Resolving a Confusion in the Bongaarts and Feeney’s Tempo-Adjusted Total Fertility Rate

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(Abstract)
In an article, Bongaarts and Feeney proposed a new method for adjusting the period total fertility rate to eliminate the effects of the timing of fertility behavior. Though their method have raised some criticism among demographers, related controversies throw some lights to understand effects of age-shift in demographic variables. In this short note, we do not concern the empirical problem raised by the Bongaarts and Feeney method, but mainly study the logical structure of that method to reveal its hidden hypothesis. As a result, we show that its hidden hypothesis is too difficult to hold in the real world.

1. Introduction

In an article, Bongaarts and Feeney proposed a new method (in the following, we call it as the B-F method for simplicity) for adjusting the period total fertility rate to eliminate the effects of the timing of fertility behavior (Bongaarts and Feeney 1998). Their tempo adjusted total fertility rate by the birth order $i$ is given by

$$TFR_{ad}(i) = \frac{TFR(i)}{1 - \pi_i},$$

(1.1)

where $TFR(i)$ is the total fertility rate of $i$-th birth and $\pi_i$ is the constant speed of age-shift of the $i$-th birth rate along the age axis. Bongaarts and Feeney claimed that their tempo-adjusted total fertility rate provides a better indication of the level of completed fertility implied by current fertility behavior and some authors have extended their method and applied it to real data (Kohler and Philipov 2001, Yi and Land 2002).

On the other hand, the B-F method has been criticized by several authors (Van Imhoff and Keilman 2000, Kim and Schoen 2000, Van Imhoff 2001). Kim and Schoen have insisted that the mathematical basis of $TFR_{ad}$ holds only under very restrictive conditions and that with those restrictions even slightly relaxed, $TFR_{ad}$ is quite volatile in the presence of modest fertility fluctuations. Van Imhoff and Keilman have argued that Bongaarts and Feeney adjustment procedure is based on fertility measures unsuitable for the purpose of tempo adjustment because the use of frequencies exaggerates the effect of tempo distortions and the assumption underlying their method that period-by-period timing changes are independent of age and cohort is not supported by the data.
In reply to Kim and Schoen's critique, Bongaarts and Feeney (2000) have written that their criticism based on the assumption that the B-F method attempts to estimate the completed fertility of actual cohorts is not relevant because they did not attempt to estimate the fertility of actual cohorts. If the B-F adjusted fertility rate does not refer to actual cohorts of women, according to Van Imhoff (2000, 2001), we would like to ask "what does it refer to?"

In order to resolve a confusion for interpretations of the B-F method, we can clearly distinguish three aspects of the problem. The first problem is what is the correct interpretation for the (original) B-F formula, and whether it is mathematically self-consistent. This is a formal aspect of the problems. The second problem is, as criticized by Van Imhoff and Keilman, whether the basic assumed situation that women of all ages bearing children in year t defer or advance their births to the same extent independently of their age or cohort identification is empirically observed or not. The third aspect is, even though the above situation could be observed, whether that situation of age-shift could justify the B-F method. In fact, in my opinion it seems that the B-F method implicitly assumes more than the age-shift. In this short note, we do not concern the second empirical problem, but mainly study the logical structure of the B-F method to clear its hidden hypothesis.

2. The tempo adjustment formula by Bongaarts and Feeney

Bongaarts and Feeney have written their definition for the quantum and the tempo as follows (Bongaarts and Feeney 1998, p. 272):

**Quotation 2.1** We define the quantum component as the TFR that would have been observed in the absence of changes in the timing of childbearing during the period in which the TFR is measured. The tempo component equals the distortion that occurs due to timing changes. Our objective is to measure the quantum component by eliminating the tempo distortion from the conventional TFR. The resulting quantum measure will be called the tempo-adjusted TFR.

According to scenarios in Appendix of Bongaarts and Feeney's paper, let us introduce their Scenarios 1 and 2 to derive the tempo adjustment formula (1.1). Though they have proposed other two scenarios to calculate the tempo-adjusted TFR, the essence of their procedure is shown in Scenarios 1 and 2. Moreover, for simplicity, here we do not consider the parity-specific rate, since it does not affect the essence of our interpretation.

