Duration-Dependent Multistate Population Dynamics

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This paper was presented at the International Conference on Multistate Demography:
Duration-Dependent Multistate Population Dynamics

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1. INTRODUCTION
During the past decade, multistate mathematical demography has been widely developed and applied to various kinds of demographic phenomena. In general, multistate demographic models intend to describe the dynamics of populations which are divided into subpopulations according to "states" of individuals. The states may correspond to region of residence, labor status, marital status, parity or other classifications. In most of usual multistate models, it is assumed that transition intensities from one state to another depend only on age and the state in which the person is currently living. In other words, the subpopulation within a state is assumed to be homogeneous about their age-specific demographic behavior. This assumption implies that the transition intensities are independent of the past history of the individual in a given state (the Markovian assumption).

Although the models adopting the Markovian assumption have shown their usefulness in various applications, it is also clear that the Markovian assumption is too restrictive, because we often encounter the situation that heterogeneity of individuals in a state cannot be neglected. In particular, duration dependence is one of the most important factors that make demographic processes non-Markovian. For example, in the field of migration study, it has been well known that it is necessary to introduce duration-of-residence effect in order to understand migration histories (Morrison, 1967; McGinnis, 1968;

* This paper was presented at the International Conference on Multistate Demography: Measurement, Analysis, Forecasting, Zeist, October 31 - November 4, 1988.
Land, 1969; Ginsberg, 1978, 1979; Hingstman and Harts, 1984). The effect of duration dependence has been considered by several authors in some kinds of demographic phenomena; contraceptive-pregnancy histories (Littman and Mode, 1977), work histories (Hennessey, 1980), marital behavior (Rajulton, 1985), multistate life table (Wolf, 1988), birth interval dynamics (Lamas, 1985). Many of the above mentioned authors use stochastic processes, especially, a semi-Markov process in order to formulate the duration-dependent demographic process. However, until now, the extension of the classical stable population model which takes into account the effect of duration-dependence has been lacking. Main aim of this paper is to construct the stable growth theory for the duration-dependent multistate populations, which will make it possible to combine the birth process with the interstate duration-dependent transition process.

In the following, we provide two theoretical models which deterministically describe the dynamics of duration-dependent multistate populations. The models are formulated respectively by a system of first order partial differential equations with integral boundary conditions. The boundary condition gives the birth law of the population. For the first model, we shall use the age-specific birth rate function to give the birth law, which implies that childbirth is independent of the transition between states. Hence, the first model can be seen as an expansion of the classical multi-regional stable population model (Rogers, 1975). On the other hand, if childbirth causes the transition between states, we must formulate a parity-structured population, which is our second model. Reducing the differential equation system into a renewal integral equation system, we can give a solution of the system and analyze its asymptotic behavior. In either case, we will be able to prove that the strong ergodic theorem holds. Finally, we briefly mention the relationship between our deterministic approach and the semi-Markov model.

2. The duration-dependent multistate model.
We consider here a large scale closed one-sex multistate population. The individuals are characterized by two discrete parameters; a number representing the state which is occupied by a person and times-of-movement between states, and three continuous parameters; age, time and the duration since the last movement. Let \( p_j(t, a, s) \, da \, ds \) (1 \( \leq j \leq N \), \( 1 \leq i \)) be the number of female population at time \( t \) and state \( j \) in the age interval \((a, a + da)\) who have spent time between \( s \) and \( s + ds \) since the \( i \)-th movement. Let \( p^0_j(t, a) \, da \) be the female population at time \( t \) and state \( j \) in the age interval \((a, a + da)\) who have never moved, that is, the population was born in the state \( j \) and the duration at the state \( j \) equals to age \( a \). Let \( \lambda^i_{jk}(a, s) \) (\( i \geq 1 \), \( 1 \leq j, k \leq N \)) be the probability per unit time of a female population of age \( a \) at state \( k \) who have spent time \( s \) since the \( i \)-th movement moving to state \( j \) and let \( \lambda^0_{jk}(a) \) be the probability per unit time of a female population of age \( a \) at state \( k \) who have never moved moving to state \( j \). In other words, \( \lambda^i_{jk}(a, s) (\lambda^0_{jk}(a)) \) is the force of the \((i + 1)\)-th transition (the first transition) from state \( k \) to state \( j \) at age \( a \) and the duration \( s \) (at age \( a \)). Since we disregard external migration, then we can formulate the
duration-dependent multistate population dynamics by the following system:

\[
\frac{\partial}{\partial a} + \frac{\partial}{\partial s} + \frac{\partial}{\partial t} p_k^i(a,s,t) = (-\mu_k(a) - \sum_j \lambda_{jk}^i(a,s)) p_k^i(a,s,t), \quad i \geq 1, \tag{2.1a}
\]

