A Mathematical Model for Human Population Reproduction by Iterative Marriage

by

Hisashi INABA

INSTITUTE OF POPULATION PROBLEMS

Ministry of Health and Welfare
1-2-2 Kasumigaseki, Chiyoda-ku
Tokyo 100-45
Japan
A MATHEMATICAL MODEL FOR HUMAN POPULATION REPRODUCTION BY ITERATIVE MARRIAGE*

Hisashi INABA
Institute of Population Problems
2-2, 1-Chome, Kasumigaseki, Chiyoda-ku,
Tokyo 100-45, Japan

Abstract

We develop a linear one-sex dynamical model for human population reproduction by marriage. In our model, it is assumed that a woman can repeat marriage and divorce but only married women produce children. The iterative marriage process is described by a semi-Markov process and the birth law is given by using a duration-specific birth rate by age at marriage. We show that the dynamics of this population system can be essentially expressed by a renewal integral equation and hence the strong ergodicity theorem holds. In order to see the effect of nuptiality on fertility, new formulae of reproduction indices are given. In particular, total fertility rate is considered as a product of the indices of marriage and marital fertility.

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1 INTRODUCTION

In modern societies, it is usual that individuals produce their children within a stable persistent union with the other sex. If we call making the persistent union for reproduction as marriage, no matter whether the union is legally sanctioned or not, realization of human fertility mainly depends on two factors; frequency of marriage and fertility schedule within marriage. Hence it is obviously important to develop a reproduction theory that can take into account marriage phenomena in order to factor out the causes of changes in human fertility. Nevertheless, so far dynamic marriage models in demography have not necessarily been fully developed.

In this paper, we formulate a one-sex linear dynamical model for human population reproduction by marriage. Of our concern here is to establish relations between nuptiality and fertility indices which could allow us to understand the effect of nuptiality on fertility. In the previous papers (Inaba, 1992a, 1993a, 1993b), the author have developed dynamic marriage models such that newborns are produced only by first marriage couples. In the following, we construct a marriage model in which not only first marriage couples but also remarriage couples can produce children as in the real. However, the interaction of both sexes is neglected and so we only treat one sex (female dominant) model. Further, we assume that marriage occurs only for woman in marriage market and newborns are produced only by married women. All of widowed or divorced women again belong to the marriage market. We assume that the forces of marriage and divorce for women in the marriageable state is not age-dependent but duration-dependent, so the process of repeating marriage and its dissolution is described by a semi-Markov process. Marital fertility schedule is also given by a duration-specific marital fertility rate by age at marriage. We disregard the effect of parity structure.

The basic model is formulated as a partial differential equation system with integral boundary condition. In spite of its complex appearance, it will be shown that the birth rate of the population satisfies Lotka's renewal integral equation and so strong ergodicity holds for the population system. Subsequently we show new formulae for reproduction indices. Total fertility rate is expressed by a product of the indices of marriage and marital fertility. Such a decomposition formula of reproduction indices would become most useful tool to understand the effect of nuptiality on fertility.

2 THE BASIC MODEL

In the following, we consider a closed female population such that all births occur within marriage. Divide the population into three groups; \( u \) (singles out of marriage market), \( v \) (singles in marriage market) and \( w \) (married). Let \( u(t, a) \) be the age-density of single population out of marriage market at time \( t \) and
age \( a \), let \( u(t, \tau; \zeta) \) be the density of single population in marriage market at time \( t \) and duration \( \tau \) who entered into the marriage market at age \( \zeta \) and let \( w(t, \tau; \zeta) \) be the density of married population at time \( t \) and marital duration \( \tau \) by age at marriage \( \zeta \). Let \( \lambda(a) \) be the force of entering into marriage market at age \( a \), let \( \gamma(\tau) \) be the force of marriage at duration \( \tau \) in marriage market, let \( \mu(a) \) be the force of mortality at age \( a \) (i.e., for simplicity, we do not assume differential mortality by status of individuals), \( m(\tau; \zeta) \) be the marital fertility rate at duration \( \tau \) by age at marriage \( \zeta \), let \( \ell(\tau) \) be the force of dissolution of couples at marital duration \( \tau \) and let \( \kappa \) be the proportion of female newborns. Then we obtain the following dynamic one-sex marriage model as follows:

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) u(t, a) = -[\mu(a) + \lambda(a)]u(t, a), \quad (2.1)
\]

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) v(t, \tau; \zeta) = -[\mu(\tau + \zeta) + \gamma(\tau)]v(t, \tau; \zeta), \quad (2.2)
\]

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) w(t, \tau; \zeta) = -[\mu(\tau + \zeta) + \delta(\tau)]w(t, \tau; \zeta), \quad (2.3)
\]

\[
u(t, 0; \zeta) = \kappa \int_0^\infty \int_0^\infty m(\tau; \zeta)w(t, \tau; \zeta)d\tau d\zeta, \quad (2.4)\]

\[
v(t, 0; \zeta) = \lambda(\zeta)u(t, \zeta) + \int_0^\zeta \delta(\tau)w(t, \tau; \zeta - \tau)d\tau, \quad (2.5)\]

\[
w(t, 0; \zeta) = \int_0^\zeta \gamma(\tau)v(t, \tau; \zeta - \tau)d\tau. \quad (2.6)\]

McKendrick-von Foerster equations (2.1)-(2.3) are easily integrated along the characteristic line and we have

\[
u(t, a) = \ell(a)\Lambda(a)u(t - a, 0), \quad (2.7)\]

\[
v(t, \tau; \zeta) = \frac{\ell(\zeta + \tau)}{\ell(\zeta)}\Gamma(\tau)v(t - \tau, 0; \zeta), \quad (2.8)\]

\[
w(t, \tau; \zeta) = \frac{\ell(\zeta + \tau)}{\ell(\zeta)}\Delta(\tau)w(t - \tau, 0; \zeta), \quad (2.9)\]

where \( \ell(a) \) is the survival rate given by

\[
\ell(a) := e^{-\int_0^a s(\sigma)d\sigma}, \quad (2.10)\]

and

\[
\Lambda(a) := e^{-\int_0^a \lambda(\sigma)d\sigma}, \quad \Gamma(\tau) := e^{-\int_0^\tau \gamma(\sigma)d\sigma}, \quad \Delta(\tau) := e^{-\int_0^\tau \delta(\sigma)d\sigma}. \quad (2.11)\]
\( \Lambda(a) \) denotes the probability of remaining in \( u \)-state at age \( a \), \( \Gamma(\tau) \) the probability of remaining in \( v \)-state at duration \( \tau \) and \( \Delta(\tau) \) the probability of remaining in \( w \)-state at duration \( \tau \) respectively. From (2.7)-(2.9), we know that distributions \( u, v \) and \( w \) can be determined by the knowledge of their boundary values \( u(t, 0), v(t, 0; \zeta) \) and \( w(t, 0; \zeta) \).

Next let \( n(t, a) \) be the age-density of total female population. Then it follows that

\[
n(t, a) = u(t, a) + \int_0^a v(t, a - \zeta; \zeta) d\zeta + \int_0^a w(t, a - \zeta; \zeta) d\zeta. \tag{2.12}
\]

It is easily checked from (2.1)-(2.6) that the following equation holds:

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) n(t, a) = -\mu(a)n(t, a), \quad n(t, 0) = u(t, 0). \tag{2.13}
\]

Hence the age-density of total population is given by

\[
n(t, a) = \ell(a)u(t - a, 0). \tag{2.14}
\]

3 RENEWAL INTEGRAL EQUATION AND STRONG ERGODICITY

3.1 Renewal Integral Equation

In this section, we give a general solution for system (2.1)-(2.6) and consider its asymptotic behavior. In order to solve system (2.1)-(2.6), we make use of ideas and terminologies of the semi-Markov model (Mode, 1985; Rajulton and Lee, 1988; Inaba, 1992b).