In the following, we denote \( g(t, a) \) as the age-specific fertility rate for women aged \( a \in [0, \infty) \) at time \( t \in (-\infty, \infty) \). Then the period total fertility rate for time \( t \) is given by

\[
TFR(t) = \int_0^\infty g(t, a) da,
\]

and the completed fertility rate for the cohort born at time \( T \) is given by

\[
CFR(T) = \int_0^\infty g(T + a, a) da.
\]

In Scenario 1 by Bongaarts and Feeney, there are no changes in tempo or quantum, hence \( g(t, a) \) is constant with respect to \( t \) for all \( a \), or equivalently, \( g(T + a, a) \) is constant with respect to \( T \) for all \( a \). The age-specific fertility rate is time-independent, so we can write \( g(t, a) = \phi(a) \), where \( \phi(a) \) is a given standard schedule and it has a compact support \([a, \beta]\) (reproductive age period). Then we have \( TFR=CFR \) for all periods and cohorts. In the following this constant total fertility rate is
referred as $TFR_1$ (which is denoted by $TFR$, in Bongaarts and Feeney's paper). That is,

$$TFR_1 = \int_0^\infty \phi(a) da.$$ 

Next let us consider Scenario 2 by Bongaarts and Feeney. They have written as follows (1998, p. 287):

**Quotation 2.2** Beginning with Scenario 1, suppose that cohort fertility (quantum) does not change, but that from time 0 births are deferred, so that age at childbirth rises, and that $f_0(a)$ with respect to its mean does not change. The level of the period schedules in this second scenario will fall as a result of the deferral (or rise, if births are advanced instead of deferred). We want to find a procedure to determine the TFR that would have been observed in the absence of changes in timing (i.e., the Scenario 1 $TFR_1$) from the observed TFR in Scenario 2.

Their assumption of Scenario 2 is loosely stated, it is difficult to understand its mathematically correct meaning. But from their mathematical formulation presented in page 288, we could interpret their assumption as follows:

1. CFR in Scenario 2 is constant over time and it is the same as in Scenario 1,
2. It is observed that the period fertility rate is also constant over time and the age pattern of period fertility in Scenario 2 is invariant, but age-shift occurs with constant speed for the period fertility schedule.

That is, it is assumed that the deferral of childbirth happens so that the above conditions (1) and (2) are satisfied, though we cannot imagine what kind of changes in childbearing behavior at individual level could lead such deferral.

Suppose that such deferral happens after time $t=0$, hence it is assumed that $g(t, a) = \phi(a)$ for $t < 0$. From the above B-F assumption (2), we can write

$$g(t, a) = g(0, a - rt), t \geq 0,$$ \hspace{1cm} (2.1)

where $r$ denotes the constant speed of age-shift along the age axis, so if $r > 0$ the deferral occurs and if $r < 0$ births are advanced. We formally define such that $g(t, a) = 0$ for $a < 0$. Then it follows that for $t \geq 0$

$$TFR(t) = \int_0^\infty g(t, a) da = \int_0^\infty g(0, a - rt) da = \int_0^\infty g(0, a) da.$$ \hspace{1cm} (2.2)

That is, the period total fertility rate is time invariant for $t \geq 0$. We define this period total fertility rate as $TFR_2$:

$$TFR_2 := \int_0^\infty g(0, a) da.$$ 

On the other hand, the total fertility rate for cohorts born after time $T \geq 0$, denoted by $CFR(T)$, is given by

$$CFR(T) = \int_0^\infty g(T + a, a) da = \int_0^\infty g(0, a - r(T + a)) da = \frac{1}{1-r} \int_0^\infty g(0, a) da.$$ \hspace{1cm} (2.3)

Therefore, $CFR(T)$ is also constant for all $T \geq 0$. From the assumption (2), it follows that $CFR(T) = TFR$, so we obtain
\[ TFR_i = \frac{TFR_{\text{obs}}} {1 - r}. \] (2.4)

Under the Scenario 2, the observed period TFR is \( TFR_{\text{obs}} \), the adjusted total fertility rate \( TFR_{\text{adj}} \) is given by \( TFR_i = TFR_{\text{obs}}/(1 - r) \).

After the above calculation, Bongaarts and Feeney says (Bongaarts and Feeney 1998, p. 288):

**Quotation 2.3** Since by assumption the completed fertility of cohorts in Scenario 2 is the same as in Scenario 1, the TFR in Scenario 2 may be estimated by multiplying the TFR in Scenario 1 by \((1 - r)\). Alternatively, if Scenario 2 is observed, then the Scenario 1 TFR can be estimated by dividing the observed TFR by \((1 - r)\).

Since the shape of the age pattern of period fertility in Scenario 2 is by assumption invariant and the same as in Scenario 1, it follows that

\[ g(t, a) = f_0(t, a + d)(1 - r), \] (2.5)

where \( d \) is the total number of years by which \( g_0 \) has been shifted relative to \( f_0 \) at time \( t \). In other words, in Scenario 2 at time \( t \) the original schedule of age-specific fertility rates has been moved along the age axis by an amount \( d \) and has been multiplied by \((1 - r)\).

