, d\eta.$$  \hspace{1cm} (58)

Because the solution $\Lambda$ is built by a positive iteration from the initial data $\epsilon g_0$ and $f$ is monotone increasing, $\Lambda(\infty, \xi; \epsilon)$ is monotone increasing with respect to $\epsilon$.

The intensity of the epidemic at state $\xi$ is:

$$p(\xi) := 1 - \frac{S(\infty, \xi)}{N(\xi)} = 1 - \frac{S_0(\xi)}{N(\xi)} e^{-\Lambda(\infty, \xi; \epsilon)},$$  \hspace{1cm} (59)

and the cumulative force of infection is:

$$\Lambda(t, \xi; \epsilon) := \int_0^t \lambda(x, \xi) \, dx = -\ln \frac{S(t, \xi)}{S_0(\xi)}.$$  \hspace{1cm} (60)

Then $p(\xi)$ gives the ultimate proportion of recovered individuals at trait $\xi$. This proportion is the final size of the epidemic at $\xi$ with the initial distribution of infected $i_0 = \epsilon u_0$. 

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**PANDEMIC THRESHOLD THEOREM**
The function \( z \in L^1_+(\Omega) \):

\[
z(\xi; \epsilon) := N(\xi)(1 - \exp(-\Lambda(\infty, \xi; \epsilon)) = N(\xi)\left(1 - \frac{S(\infty, \xi)}{S_0(\xi)}\right)
\]

(61)

is monotone increasing with \( \epsilon \), because \( \Lambda(\infty, \xi; \epsilon) \) is monotone increasing with respect to \( \epsilon \).

From Eq. (58),

\[ z \geq U(1 - \exp(-U^{-1}KI, z)), \]

(62)

where \( I_c : L^1 \rightarrow L^1 \) is a multiplication operator defined by

\[ (I_c \phi)(\xi) := \frac{S_0}{N} \phi = \left(1 - \frac{\epsilon}{N(\xi)} \int_0^\infty u_0(\tau, \xi) d\tau\right) \phi(\xi). \]

(63)

Consider the associated operator equation in \( L^1_+(\Omega) \):

\[ y = U(1 - \exp(-U^{-1}KI, y)) =: F_c(y). \]

(64)

**Lemma 5.1.** If \( R_0 > 1 \), for sufficiently small \( \epsilon > 0 \), Eq. (64) has a unique positive solution \( y(\xi; \epsilon) \) in \( L^1_+(\Omega) \).

**Proof.** \( F'_c(0) \) is the Fréchet derivative of \( F_c \) at the origin. Then \( F'_c(0) \rightarrow F'(0) \) in the sense of the operator norm when \( \epsilon \downarrow 0 \) because for \( \phi \in L^1 \),

\[
\|F'(0)\phi - F'_c(0)\phi\| \leq K\|\|I - I_c\||\phi\|
\]

(65)

where \( I \) denotes the identity operator and \( \lim_{\epsilon \downarrow 0}\|I - I_c\| = 0 \). Therefore, from \( R_0 = r(F'(0)) > 1 \), for sufficiently small \( \epsilon > 0 \), \( r(F'_c(0)) > 1 \). By using the argument presented in the proof of Proposition 4.2, Eq. (64) has a unique positive solution \( y(\xi; \epsilon) \) in \( L^1_+(\Omega) \).

**Lemma 5.2.** If \( R_0 > 1 \),

\[
\lim_{\epsilon \downarrow 0} z(\xi; \epsilon) = \lim_{\epsilon \downarrow 0} y(\xi; \epsilon) = N(\xi)p_\infty(\xi).
\]

(66)

**Proof.** For a sufficiently small \( \epsilon > 0 \), from Lemma 5.1, the positive solution \( y \) of Eq. (64) exists for all \( \epsilon \in (0, \epsilon') \). Define a sequence \( \{y_n\}_{n=0,1,2,...} \) by \( y_n = F_c(y_{n-1}) \) with \( y_0 = z \). Then \( y_0 = z \geq F_c(y_0) = y_1 \). Because \( F_c \) is a monotone operator, the series \( \{y_n\}_{n=0,1,2,...} \) is positive monotone decreasing. Because \( F_c \) is a monotone concave operator having a unique nonzero fixed point in the normal cone, \( y_n \) converges to the unique nonzero fixed point \( y = y(\xi; \epsilon) \) of \( F_c \) (Krasnoselskii, 1964, theorem 6.6). Then \( z \geq \lim_{n \rightarrow \infty} y_n = y > 0 \). As \( \lim_{\epsilon \downarrow 0} y = \lim_{\epsilon \downarrow 0} F_c(y) = F(\lim_{\epsilon \downarrow 0} y) \), \( \lim_{\epsilon \downarrow 0} y = p_\infty N \).
Moreover,