First let us define the \textit{one step transition density} matrix \( F(\tau) \) as

\[
F(\tau) := \begin{pmatrix} 0 & \delta(\tau)\Delta(\tau) \\ \gamma(\tau)\Gamma(\tau) & 0 \end{pmatrix}. \tag{3.1}
\]

In the following, state 1 of individuals denotes the single status in the marriage market (\( v \)-state) and state 2 denotes the marital status (\( w \)-state). We call state 1 and 2 as \textit{marriageable state}. Then the \((i, j)\)-th element \((i \neq j)\) of \( F(\tau) \) is the probability of making one-step transition from state \( j \) to state \( i \) at duration \( \tau \) in the state \( j \). Once an individual enters into \( v \)-state, she repeats \textit{jump} between \( v \)-state and \( w \)-state. This transition process between two states is described by a semi-Markov process, because the transition intensities \( \gamma \) and \( \delta \) are duration-dependent.

Let us consider the solution matrix \( R(\tau) \) of the following integral equation

\[
R(\tau) = F(\tau) + \int_0^\tau F(\tau - \rho)R(\rho)d\rho. \tag{3.2}
\]
Then it follows that the \((i, j)\)-th element \(R_{ij}(\tau)\) of \(R(\tau)\) denotes the probability density that an individual in the state \(j\) at time zero makes a jump to state \(i\) at time \(\tau\).

In particular, \(R_{21}(\tau)\) is the probability density that an individual in the marriage market at time zero marries (i.e., makes a jump to \(w\)-state) at time \(\tau\). Thus we obtain

\[
 w(t, 0; \zeta) = \int_0^\zeta R_{21}(\sigma) \frac{\ell(\zeta)}{\ell(\zeta - \sigma)} \lambda(\zeta - \sigma) u(t - \sigma, \zeta - \sigma) d\sigma. \tag{3.3}
\]

Substituting (2.9) and (3.3) into (2.4), we have

\[
 u(t, 0) = \kappa \int_0^\infty \int_0^\infty m(\tau; \zeta) \frac{\ell(\tau + \zeta)}{\ell(\zeta)} \Delta(\tau) w(t - \tau, 0; \zeta) d\tau d\zeta
 = \kappa \int_0^\infty \int_0^\infty m(\tau; \zeta) \ell(\tau + \zeta) \Delta(\tau)
 \times \int_0^\zeta R_{21}(\sigma) \lambda(\zeta - \sigma) \Lambda(\zeta - \sigma) u(t - \tau - \zeta, 0) d\sigma d\tau d\zeta. \tag{3.4}
\]

Let us define a function \(\phi(a)\) as

\[
 \phi(a) := \int_0^a R_{21}(\sigma) \lambda(a - \sigma) \Lambda(a - \sigma) d\sigma. \tag{3.5}
\]

Thus \(\phi(a)\) denotes the probability that marriage occurs at age \(a\). Since marriage can happen many times for an individual, \(\int_0^\infty \phi(a) da\) could be larger than unity and it is the expected number of times of marriage per capita as far as we do not take into account mortality.

Let \(B(t) := u(t, 0)\) be the density of newborn females at time \(t\). Then (3.4) can be written as the renewal equation as

\[
 B(t) = \kappa \int_0^\infty \int_0^\infty m(\tau; \zeta) \ell(\tau + \zeta) \Delta(\tau) \phi(\zeta) B(t - \tau - \zeta) d\tau d\zeta. \tag{3.6}
\]

By changing the order of integrals, we have

\[
 B(t) = \kappa \int_0^\infty \int_0^a m(a - \zeta; \zeta) \Delta(a - \zeta) \phi(\zeta) \ell(\zeta) \beta(a) B(t - a) da. \tag{3.7}
\]

If we define

\[
 \psi(a) := \kappa \beta(a) \ell(a), \tag{3.8}
\]

\[
 \beta(a) := \int_0^a m(a - \zeta; \zeta) \Delta(a - \zeta) \phi(\zeta) d\zeta, \tag{3.9}
\]
then $\beta(a)$ denotes the age-specific birth rate at age $a$ and $\psi(a)$ is the net maternity function. From (3.7)-(3.9), we arrive at the Lotka's renewal integral equation:

$$B(t) = \int_0^\infty \psi(a)B(t - a)da.$$  \hspace{1cm} (3.10)