\[
p_j^i(a,0,t) = \sum_k \int_0^a \lambda_{jk}^{i-1}(a,s) p_k^{i-1}(a,s,t) ds, \quad i \geq 2, \tag{2.1b}
\]

\[
p_j^1(a,0,t) = \sum_k \lambda_{jk}^0(a) p_k^1(a,t), \tag{2.1c}
\]

\[
\frac{\partial}{\partial a} + \frac{\partial}{\partial t} p_k^1(a,t) = (-\mu_k(a) - \sum_j \lambda_{jk}^0(a)) p_k^1(a,t), \tag{2.1d}
\]

\[
p_j^0(0,t) = \sum_k \int_0^\infty m_{jk}(a) p_k^0(a,t) + \sum_{i=1}^a \int p_k^i(a,s,t) ds \, da, \tag{2.1e}
\]

\[
p_j^0(a,s,0) = f_j^0(a,s), \quad p_j^0(a,0) = f_j^0(a), \tag{2.1f}
\]

where \( t > 0 \), \( \mu_k(a) \) is the force of mortality in \( k \)-th state, \( m_{jk}(a) \) da is the annual rate at which women aged \( a \) to \( a + da \) in state \( k \) bear female children of state \( j \) and (2.1f) is a set of initial conditions. Equation (2.1e) gives the birth law of the population system.

In the above model, there exists the possibility that an individual moves from one location to another in the same state, if we assume that \( \lambda_{ji}^j(a,s) \neq 0 \). Such a situation occurs when the states of the population correspond to regions of residence. On the other hand, if the states correspond to social status, for example, marital status, labor status and so on, then there is no movement within a state, and we assume that \( \lambda_{ji}^j(a,s) = 0 \). In addition, it is assumed that \( m_{jk}(a) = 0, j \neq k \), for a multiregional population, since new born children belong to the same region as their mothers live in.

If we do not decompose the population by times-of-movement between states, instead of the system (2.1), we can formulate an aggregated system as

\[
\frac{\partial}{\partial a} + \frac{\partial}{\partial s} + \frac{\partial}{\partial t} p_k(a,s,t) = (-\mu_k(a) - \sum_j \lambda_{jk}(a,s)) p_k(a,s,t), i \geq 1, \tag{2.2a}
\]

\[
p_j(a,0,t) = \sum_k \left[ \lambda_{jk}^0(a) p_k^0(a,t) + \int_0^a \lambda_{jk}(a,s) p_k(a,s,t) ds \right], \tag{2.2b}
\]

\[
\frac{\partial}{\partial a} + \frac{\partial}{\partial t} p_k^1(a,t) = (-\mu_k(a) - \sum_j \lambda_{jk}^0(a)) p_k^1(a,t), \tag{2.2c}
\]

\[
p_j^0(0,t) = \sum_k \int_0^\infty m_{jk}(a) p_k^0(a,t) + \int p_k(a,s,t) da, \tag{2.2d}
\]

\[
p_j(a,s,0) = f_j^0(a,s), \quad p_j^0(a,0) = f_j^0(a), \tag{2.2e}
\]

where \( \lambda_{jk}(a,s) \) is the force of transition from state \( k \) to state \( j \) at age \( a \) and duration \( s \), which is independent of times-of-movement between states and \( p_k(a,s,t) \) da ds is the female population in the state \( k \) at age \( a \), duration \( s \) and
time \( t \). The nature of solutions of the system (2.2) is easily induced from that of the system (2.1) by letting

\[
p_j(a,s,t) = \sum_{i \geq 1} p_j^i(a,s,t),
\]

and replacing the force of transition by \( \lambda_{ij}(a,s) \). From the practical point of view, the system (2.2) is often more important than the system (2.1), though the system (2.1) is more general and easy to treat theoretically. In fact, it seems to be overwork for practical use to make the force of transition depend on times-of-movement. On the other hand, since \( \lambda_{ij}(a,s) \) can be interpreted as the latent risk function used in the semi-Markov model, the model (2.2) provides an adequate framework to integrate the deterministic approach and the stochastic approach (see section 6).

3. Solutions of the System

In this section, we shall construct the solution of the system (2.1). First we define the cohort functions \( u_k^i(a+h,s+h;a,s), \) \( u_k^0(a+h;a), \) \( i \geq 1, \) \( 1 \leq k \leq N, \) \( h \geq 0 \) as

\[
u_k^i(a+h;a) = \exp(-\int_0^h \mu_k(a+\rho) d\rho - \sum_j^h \int_0^h \lambda_{jk}^0(a+\rho) d\rho),
\]

(3.1)

\[
u_k^i(a+h,s+h;a,s) = \exp(-\int_0^h \mu_k(a+\rho) d\rho - \sum_j^h \int_0^h \lambda_{jk}^i(a+\rho,s+\rho) d\rho), \quad i \geq 1,
\]