\[ N(\xi)p(\xi) = N(\xi) - S(\infty, \xi) \geq \frac{N(\xi)}{S_0(\xi)} (S_0(\xi) - S(\infty, \xi)) = z(\xi; \epsilon) \]  

(67)

so that \( \lim_{\epsilon \downarrow 0} z = \lim_{\epsilon \downarrow 0} y = p_\infty N \geq \lim_{\epsilon \downarrow 0} z \), which proves Eq. (66).

\[ \text{Proposition 5.3. For the intensity of the epidemic } p(\xi) \text{ given by Eq. (59),} \]

\[ \lim_{\epsilon \downarrow 0} p(\xi) \geq p_\infty(\xi), \]  

(68)

where \( p_\infty \) is the final size of the limit epidemic satisfying Eq. (42).

\[ \text{Proof. If } R_0 > 1, \]

\[ p(\xi) = 1 - \frac{S_0(\xi)}{N(\xi)} e^{-\Lambda(\infty, \xi; \epsilon)} \geq 1 - e^{-\Lambda(\infty, \xi; \epsilon)} = \frac{z(\xi, \epsilon)}{N(\xi)}. \]  

(69)

Eq. (68) results from Eq. (66) when \( \epsilon \downarrow 0 \). If \( R_0 \leq 1, p(\xi) \geq p_\infty(\xi) \), because \( p_\infty = 0 \) when \( R_0 \leq 1 \). \( \square \)

6. CONCLUSION

In the threshold theorem of Kermack–McKendrick (1927), the lower bound of the final size of an epidemic is given by the final size of the limit epidemic (Metz and Diekmann, 1986). My Proposition 5.3 is an extension to recognize individual heterogeneity. Instead of assuming connectivity and compactness of the heterogeneity parameter domain or assuming separable mixing for the transmission kernel, condition 2.1 allows the next generation operator to be a compact nonsupporting operator, which guarantees the existence of the basic reproduction number \( R_0 \). The functional analytic framework can be applied to the case of a noncompact domain of the heterogeneity parameter without assuming the separable mixing transmission coefficient. For example, it is convenient to use a noncompact domain to represent evolutionary changes or to consider the spread of the infectious disease in an abstract risk space. The definition of pandemic by Kendall and Bailey originally used a noncompact domain to consider the spatial spread of infectious diseases, although practically it is used in a broader sense.

I proved that the final size equation of the limit epidemic starting from a completely susceptible steady state at \( t = -\infty \) has a unique positive solution if and only if \( R_0 > 1 \). The main result is that the positive solution of the final size equation gives the lower bound of the intensity of any epidemic starting from a host population composed of susceptible and a few infected individuals with a noncompact domain of heterogeneity. I conjecture that the main result holds even when the size of the initial infected is not necessarily small.
The main result does not cover the case where the next generation operator is noncompact and nonsupporting or the case where the transmission coefficient $\beta$ is not comparable with a separable mixing function. Even in those situations, the basic reproduction number $R_0$ can be defined and operate as a threshold value (Inaba, 2012). Although I assume that there is no trait change along individual survival, it will take another work to examine the case that individuals are able to move with a random walk or with a migration in the trait space (Ducrot et al., 2010).

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REFERENCES


**APPENDIX**

I summarize definitions and results for positive operators. $E$ is a real or complex Banach space and $E^*$ is its dual space. $E^*$ is a space of all linear functionals on $E$. The value of $f \in E^*$ at $\psi \in E$ is $\langle f, \psi \rangle$. A closed subset $C \subseteq E$ is called cone (or positive cone) if $C + C \subseteq C$, $\hat{\lambda} C \subseteq C$, $C \cap (-C) = \{0\}$, and $C \neq \{0\}$. With respect to the cone $C$, $x \leq y$ if $y - x \in C$, and $x < y$ if $y - x \in C^+ := C \setminus \{0\}$. If the set $\{\psi - \phi, \psi, \phi \in C\}$ is dense in $E$, the cone $C$ is called total. If $E = C - C$, $C$ is called a reproducing cone.