If we give an initial data $B(-t)$ for $t \in [0,\omega]$ ($\omega$ is the upper bound of reproductive age), (3.10) becomes a Volterra integral equation and existence and uniqueness of its solution is widely established under appropriate conditions (Bellman-Cooke, 1963). Once $B(t)$ is determined, $w(t,0;\zeta)$ and $v(t,0;\zeta)$ are calculated by using (2.5), (2.7) and (3.3) as follows:

$$v(t,0;\zeta) = \ell(\zeta)\pi(\zeta)B(t - \zeta),$$  \hspace{1cm} (3.11)

$$w(t,0;\zeta) = \ell(\zeta)\phi(\zeta)B(t - \zeta),$$  \hspace{1cm} (3.12)

where $\pi(\zeta)$ is the probability of making a jump to $v$-state at age $\zeta$ given by

$$\pi(\zeta) = \lambda(\zeta)A(\zeta) + \int_0^\zeta \delta(\eta)\Delta(\eta)\phi(\zeta - \eta)d\eta.$$  \hspace{1cm} (3.13)

Finally, using the above boundary values, all of distributions $u$, $v$ and $w$ can be determined by relations (2.7)-(2.9) as follows:

$$u(t,a) = \ell(a)A(a)B(t - a),$$  \hspace{1cm} (3.14)

$$v(t,\tau;\zeta) = \ell(\zeta + \tau)\Gamma(\tau)\pi(\zeta)B(t - \tau - \zeta),$$  \hspace{1cm} (3.15)

$$w(t,\tau;\zeta) = \ell(\zeta + \tau)\Delta(\tau)\phi(\zeta)B(t - \tau - \zeta).$$  \hspace{1cm} (3.16)

### 3.2 Strong Ergodicity

For Lotka's renewal integral equation (3.10), it is well known that its solution (birth rate) $B(t)$ is asymptotically growing with the *Malthusian parameter* $r_0$ (Pollard, 1973; Hoppensteadt, 1975; Keyfitz, 1977; Inaba, 1987). More precisely speaking, there exists a small number $\epsilon > 0$ such that

$$B(t) = B_0e^{r_0t} + o(e^{(r_0-\epsilon)t}),$$  \hspace{1cm} (3.17)

where $\lim_{t \to \infty} e^{-(r_0-\epsilon)t} = 0$, $B_0$ is a constant calculated by the initial data as

$$B_0 := \frac{\int_0^\infty e^{-ra}G(s)ds}{\int_0^\infty ae^{-ra}\psi(a)da}, \hspace{1cm} G(t) := \int_0^\infty \psi(a)B(t - a)da,$$  \hspace{1cm} (3.18)

and the intrinsic growth rate $r_0$ is given as a unique dominant real root of the characteristic equation.
\begin{equation}
\int_0^{\infty} e^{-r_0 a} \psi(a) da = 1. \tag{3.19}
\end{equation}

From (2.14) and (3.17), we know that
\begin{equation}
n(t, a) \sim B_0 \ell(a) e^{r_0(t-a)}, \quad (t \to \infty). \tag{3.20}
\end{equation}

Using (3.14)-(3.16) and (3.17), it follows that when \( t \to \infty \)
\begin{equation}
u(t, \tau; \zeta) \sim B_0 e^{r_0(t-\tau-\zeta)} \ell(\zeta + \tau) \Gamma(\tau) \pi(\zeta), \tag{3.22}
\end{equation}
\begin{equation}
w(t, \tau; \zeta) \sim B_0 e^{r_0(t-\tau-\zeta)} \ell(\zeta + \tau) \Delta(\tau) \phi(\zeta). \tag{3.23}
\end{equation}

Therefore we conclude that age-duration-distribution by status converges to a stable distribution as time evolves and the asymptotic growth rate is given by \( r_0 \). In summary, the population evolution process described by system (2.1)-(2.6) is strongly ergodic (Inaba, 1989).