Then \( u_k^i(a+h,s+h;a,s) \) denotes the probability that an individual at age \( a \) and state \( k \) who has spent time \( s \) since the \( i \)-th movement will survive and be in state \( k \) at age \( a+h \) and duration \( s+h \) and \( u_k^i(a+h;a) \) is the probability that an individual at age \( a \) and state \( k \) who has never moved will not move to other states and be in state \( k \) at age \( a+h \). Next, we define the probability densities of interstate movements. Let \( \omega_{jk}^i(b;a), \) \( b \geq a, \) \( i \geq 0, \) \( 1 \leq j, \) \( k \leq N \) be the functions defined successively as

\[
\omega_{jk}^0(b;a) = \lambda_{jk}^0(b) u_k^0(b;a),
\]

(3.2)

\[
\omega_{jk}^i(b;a) = \sum_j^b \int_0^{b-a} \lambda_{ij}^j(b,s) u_j^i(b,s;b-s,0) \omega_{jk}^{i-1}(b-s;a) ds, \quad i \geq 1.
\]

Then \( \omega_{jk}^i(b;a) \) is the probability density that an individual at age \( a \) and state \( k \) who have never moved will move to state \( j \) at age \( b \) by the \((i+1)\)-th movement. Let \( \xi_j^i(a) \), \( \xi_{jk}^i(a) \), \( i \geq 1, \) \( 1 \leq j, \) \( k \leq N \) be the functions defined by

\[
\xi_j^0(a) = u_j^0(a;0),
\]

(3.3)

\[
\xi_{jk}^i(a) = \int_0^a u_j^i(a,s;a-s,0) \omega_{jk}^{i-1}(a-s;0) ds, \quad i \geq 1.
\]

Thus \( \xi_{jk}^i(a) \) denotes the probability that an individual born in state \( k \) will be in
state \( j \) at age \( a \) after \( i \)-th movement, that is, \( \ell^j_{jk}(a) \) is the survival rate which depends on times-of-movement between states. For simplicity we introduce matrix notations; Let \( \Lambda^j_0(a; s) \) \((\Lambda^0(a))\) be a \( N \times N \) matrix whose \((j,k)\)-th element is \( \lambda^j_{jk}(a; s) \) \((\lambda^0_{jk}(a))\), let \( U^j(a + h, s + h; a, s) \) be a \( N \times N \) diagonal matrix whose \( j \)-th diagonal element is \( u^j_j(a + h, s + h; a, s) \), let \( \Omega^j(b; a) \) be a \( N \times N \) matrix whose \((j,k)\)-th element is \( \omega^j_{jk}(b; a) \) and let \( L^j(a) \) \((L^0(a)) \) be a \( N \times N \) matrix whose \((j,k)\)-element \((j\)-th diagonal element\) is \( \ell^j_{jk}(a) \) \((\ell^j_{0j}(a)) \). Using the above definitions, we can rewrite (3.2) and (3.3) into matrix representations as follows:

\[
\begin{align*}
\Omega^0(b; a) &= \Lambda^0(b)U^0(b; a), \\
\Omega^j(b; a) &= \int_0^b \Lambda^j(b, s)U^j(h, s; b - s, 0)\Omega^{-1}(b - s; a)ds, \quad i \geq 1, \\
L^0(a) &= U^0(a; 0), \\
L^j(a) &= \int_0^a U^j(a, s; a - s, 0)\Omega^{-1}(a - s; 0)ds, \quad i \geq 1.
\end{align*}
\]

Note that if we define \( L(a) \) \(\overset{\text{def}}{=} \sum_{i \geq 0} L^i(a) \), then \( L(a) \) is the survival rate matrix of the multistate population. Furthermore, we can calculate the expected sojourn time of individuals as follows:

\[
\begin{align*}
T^0_i &= \int_0^\infty a[\mu_i(a) + \sum_{j} \lambda^j_{ji}(a)]u^0_j(a; 0)da = \int_0^\infty \ell^0_j(a)da, \\
T^j_{jk} &= \int_0^\infty \int_0^\infty s[\mu_j(a + s) + \sum_{k} \lambda^j_{kj}(a + s, s)]u^j_j(a + s, s; a, 0)\omega^{-1}_j(a; 0)dsda \\
&= \int_0^\infty \int_0^\infty u^j_j(a + s, s; a, 0)\omega^{-1}_j(a; 0)dsda = \int_0^\infty \ell^j_j(a)da.
\end{align*}
\]

Here \( T^0_i \) denotes the expected sojourn time at \( i \)-th state that individuals born in the state \( i \) spend before the first movement, and \( T^j_{jk} \) denotes the expected sojourn time since the \( i \)-th movement at \( j \)-th state of individuals born in the state \( k \).