$B(E)$ is a set of bounded linear operators from $E$ into itself, $r(T)$ is the spectral radius of $T \in B(E)$, and $P_\sigma(T)$ is the point spectrum of $T$. The dual cone $C^*$ is a subset of $E^*$ composed of all positive linear functionals. $f \in C^*$ is called a quasi-interior point or nonsupporting point provided that $\langle f, \psi \rangle > 0$ for all $f \in C^* \setminus \{0\}$. A positive linear functional $f \in C^*$ is called strictly positive if $\langle f, \psi \rangle > 0$ for all $\psi \in C^+$. $T \in B(E)$ is called positive if $T(C) \subseteq C$ and $T \in B(E)$ is called strictly positive if $T(C^+) \subseteq C^+$. If $(T - S)(C) \subseteq C$ for $T, S \in B(E)$, then $S \leq T$.

**Definition 7.1. (Sawashima, 1964; Marek, 1970)** A positive operator $T \in B(E)$ is called semi-nonsupporting if for any $\psi \in C^+$ and $f \in C^* \setminus \{0\}$, there exists an integer $p = p(\psi, f)$ such that $\langle f, T^n \psi \rangle > 0$. A positive operator $T \in B(E)$ is called nonsupporting if for any $\psi \in C^+$ and $f \in C^* \setminus \{0\}$, there exists an integer $p = p(\psi, f)$ such that $\langle f, T^n \psi \rangle > 0$ for all $n \geq p$. A positive operator $T \in B(E)$ is called strictly nonsupporting if for any $\psi \in C^+$, there exists a positive integer $p = p(\psi)$ such that $T^n \psi$ is a quasi-interior point of $C$ for all $n \geq p$. 

...
Semi-nonsupporting extends the indecomposability of nonnegative matrices to the infinite dimension. It is similar to irreducibility (Krasnosel’skij et al., 1989, theorem 11.2).

**Proposition 7.2. (Sawashima, 1964; Marek, 1970)** If the cone $C$ is total, $T \in B(E)$ is semi-nonsupporting with respect to $C$, and $r(T)$ is a pole of resolvent $R(\lambda, T) = (\lambda - T)^{-1}$, then

1. $r(T) \in P_\sigma(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent $R(\lambda, T)$;
2. The eigenspace corresponding to $r(T)$ is one-dimensional and its eigenvector $\psi \in C$ is a quasi-interior point. Any eigenvector in $C$ is proportional to $\psi$;
3. The adjoint eigenspace corresponding to $r(T)$ is one-dimensional and its eigenfunctional $f \in C^* \setminus \{0\}$ is strictly positive.

**Proposition 7.3. (Sawashima, 1964; Marek, 1970)** If the cone $C$ is total, $T \in B(E)$ is nonsupporting with respect to $C$, and $r(T)$ is a pole of resolvent $R(\lambda, T) = (\lambda - T)^{-1}$, then 1–3 of Proposition 7.2 hold and

1. $r(T)$ is a dominant point of the spectrum $\sigma(T)$, that is, $|\mu| < r(T)$ for all $\mu \in \sigma(T) \setminus \{r(T)\}$;
2. $B_1 := \lim_{n \to \infty} r(T)^{-n} T^n$ converges in the operator norm and $B_1$ is a strictly nonsupporting operator given by
   \[ B_1 = \frac{1}{2\pi i} \int_{\Gamma_0} R(\lambda, T)d\lambda, \] (70)
   where $\Gamma_0$ is a positively oriented circle with center at $r(T)$ such that no point of the spectrum $\sigma(T)$ except $r(T)$ lies on or inside the circle $\Gamma_0$.

**Proposition 7.4. (Marek, 1970)** $S$ and $T$ are positive bounded linear operators in a Banach lattice $E$. Then

1. If $S \leq T$, then $r(S) \leq r(T)$.
2. If $S$ and $T$ are semi-nonsupporting and compact, $S \leq T$, $S \neq T$, and $r(T) \neq 0$, then $r(S) < r(T)$.