### 3.3 Stable Age Distribution

Using (3.20)-(3.23), it is easily observed that the stable age distributions by status are given by
\begin{equation}
\lim_{t \to \infty} \frac{u(t, a)}{\int_0^{\infty} n(t, a) da} = C(a) \Lambda(a), \tag{3.24}
\end{equation}
\begin{equation}
\lim_{t \to \infty} \frac{\int_0^a v(t, \tau; a-\tau) d\tau}{\int_0^{\infty} n(t, a) da} = C(a) C_v(a), \tag{3.25}
\end{equation}
\begin{equation}
\lim_{t \to \infty} \frac{\int_0^a w(t, \tau; a-\tau) d\tau}{\int_0^{\infty} n(t, a) da} = C(a) C_w(a), \tag{3.26}
\end{equation}

where \( C(a) \) is the normalized stable age distribution of total population as
\begin{equation}
C(a) := \lim_{t \to \infty} \frac{n(t, a)}{\int_0^{\infty} n(t, a) da} = \frac{e^{-r_0 a} \ell(a)}{\int_0^{\infty} e^{-r_0 a} \ell(a) da}, \tag{3.27}
\end{equation}

\( C_v(a) \) is the proportion that individuals are in the marriage market at age \( a \) under stability given by
\begin{equation}
C_v(a) := \int_0^a \Gamma(a-\tau) \pi(\tau) d\tau, \tag{3.28}
\end{equation}

and \( C_w(a) \) is the proportion married at age \( a \) under stability given by
\[ C_w(a) := \int_0^a \Delta(a - \tau) \phi(\tau) d\tau. \] (3.29)

Observe that
\[ \phi(a) = \int_0^a \gamma(a - \tau) \Gamma(a - \tau) \pi(\tau) d\tau. \] (3.30)

Using (3.30), it is easy to verify that the following relation holds:
\[ \Lambda(a) + C_v(a) + C_w(a) = 1. \] (3.31)

**Remark.** From (3.13) and (3.30), we obtain a Volterra integral equation as
\[ X(a) = Y(a) + \int_0^a F(\tau) X(a - \tau) d\tau, \] (3.32)

where 2-dimensional vectors \(X\) and \(Y\) are given by
\[ X(a) := \begin{pmatrix} \pi(a) \\ \phi(a) \end{pmatrix}, \quad Y(a) := \begin{pmatrix} \lambda(a) \Lambda(a) \\ 0 \end{pmatrix}. \] (3.33)

Then equation (3.2) is the resolvent equation corresponding to (3.32). It is well known that using solution \(R(\tau)\) of (3.2), solution \(X(a)\) of (3.32) is given as
\[ X(a) = Y(a) + \int_0^a R(\tau) Y(a - \tau) d\tau. \] (3.34)

From the second component of (3.34), we obtain relation (3.5). By using the fact that \(X(a)\) is the solution of (3.32), it is easily checked that (3.14)-(3.16) satisfies (2.1)-(2.6) when \(B(t)\) is given by (3.10).

### 4 NEW FORMULÆ FOR REPRODUCTION INDICES

#### 4.1 Reproduction Indices

From the renewal integral equation (3.10), we know that the **basic reproduction number** \(R_0\) (net reproduction ratio: NRR) for our population system can be calculated as
\[ R_0 = \int_0^\infty \psi(a) da = \kappa \int_0^\infty S(\zeta) \phi(\zeta) \ell(\zeta) d\zeta, \] (4.1)

where
\[ S(\zeta) := \int_0^\infty m(\tau, \zeta) \Delta(\tau) \frac{\ell(\tau + \zeta)}{\ell(\zeta)} d\tau. \] (4.2)
Then \( S(\zeta) \) denotes the total number of children produced per marriage by age at marriage \( \zeta \). \( R_0 \) denotes the total number of female newborns produced per woman. Just the same as the standard stable population theory, it follows that 

\[ r_0 > 0 \text{ if } R_0 > 1; \quad r_0 = 0 \text{ if } R_0 = 1; \quad r_0 < 0 \text{ if } R_0 < 1. \]

Moreover, total fertility rate (TFR) is given by

\[ TFR = \int_0^\infty \beta(a)da = \int_0^\infty T(\zeta)\phi(\zeta)d\zeta, \quad (4.3) \]

where

\[ T(\zeta) := \int_0^\infty m(\tau;\zeta)\Delta(\tau)d\tau. \quad (4.4) \]