By using the functions defined above, we can immediately integrate the McKendrick-Von Foerster equations (2.1a), (2.1d) along the life line \( t = a + \text{const.} \) as

\[
\begin{align*}
p^0_k(a + h, t + h) &= u^0_k(a + h; a)p^0_k(a, t), \\
p^0_k(a + h, s + h, t + h) &= u^0_k(a + h, s + h; a, s)p^0_k(a, s, t), \quad i \geq 1.
\end{align*}
\]

Thus we have the following representations,

\[
p^0_k(a, t) = \begin{cases} 
  u^0_k(a; 0)p^0_k(0, t - a), & t - a \geq 0, \\
  u^0_k(a; a - t)p^0_k(a - t), & a - t \geq 0.
\end{cases}
\]
\[ p^i_k(a,s,t) = \begin{cases} 
 u^i_k(a,s;a-s,0)p^i_k(a-s,0,t-s), & t-s>0, \\
 u^i_k(a,s;a-t,s-t)f^i_k(a-t,s-t), & a\geq s \geq t. 
\end{cases} \tag{3.10} \]

By the above representation (3.9) and (3.10), we know that the age-duration density functions \( p^i_k(a,s,t), p^i_k(a,t) \) are determined by boundary values \( p^i_k(a,0,t) \) and \( p^i_k(0,t) \). Furthermore, we can prove the following:

**Proposition 3.1.** The following representation holds for \( i \geq 1 \),

\[ p^i_j(a,0,t) = \begin{cases} 
 \sum_k \omega^i_{jk}^{-1}(a;0)p^0_k(0,t-a), & t-a>0, \\
 \sum_{n=0}^{i-1} g^n_{i-n-1,j}(a,t), & a-t\geq 0, 
\end{cases} \tag{3.11} \]

where \( g^i_{j,k}(a,t), i \geq 0, j \geq 0, 1 \leq k < N, a-t \geq 0 \) are given as

\[
\begin{align*}
g^0_{0,j}(a,t) & \overset{\text{def}}{=} \sum_k \omega^0_{jk}(a;a-t)f^0_k(a-t), \\
g^0_{0,j}(a,t) & \overset{\text{def}}{=} \sum_k \int_0^1 \lambda^0_{jk}(a,s)u^i_k(a,s;a-t,s-t)f^i_k(a-t,s-t)ds, \quad i \geq 1, \\
g^i_{j,k}(a,t) & \overset{\text{def}}{=} \sum_k \int_0^a \lambda^{i+j}_{jk}(a,s)u^{i+j}_k(a,s;a-s,0)p^i_j(a-s,t-s)ds, \quad j \geq 1.
\end{align*}
\]

**Proof.** From (2.1c) and (3.9), we obtain

\[ p^1_j(a,0,t) = \begin{cases} 
 \sum_k \omega^0_{jk}(a;0)p^0_k(0,t-a), & t-a>0, \\
 \sum_k \omega^0_{jk}(a;a-t)f^0_k(a-t), & a-t\geq 0, 
\end{cases} \tag{3.12} \]

which shows that (3.11) holds for \( i = 1 \). Suppose that (3.11) holds for an integer \( n \geq 1 \). From (2.1b), we have for \( t-a>0 \),

\[
p^{n+1}_j(a,0,t) = \sum_k \int_0^a \lambda^n_{jk}(a,s)p^n_k(a,s,t)ds
\]

\[
= \sum_k \int_0^a \lambda^n_{jk}(a,s)u^n_k(a,s;a-s,0)p^n_j(a-s,0,t-s)ds
\]

\[
= \sum_l \sum_k \int_0^a \lambda^n_{jk}(a,s)u^n_k(a,s;a-s,0)\omega^n_{kl}^{-1}(a-s;0)ds \cdot p^n_l(0,t-a)
\]

\[
= \sum_k \omega^n_j(a;0)p^n_j(0,t-a).
\]

On the other hand, if \( a-t \geq 0 \), we obtain
\[ p_{j+1}^n(a, 0, t) = \sum_k^t \int \lambda_{jk}(a, s)p_k^n(a, s, t)ds + \sum_k^a \int \lambda_{jk}(a, s)p_k^n(a, s, t)ds \]

\[ = \sum_k^t \int \lambda_{jk}(a, s)u_k^n(a, s; a-s, 0)p_k^n(a-s, 0, t-s)ds \]

\[ + \sum_k^a \int \lambda_{jk}(a, s)u_k^n(a, s; a-t, s-t)f_k^n(a-t, s-t)ds, \]

where

\[ \sum_k^t \int \lambda_{jk}(a, s)u_k^n(a, s; a-s, 0)p_k^n(a-s, 0, t-s)ds \]

\[ = \sum_k^t \int \lambda_{jk}(a, s)u_k^n(a, s; a-s, 0)(\sum_{l=0}^{n-1} g_{n-l-1,k}^l(a-s, t-s))ds \]

\[ = \sum_{l=0}^{n-1} g_{n-l, j}^l(a, t), \]

\[ \sum_k^a \int \lambda_{jk}(a, s)u_k^n(a, s; a-t, s-t)f_k^n(a-t, s-t)ds = g_0^{0,j}(a, t). \]