Though Bongaarts and Feeney’s statement above is not so clear again, we can understand that they here tried to determine the unknown function \( g(t, a) \) for \( t \geq 0 \) explicitly by using the standard schedule \( \phi \). Since the age pattern of \( g(t, a) \) is the same as \( \phi \), we can write as

\[ g(t, a) = g(0, a - rt) = a\phi(a - rt), \] (2.6)

where \( \alpha > 0 \) is unknown constant. For cohorts born at time \( T \geq 0 \), we have

\[ g(T + a, a) = g(0, a - r(a + T)) = a\phi((1 - r)a - rT) \]

\[ = a\phi((1 - r)(a - \frac{r}{1 - r} T)) = h(a - \frac{r}{1 - r} T). \] (2.7)

That is, it is observed for cohorts born at time \( T \geq 0 \) that the age-specific fertility function \( h(a) := a\phi((1 - r)a) \) is shifted by \( r/(1 - r) \) per year. From assumption (1) of the Scenario 2, we obtain

\[ \int_0^\infty h(a - \frac{r}{1 - r} T)da = TFR_i = \int_0^\infty \phi(a)da. \] (2.8)

Hence it follows that \( \alpha = 1 - r \), then we know that

\[ g(t, a) = (1 - r)\phi(a - rt), \quad t \geq 0. \] (2.9)

This is a correct formula that Bongaarts and Feeney would have tried to show in their equation (2.5). Since we assume that the fertility is dominated by the Scenario 1 for \( t < 0 \), we can write

\[ g(t, a) = \begin{cases} \phi(a), & t < 0 \\ (1 - r)\phi(a - rt), & t \geq 0. \end{cases} \] (2.10)

(2.10) completely determines the fertility in the whole age-time plane, which is implied by the B-F assumptions. Then it follows that for cohorts born at time \( T \), its fertility schedule is given by

\[ g(a + T, a) = \begin{cases} \phi(a), & a < -T \\ (1 - r)\phi((1 - r)a - rT), & a \geq -T. \end{cases} \] (2.11)
Then it is easy to see that for any \( T \in (-\infty, \infty) \), it follows that

\[
CFR(T) = \int_{0}^{\infty} g(a + T, a) da = \int_{0}^{\infty} \phi(a) da = TFR_i.
\]  

(2.12)

That is, the cohort total fertility rate is always constant given by \( TFR_i \).

From the formula (2.10), we can reveal the essence of Bongaarts-Feeney adjustment that has been hidden in their vague statements. Certainly their adjustment procedure can recover \( TFR_i \) which would be observed if there were no deferral, but their assumption implies that the deferral in the period age-specific fertility rate suddenly occurs at time zero such that the shift speed \( r \) equals to the reduction rate of the fertility rate! This is a very difficult assumption to be realized in the real world, and we can not imagine what kind of individual behavioral change could produce such a change in the period fertility schedule.

3. Other interpretations

As we have seen above, the original interpretation for the B-F method based on period approach has difficulty, so let us consider other possible interpretations that could induce the B-F formula. The reader may refer to more comprehensive argument for this problem in Van Imhoff (2001) and Yi and Land (2002).

We suppose that the age-specific fertility rate for cohorts born at time \( T \geq 0 \) will be shifted by constant speed, but their quantum are constant for all \( T \geq 0 \). Since the age-specific fertility schedule is given by \( \phi(a) \) before time \( t = 0 \), that is,

\[
g(t, a) = \phi(a), \ t < 0.
\]  

(3.1)

and the fertility rate for \( t \geq 0 \) is given by

\[
g(t, a) = \begin{cases} 
\phi(a - \nu(t - a)), & t - a \geq 0 \\
\phi(a), & a - t > 0.
\end{cases}
\]  

(3.2)

where \( \nu \) is a constant speed of age-shift along the life line.