Then \( T(\zeta) \) denotes the total number of children produced per marriage by age at marriage \( \zeta \) under the condition that the reproduction process is not terminated by death of female partner. Following Song and Yu (1988), let us introduce female critical fertility rate (CFR), denoted by \( \beta_{cr} \), as

\[ \beta_{cr} = \left( \kappa \int_0^\infty \beta_0(a)\ell(a)da \right)^{-1}, \quad (4.5) \]

where \( \beta_0(a) \) is the normalized age-specific birth rate given by

\[ \beta_0(a) = \frac{\beta(a)}{\int_0^\infty \beta(a)da}. \quad (4.6) \]

Then we obtain the criterion as \( r_0 > 0 \) if \( TFR > \beta_{cr} \); \( r_0 = 0 \) if \( TFR = \beta_{cr} \); \( r_0 < 0 \) if \( TFR < \beta_{cr} \). This means that comparing \( R_0 \) with unity is equivalent to comparing TFR with \( \beta_{cr} \), that is, \( R_0 \) and TFR are mutually equivalent indices to measure the reproductivity of a population. Among developed countries, \( \beta_{cr} \) has a stable value about 2.08 ~ 2.1. In such a case, comparing TFR with \( \beta_{cr} \) is a convenient way to estimate reproductivity of the population, because calculating TFR is much easier.

Define the normalized frequency distribution of age at marriage \( \Phi(a) \) in the reproductive age period as

\[ \Phi(a) := \frac{\phi(a)}{\int_0^\omega \phi(z)dz}, \quad 0 \leq a \leq \omega. \quad (4.7) \]

Let us define nuptiality level \( N_\omega \) and marital fertility level \( MF \) as

\[ N_\omega := \int_0^\omega \phi(a)da, \quad MF := \int_0^\omega T(\zeta)\Phi(\zeta)d\zeta. \quad (4.8) \]

Then \( N_\omega \) denotes the expected total number of times of marriage in age interval \([0, \omega]\) per woman and \( MF \) denotes the average number of children produced per marriage in case that we neglect the effect of mortality. Since \( T(\zeta) = 0 \) for \( \zeta > \omega \), we have the following decomposition for TFR:
\[ TFR = N_\omega \times MF = \int_0^\omega \phi(a)da \times \int_0^\omega T(\zeta)\Phi(\zeta)d\zeta. \] (4.9)

Decomposition formula as (4.9) is much useful to see the effect of changes in nuptiality on fertility (see Inaba, 1992a, 1993b). It should be noted that \(N_\omega\) and \(MF\) are age-distribution-free scalar measures for nuptiality and marital fertility (Trussell, et al., 1982).

### 4.2 Calculation of Nuptiality Level \(N_\omega\)

For simplicity, here we assume that for \(\tau > \omega\)

\[ \lambda(\tau) = 0, \quad \gamma(\tau) = 0, \quad \delta(\tau) = 0. \] (4.10)

Then it follows that for \(\tau > \omega\)

\[ \Lambda(\tau) = \Lambda(\omega), \quad \Gamma(\tau) = \Gamma(\omega), \quad \Delta(\tau) = \Delta(\omega). \] (4.11)

Under assumption (4.10), marriage does not occur after age \(\omega\), we have \(N_\omega = N_\infty\). From (3.5), it is easily observed that

\[ \int_0^\infty \phi(a)da = (1 - \Lambda(\infty)) \int_0^\infty R_{21}(\tau)d\tau. \] (4.12)

On one hand, from (3.2), we obtain

\[ \int_0^\infty R(\tau)d\tau = \int_0^\infty F(\tau)d\tau + \int_0^\infty F(\tau)d\tau \int_0^\infty R(\tau)d\tau. \] (4.13)

Thus it follows that

\[ \int_0^\infty R(\tau)d\tau = \left( I - \int_0^\infty F(\tau)d\tau \right)^{-1} \int_0^\infty F(\tau)d\tau. \] (4.14)

Since \(F(\tau)\) is a \(2 \times 2\) matrix given by (3.1), the right hand side of (4.14) is easily calculated and we have

\[ \int_0^\infty R_{21}(\tau)d\tau = \frac{1 - \Gamma(\infty)}{1 - (1 - \Gamma(\infty))(1 - \Delta(\infty))}. \] (4.15)

Using (4.11), we arrive at the following expression:

\[ N_\omega = N_\infty = \int_0^\infty \phi(a)da = \frac{xy}{1 - yz}. \] (4.16)

Here \(x := 1 - \Lambda(\omega)\) is the proportion of ever entering into the marriage market, \(y := 1 - \Gamma(\omega)\) is the proportion ever married for women in the marriage market and \(z := 1 - \Delta(\omega)\) is the probability of ever becoming widowed or divorced. We know that \(N_\infty\) is an monotonic increasing function of \(x, y, z \in [0, 1]\).