Therefore, (3.11) holds for \( i = n + 1 \). By mathematical induction, we know that (3.11) holds for every integer \( n \geq 1 \). This completes our proof. \( \Box \)

**Remark.** The function \( G_{i,k}^j(t, t) \) denotes the density of individuals who have already experienced \( i \)-th movement at time zero and will move to state \( k \) at age \( a \) and time \( t \) by \( (i+j+1) \)-th movement.

**Corollary 3.2.** Let \( p_j(a, 0, t) \) be the boundary value at age \( a \) and time \( t \) of the system (2.2). Then the following representation holds,

\[ p_j(a, 0, t) = \begin{cases} \sum_k \omega_{jk}(a; 0)p_k^n(0, t-a), & t-a > 0, \\ \sum_i g_i^{(j)}(a, t), & a-t \geq 0, \end{cases} \]

(3.13)

where \( \omega_{jk}(b; a) \), \( b \geq a \) is the \((i, j)\)-th element of the solution matrix \( \Omega(b; a) \) of a renewal integral equation

\[ \Omega(b; a) = \Lambda^0(b)U^0(b; a) + \int_0^{b-a} \Lambda(b, s)U(b, s; b-s, 0)\Omega(b-s; 0)ds, \]

(3.14)

and \( g_i^{(j)}(a, t), i \geq 0, 1 \leq j \leq N, a-t \geq 0 \) are given as

\[ g_i^{(j)}(a, t) \overset{\text{def}}{=} \sum_k \omega_{jk}^0(a; a-t)f_k^n(a-t), \]

\[ g_i^{(j)}(a, t) \overset{\text{def}}{=} \sum_k^a \int \lambda_{jk}(a, s)u_k^n(a, s; a-t, s-t)f_k^n(a-t, s-t)ds, i \geq 1, \]

\[ \]
\[ g_j^{(i+1)}(a,t) = \sum_{k=0}^{t} \int_{\lambda_j(a,s)} u_k(a,s;a-s,0) g_k^{(i)}(a-s,t-s) ds, \quad i \geq 1. \]

Using the above proposition, we can induce the renewal system for \( p_j^0(0,t) \), which denotes the number of female newborn children produced per unit time at state \( j \). From (2.1e), it follows that

\[
p_j^0(0,t) = \sum_{k=0}^{\infty} \int_{0}^{t} m_j(a)p_j^0(a,t) da + \sum_{i=1}^{t} \int_{0}^{a} p_j^i(a,s,t) ds da
\]

\[
= \sum_{k=0}^{t} \int_{0}^{a} m_j(a)p_j^0(a,t) da + \sum_{i=1}^{t} \int_{0}^{a} p_j^i(a,s,t) ds da
\]

\[
+ \sum_{k=0}^{\infty} \int_{0}^{t} m_j(a)p_j^0(a,t) da + \sum_{i=1}^{t} \int_{0}^{a} p_j^i(a,s,t) ds da \overset{\text{def}}{=} I + J. \quad (3.15)
\]

Observe that

\[
I = \sum_{k=0}^{t} \int_{0}^{a} m_j(a)p_j^0(a,t) da + \sum_{i=1}^{t} \int_{0}^{a} p_j^i(a,s,t) ds da
\]

\[
= \sum_{k=0}^{t} \int_{0}^{a} m_j(a)u_j^0(a;0)p_j^0(0,t-a) da
\]

\[
+ \sum_{k=0}^{t} \int_{0}^{a} \int_{0}^{b} m_j(a)u_j^0(a;0;0) \omega_{kt}^{-1}(a-s;0) ds da p_j^0(0,t-a) da
\]

\[
= \sum_{k=0}^{t} \int_{0}^{a} m_j(a)[p_j^0(a;0,t-a)] + \sum_{n,i=1}^{t} \int_{0}^{a} \left[ \sum_{k=0}^{t} u_k(a,s;0;0) \omega_{kt}^{-1}(a-s;0) ds \right] p_j^0(0,t-a) da.
\]

On the other hand, we have

\[
J = \sum_{k=0}^{t} \int_{0}^{a} \int_{0}^{b} m_j(a)p_j^i(a,s,t) ds da + \sum_{i=1}^{t} \int_{0}^{a} p_j^i(a,s,t) ds da
\]

\[
= \sum_{k=0}^{t} \int_{0}^{a} m_j(a)u_j^0(a;0-t) p_j^0(a-t) da + \sum_{k=1}^{t} \sum_{i=0}^{a} \int_{0}^{a} m_j(a)p_j^i(a,s,t) ds da,
\]

where

\[
\int_{0}^{a} \int_{0}^{a} m_j(a)p_j^i(a,s,t) ds da = \int_{0}^{a} ds \int_{0}^{a} m_j(a)p_j^i(a,s,t) da
\]

\[
+ \int_{s}^{t} ds \int_{0}^{a} m_j(a)p_j^i(a,s,t) da \overset{\text{def}}{=} J_1 + J_2.
\]