Then if \( \beta \) is the upper bound of reproductive age, we know that all reproductive women at time \( t > \beta \) adopt the fertility schedule \( \phi(a - \nu(t - a)) \). Observe that

\[
\phi(a - \nu(t - a)) = \phi((1 + \nu)a - \nu t) = \phi((1 + \nu)(a - \frac{\nu}{1 + \nu} t)) = k(a - \frac{\nu}{1 + \nu} t).
\]

where \( k(a) = \phi((1 + \nu)a) \). That is, it is observed at time \( t > \beta \) as if the age-specific fertility schedule \( k(a) \) is shifted along the age axis with speed \( \nu/(1 + \nu) \), which is different from \( \nu \) the speed of age-shift along life line. For \( t > \beta \), the period age specific schedule \( k(a) \) is different from the schedule \( \phi(a) \) for time \( t < 0 \), and the period TFR (\( TFR_0 \)) is also different from the cohort completed fertility rate (\( TFR_i \)), which is also the period TFR for \( t < 0 \):

\[
TFR_0 = \int_{0}^{\infty} \phi((1 + \nu)a) da = \frac{1}{1 + \nu} \int_{0}^{\infty} \phi(a) da = \frac{CFR}{1 + \nu}.
\]  

(3.3)

If we denote the speed of age-shift along the age axis of period fertility schedule as \( r \), we have \( r = \nu/(1 + \nu) \), then it follows that \( \nu = r/(1 - r) \) and
\[ TFR_t = (1-r) \int_0^\infty \phi(a) da = (1-r) \times TFR_t. \] (3.4)

(3.3) and (3.4) have been shown also in Van Imhoff (2001, Appendix). From the above observations, if we observe that the period TFR is \( TFR_t \) and the period schedule is shifted with speed \( r \), \( TFR_t = TFR_0 / (1-r) \) is seen to be the TFR that would have been observed in the absence of changes in timing occurring on cohorts.

However it should be noted that the observed shape of the age pattern of period fertility is \( k(a) = \phi((1+r)a) \), and it is different from \( \phi(a) \). This fact contradicts the basic assumption of the B-F method (see Quotation 2.3), hence we could not adopt the cohort approach as the original interpretation of the B-F method, even though it would be a more reasonable re-interpretation or modification for the B-F method.

Finally, let us consider another possible scenario based on period shift. If we assume that the age shift with constant speed \( r \) occurs along the age axis after time zero, the fertility schedule on the age-time plane can be written as follows:

\[
g(t, a) = \begin{cases} 
\phi(a), & t < 0 \\
\phi(a - rt), & t \geq 0.
\end{cases} \] (3.5)

In this case, it is clear that the period TFR is always constant given by

\[
TFR(t) = \int_0^\infty g(t, a) da = \int_0^\infty \phi(a) da, \quad \forall t \in (-\infty, \infty).
\] (3.6)

Therefore, there is no adjustment problem between period total fertility rates. On the other hand, the cohort completed fertility rate is given by

\[
CFR(T) = \int_0^\infty g(a + T, a) da = \begin{cases} 
\int_0^{-T} \phi(a) da + \frac{1}{1-r} \int_{-T}^\infty \phi(a) da, & T < 0 \\
\frac{1}{1-r} \int_0^\infty \phi(a) da, & T \geq 0
\end{cases}
\] (3.7)

In this case, we have

\[
CFR(T) = \frac{\text{period} \ TFR}{1-r}, \quad T \geq 0.
\] (3.8)

That is, the formal application of the B-F method to the period TFR gives the cohort completed fertility rate born at \( T \geq 0 \). This is a very clear and simple result, but it gives a translation between cohort completed fertility rate and period TFR, so again it does not hold as a possible interpretation for the B-F method as long as we accept Bongaarts and Feeney’s statement that they did not attempt to estimate the fertility of actual cohorts.

### 4. Conclusion

If we adopt our interpretation for the B-F method given in Section 2, though we believe that it is the only possible interpretation for the original formulation by Bongaarts and Feeney, it is a mathematically self-consistent theory but based on the hypothesis that is too difficult to be realized in the real world. Even if we could observe the period age-shift of the age-specific fertility rate, to consider as it is occurred by the Scenario 2 is no more than to assume almost impossible situation.
On the other hand, if we adopt the cohort approach interpretation, we again go back to the old problem first analysed by Ryder (1956), and as pointed out by Van Imhoff (2001) (3.4) can be seen as a special case of the Ryder translation equation, which would contradict the basic idea behind the B-F formula that the period rate is the more important determinant in the change of fertility.

For any possible interpretations, we conclude that the translation equation such as (1.1) is theoretically interesting itself to see effects of age-shift in demographic variables, but there is no reason to accept the adjusted total fertility rate as a reliable indicator of the level of fertility, since its assumption expressed in (2.10) is too restrictive to be observed in the real world, even though we may accept the linear age-shift of fertility function.

References


Keywords: tempo-adjusted total fertility rate, translation equation, age-shift, cohort fertility, period fertility.