**Definition 7.5.** For a cone $C$ of a real Banach space $E$ with the partial ordering denoted $\leq$ induced by $C$, a positive operator $A: C \to C$ is called a concave operator if there exists $\psi_0 \in C^+$ satisfying

1. For any $\psi \in C^+$, there exist $\alpha = \alpha(\psi) > 0$ and $\beta = \beta(\psi) > 0$ such that $\alpha \psi_0 \leq A \psi \leq \beta \psi_0$. That is, $A \psi$ for $\psi \in C$ is comparable with $\psi_0$.
2. For any $\psi \in C$ which is comparable with $\psi_0$ and $0 \leq t \leq 1$, $A(t \psi) \geq t A \psi$.

**Proposition 7.6.** Suppose that a positive operator $A: C \to C$ is monotone and concave. If for any $\psi \in C$ satisfying $z_1 \psi_0 \leq \psi \leq \beta_1 \psi_0$ ($z_1 = z_1(\psi) > 0$, $\beta_1 = \beta(\psi) > 0$) and any $0 < t < 1$, there exists $\eta = \eta(\psi, t) > 0$ such that
then $A$ has at most one positive fixed point.

**Proof.** If $A$ has two positive fixed points $\psi_1, \psi_2 \in C^+$, from the concavity assumption, we can choose positive constants $\alpha_1 = \alpha_1(\psi_1) > 0$, $\beta_2 = \beta_2(\psi_2) > 0$ such that $\psi_1 = A\psi_1 \geq \alpha_1 \psi_0 \geq \alpha_1 \beta_2^{-1} A\psi_2 = \alpha_1 \beta_2^{-1} \psi_2$, which implies that $k := \sup \{ \mu : \psi_1 \geq \psi_2 \} > 0$. If $0 < k < 1$, there exists $\eta = \eta(\psi_2, k) > 0$ such that

$$
\psi_1 = A\psi_1 \geq A(k\psi_2) \geq kA\psi_2 + \eta \psi_0 \geq k\psi_2 + \eta \beta_2^{-1} A\psi_2 = (k + \eta \beta_2^{-1}) \psi_2,
$$

which contradicts the definition of $k$. Hence $k \geq 1$ and $\psi_1 \geq \psi_2$. By changing the role of $\psi_1$ and $\psi_2$, the same argument leads to $\psi_2 \geq \psi_1$. Thus $\psi_1 = \psi_2$. □

Because the strong asymptotic derivative $\Psi$ is zero if its range $\Psi(C)$ is bounded, a special case of Krasnosel’skii’s theorem (Krasnosel’skii, 1964, theorem 4.11; Inaba, 1990) is:

**Proposition 7.7.** For a positive operator $\Psi$ from a cone $C$ in a real Banach lattice into itself, if $\Psi(0) = 0$, $\Psi$ has the Fréchet derivative $T := \Psi'(0)$ and $T$ has a positive eigenvector $v_0 \in E_+$ associated with the eigenvalue $\lambda_0 > 1$, but it has no eigenvector in $C$ associated with unity, $\Psi$ is compact and $\Psi(C)$ is bounded, then $\Psi$ has at least one non-zero fixed point in $E_+$.

For a nonlinear positive operator $\Psi$ with $\Psi(0) = 0$, a majorant of the operator $\Phi$ is a linear positive operator $T$ such that $\Psi(x) \leq Tx$ for any $x \in C$. Then,

**Proposition 7.8.** For a positive operator $\Psi$ on a positive cone $C$ of a real Banach space $E$, if there exists a compact and nonsupporting majorant $T$ such that $Tf - \Psi(f) \in C^+$ for any $f \in C$, then $\Psi$ has no fixed point in $C^+$ when $r(T) \leq 1$.

**Proof.** If there exists a nonzero fixed point $x \in C^+$, then $x = \Psi(x) \leq Tx$. The adjoint eigenfunctional $F^* \in C^c \setminus \{0\}$ of $T$ associated with the positive eigenvalue $r(T)$ is strictly positive. Then $\langle F^*, x \rangle > 0$, $\langle F^*, Tx - x \rangle = (r(T) - 1) \langle F^*, x \rangle > 0$, and $r(T) > 1$. □