Hence,

\[
J_1 = \int_{0}^{a} ds \int_{0}^{a} m_j(a)p_j^i(a,s,t) da
\]
\[
\begin{align*}
&= \int_0^t ds \int s m_{jk}(a)u_k(a,s;a-s,0)p_k^l(a-s,0,t-s)da \\
&= \int_0^t ds \int s m_{jk}(a)u_k(a,s;a-s,0)\left[ \sum_{n=0}^{i-1} g_{n-i-n-1,k}(a-s,t-s) \right] da, \\
J_2 &= \int_0^t ds \int s m_{jk}(a)p_k^l(a,s,t)da \\
&= \int_0^t ds \int s m_{jk}(a)u_k(a,s;a-t,s-t)f_k(a-t,s-t)da.
\end{align*}
\]

If we define

\[
G_j(t) = \sum_k \int_0^t m_{jk}(a)u_k^0(a;a-t)f_k(a-t)da \\
+ \sum_k \sum_{i=1}^\infty \int_0^t ds \int s m_{jk}(a)u_k(a,s;a-t,s-t)f_k(a-t,s-t)da \\
+ \sum_k \sum_{i=1}^\infty \int_0^t ds \int s m_{jk}(a)u_k(a,s;a-s,0)f_k(a-s,t-s)da,
\]

\[
\psi_{jn}(a) = \delta_{jn}m_j(a)p_j^0(a) + \sum_k \sum_{i=1}^\infty m_{jk}(a)p_k^l(a),
\]

\[
B_j(t) = \sum_0^t \int s \psi_{jk}(a)B_k(t-a)da,
\]

then from (3.15) we arrive at the renewal system for \(B_j(t)\) as

\[
B_j(t) = G_j(t) + \sum_k \int_0^t \psi_{jk}(a)B_k(t-a)da, \quad 1 \leq j \leq N. \tag{3.16}
\]

Defining vector and matrix notations;

\[
B(t) = \tau(B_1(t),...,B_N(t)), \quad G(t) = \tau(G_1(t),...,G_N(t)),
\]

\[
\Psi(a) \equiv \begin{bmatrix}
\psi_{11}(a) & \cdots & \psi_{1N}(a) \\
\vdots & \ddots & \vdots \\
\psi_{N1}(a) & \cdots & \psi_{NN}(a)
\end{bmatrix},
\]

where \(\tau\) denotes the transpose of the vector. The \(N \times N\) matrix \(\Psi(a)\) is the generalized net maternity function, which can be decomposed as

\[
\Psi(a) = M(a) \sum_{i \geq 0} L^{(i)}(a), \tag{3.17}
\]

where \(M(a)\) is a \(N \times N\) matrix whose \((j,k)\)-th element is \(m_{jk}(a)\). Thus we can
rewrite (3.16) as a vector-type Lotka’s renewal integral equation
\[ B(t) = G(t) + \int_0^t \Psi(a)B(t-a)da. \] (3.18)

Since (3.18) is a Volterra-type integral equation, the existence and uniqueness of its solution is widely established under various conditions (Bellman and Cooke, 1963). It is clear that once we have the solution \( B(t) \) of (3.18), we immediately obtain the solution of the system (2.1) as follows, though we omit the proof. Of course, it is easy to confirm the similar result about the model (2.2).

**Proposition 3.3.** Let \( B_j(t), 1 \leq j \leq N \) be the solutions of the renewal system (3.16). Then the solution of the system (2.1) is given as
\[ p_j^0(a,t) = \begin{cases} u_j^0(a;0)B_j(t-a), & t-a > 0, \\ u_j^0(a,a-t)f_j(a-t), & a-t \geq 0, \end{cases} \] (3.19a)
\[ p_j^1(a,s) = \begin{cases} u_j^1(a,s;0)[\sum_{i=0}^{s-1} g_i \{ (a-s,t-s) \}, & a-t > s, \\ u_j^1(a,s;0)\sum_{k=1}^{s-1} g_k (a-s,t-s), & a-t \geq s, \end{cases} \] (3.19b)