In general, since \(\lambda, \gamma, \delta\) are in fact so small for \(\tau > \omega\), \(N_\infty\) calculated based on the assumption (4.10) would be a good approximation for the nuptiality level \(N_\omega\).
4.3 Calculation of Marital Fertility Level $MF$

Let $a_0$ be the mean age at marriage in the reproductive age period as

$$a_0 := \int_0^\omega a \Phi(a) da.$$  \hspace{1cm} (4.17)

Suppose that $T(\zeta)$ allows a polynomial approximation as

$$T(\zeta) = \sum_{j=0}^n A_j (\zeta - a_0)^j + \epsilon(\zeta),$$  \hspace{1cm} (4.18)

where $\epsilon(\zeta)$ is the residual error term. Then it follows from (4.8) that

$$MF = \sum_{j=0}^n A_j M_j + \int_0^\omega \epsilon(\zeta) \Phi(\zeta) d\zeta,$$  \hspace{1cm} (4.19)

where $M_j$ is the $j$-th moment around the mean of the distribution $\Phi(\zeta)$ given by

$$M_j := \int_0^\omega (\zeta - a_0)^j \Phi(\zeta) d\zeta.$$  \hspace{1cm} (4.20)

In most simple case such that $T(\zeta)$ has a linear pattern, we have $MF \approx A_0$, because it follows from definition that $M_1 = 0$. Surprisingly it is often observed that $T(\zeta)$ has a nearly linear pattern (Inaba, 1992a, 1993b). In such a case, the linear regression line $T(\zeta) \approx u + v\zeta$ (where $u$ and $v$ are regression coefficients) would provide a good approximation for marital fertility level as $MF \approx u + va_0$.

Since $T(\zeta)$ is a decreasing function of $\zeta$ (that is, $v < 0$), we know that the marital fertility level $MF$ is decreasing as $a_0$ increases. Moreover, if we take up to the second order of (4.18), we obtain the approximation formula as $MF \approx A_0 + A_2 \sigma^2$ where $\sigma$ denotes the variance of the distribution $\Phi(\zeta).

5 Discussion

In this paper, we extended the first marriage model by the author (Inaba, 1992a, 1993b) so that newborns can be produced also by remarriage couples. As we have shown in section 3, the dynamics of this marriage model is essentially determined by the renewal integral equation and hence the strong ergodic theorem holds. This suggests that new dynamic phenomena (for example, periodic solution) different from strong ergodicity would not appear as far as we deal with linear marriage models.

It should be noted that our model is a simplification of the complex dynamics of marriage phenomena in the real. As mentioned in Introduction, our model neglect the interaction of both sexes. If we take into account bisexual
mating process, we necessarily deal with nonlinear problems (or two-sex problems in demography; see Inaba, 1993a). Next, different from our assumption, widowed/divorced population does not necessarily return to the marriage market in the real. Thirdly, the force of marriage \( \gamma(\tau) \) for remarriage would be different from that of first marriage. Moreover, in the real, \( \gamma(\tau) \) and \( \delta(\tau) \) would depend also on chronological age.

However, it seems that most important restriction for our model within the scope of linear modeling is that it disregards parity structure of female population. In our model, the same marital fertility rate \( m(\tau; \zeta) \) is applied to every types of marriage. However, women's childbearing process is heavily affected by their parity status. That is, marital fertility rate \( m(\tau; \zeta) \) used in our model is not independent of parity distribution among married and so it cannot be expressed by a sum of parity-specific marital fertility rate defined in individual level as in the first marriage model (Inaba, 1993b). In order to remove this compositional effect and to get parity-distribution-free measures, \( m(\tau; \zeta) \) should be set up by parity status of woman. It would be an interesting problem to extend our model so that it can recognize parity structure.

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