4. **Duration-dependent multistate stable growth**

In this section we consider the asymptotic behavior of the duration-dependent multistate model (2.1). Equation (3.16) reflects the fact that the reproduction behavior is assumed to be independent of the times-of-movement and the duration. Therefore, if the survival schedules \( L^{(i)}(a), i = 0, 1, 2, \ldots \), are once given, we can calculate the intrinsic growth rate by the same way as the classical duration-independent multistate model. In what follows, we assume that \( m_{jk}(a) = 0, a > \beta \), for all integer \( j, k \), where \( \beta \) is the upper bound of reproductive age. Accordingly, we have \( G(t) = 0 \) for \( t > \beta \) and \( \Psi(a) = 0 \) for \( a > \beta \). Let \( \Sigma \) be a set of characteristic roots; \( \Sigma = \{ \lambda \in C; \det(I - \hat{\Psi}(\lambda)) = 0 \} \), where \( \hat{\Psi}(\lambda) \) is a \( \text{def} \) Laplace transform of \( \Psi(a) \); \( \hat{\Psi}(\lambda) = \int_0^\infty \exp(-\lambda a)\Psi(a)da \). If the generalized net maternity function is indecomposable, there exists a strictly dominant real characteristic root \( r_0 \), that is, \( r_0 \in \Sigma \cap R, r_0 > \max\{ Re\lambda; \lambda \in \Sigma - \{ r_0 \} \} \) (Inaba, 1987a, 1988), and the positive matrix \( \hat{\Psi}(r_0) \) has the Frobenius root one. Since the renewal theorem also holds for vector-type integral equations (Heijmann, 1986; Inaba, 1987a), it can be shown that
\[ \lim_{t \to \infty} \exp(-r_0 t)B(t) = \frac{\int_0^t \hat{G}(r_0)}{\int_0^t (-\Psi t)\hat{\Psi}_0} \psi_0, \] (4.1)
where \( f_0, \psi_0 \) are left and right-hand side eigenvectors of \( \hat{\Psi}(r_0) \) associated with the Frobenius root one and a matrix \( \Psi_1 \) is given by \( \Psi_1 = -\int_0^\infty a \Psi(a) \exp(-r_0 a) da \). From Proposition 3.2 and the above result (4.1), we immediately obtain the following proposition, though we omit the proof:

**Proposition 4.1.** Let \( r_c \) be the strictly dominant real characteristic root for the generalized net maternity function \( \Psi(a) \) and let \( Q \) be a \( N \)-vector given by

\[
Q \overset{\text{def}}{=} \frac{\int_0^\infty \hat{G}(r_0)}{\int_0^\infty (-\Psi_1) \psi_0} \psi_0.
\]

Then the following holds uniformly for any age interval of the form \( 0 \leq a \leq \omega \)

\[
l \lim_{t \to \infty} \exp(-r_0 t) p_j^0(a,t) = \exp(-r_0 a) u_j^0(a;0) q_j, \quad (4.3a)
\]

\[
l \lim_{t \to \infty} \exp(-r_0 t) p_j^0(a,s,t) = \exp(-r_0 a) u_j^0(a,s;a-s,0) \sum_k \omega_j^k (a-s;0) q_k, \quad i \geq 1, \quad (4.3b)
\]

where \( q_j \) is the \( j \)-th component of the vector \( Q \).

The above proposition tells us that age-duration distributions of the population are asymptotically independent of the initial conditions. This is a strong ergodic theorem for the duration-dependent population dynamics. In particular, if we set the initial data as

\[
f_j^0(a) = \exp(-r_0 a) u_j^0(a;0) q_j,
\]

\[
f_j^0(a,s) = \exp(-r_0 a) u_j^0(a,s;a-s,0) \sum_k \omega_j^k (a-s;0) q_k, \quad i \geq 1,
\]

then we obtain a balanced-growth solution

\[
p_j^0(a,t) = \exp(r(t-a)) f_j^0(a), \quad p_j^0(a,s,t) = \exp(r(t) f_j^0(a,s), \quad (4.4)
\]

The strong ergodic theorem tells us that any population governed by the system (2.1) will be asymptotically proportional to the balanced-growth population as time evolves.

Subsequently, let us introduce some relations under stability. Consider a population under stable growth as

\[
p_j^0(a,t) = \exp(r(t-a)) u_j^0(a;0) q_j, \quad (4.5)
\]

\[
p_j^0(a,s,t) = \exp(r(t-a)) u_j^0(a,s;a-s,0) \sum_k \omega_j^k (a-s;0) q_k, \quad i \geq 1,
\]

where \( r \) is an intrinsic growth rate. Let \( N_j^0(t) \) be the total population at time \( t \) and state \( j \) who have moved \( i \) times. Then

\[
N_j^0(t) = \int_0^\infty \exp(-ra) u_j^0(a) q_j da \cdot \exp(rt),
\]
\[ N_j(t) = \sum_k^\infty \int_0^\infty \exp(-ra)q_j(a)q_k da \cdot \exp(rt). \] \hspace{1cm} (4.6)

If \( b(r) \) denotes the crude birth rate, then we have
\[
b(r) \overset{\text{def}}{=} \frac{\sum p_j^0(0,t)}{N(t)}
= \frac{\sum q_j}{\sum_j \int_0^\infty \exp(-ra)q_j(a)q_j da + \sum_{i,j,k} \int_0^\infty \exp(-ra)q_j(a)q_k da},
\] \hspace{1cm} (4.7)

where \( N(t) \) is the total population given by \( N(t) = \sum_{i,j}^\infty N_j(t) \). If we let \( r = 0 \), we have the stationary population and a relation
\[
b(0) = \frac{\sum q_j}{\sum_i T^0_i q_i + \sum_{i,j,k} T_{jk} q_k}.
\] \hspace{1cm} (4.8)

Next, the proportions \( C_j^0(a,s) \), \( C_j^0(a) \) of the population at age \( a \) and duration \( s \) who have moved \( i \) times are given as
\[
C_j^0(a) \overset{\text{def}}{=} \frac{p_j^0(a,t)}{N(t)} = b(r)\exp(-ra)u_j^0(a;0)[\frac{q_j}{\sum_n q_n}],
\] \hspace{1cm} (4.9)
\[
C_j^0(a,s) = \frac{p_j^0(a,s,t)}{N(t)} = b(r)\exp(-ra)u_j^0(a,s;0-s,0)[\sum_k \omega_{jk}(a-s;0)[\frac{q_k}{\sum_n q_n}].
\]

Let \( s_j^i(r) \) be the proportion of \( j \)-th state population who have moved \( i \) times. Then
\[
s_j^0(r) \overset{\text{def}}{=} \frac{N_j^0(r)}{N(t)} = b(r)\int_0^\infty \exp(-ra)q_j(a) da [\frac{q_j}{\sum_n q_n}],
\] \hspace{1cm} (4.10a)
\[
s_j^i(r) \overset{\text{def}}{=} \frac{N_j^i(t)}{N(t)} = b(r)\sum_k \int_0^\infty \exp(-ra)q_j(a) da [\frac{q_k}{\sum_n q_n}].
\] \hspace{1cm} (4.10b)

If we let \( r = 0 \), we obtain a relation under the stationary population,
\[
s_j^0(0) = b(0)T^0_j[\frac{q_j}{\sum_n q_n}],
\] \hspace{1cm} (4.11a)
\[
s_j^i(0) = b(0)\sum_k T_{jk}[\frac{q_k}{\sum_n q_n}].
\] \hspace{1cm} (4.11b)
Since (4.4) is a special solution of (2.1), we obtain
\[
\frac{d}{dt} N_j^0(t) = \int_{0}^{t} \left(-\frac{\partial}{\partial a} - \mu_j(a) - \sum_k \lambda_k^0(a)\right)p_j^0(a,t)dsda
\]
\[
= p_j^0(0,t) - \sum_k O_{jk}(t) - D_j^0(t),
\]
\[
\frac{d}{dt} N_j^1(t) = \int_{0}^{t} \left(-\frac{\partial}{\partial a} - \frac{\partial}{\partial s} - \mu_j(a) - \sum_k \lambda_k^1(a,s)\right)p_j^1(a,s,t)dsda
\]
\[
= \sum_k O_{jk}^{i-1}(t) - \sum_k O_{jk}^i(t) - D_j^i(t),
\]
where \(O_{jk}(t)\) is the number of individuals who move from \(k\)-th state to \(j\)-th state by \(i\)-th movement, and \(D_j^i(t)\) is the number of death of individuals at \(j\)-th state who have moved \(i\) times. We define the crude movement rate \(o_{jk}^i(r)\), the state-specific crude birth rate \(b_j(r)\), and the state-specific crude death rate \(d_j^i(r)\) as
\[
o_{jk}^i(r) = O_{jk}(t)/N_k^i(t), \quad b_j(r) = p_j^0(0,t)/N(t), \quad d_j^i(t) = D_j^i(t)/N_j^i(t).
\]
Then it is easily seen that the following holds:
\[
r_j^0(r) = b_j(r) - \sum_k o_{jk}^0(r) + d_j^0(r)s_j^0(r),
\]
\[
r_j^i(r) = \sum_k o_{jk}^{i-1}(r)s_k^{i-1}(r) - \sum_k o_{jk}^i(r) + d_j^i(r)s_j^i(r), \quad (4.12)
\]
which is the balance law between the above mentioned demographic indices under stability.

5. PARITY-STRUCTURED POPULATION DYNAMICS
In this section, we briefly consider a parity-structured population model. Here the term parity denotes the number of births an individual has had. Although the birth law given by the age-specific birth rate function is very familiar, it cannot reflect the fact that individual decisions about child bearing is strongly affected by the parity state and the duration since the last birth. It is crucial when we want to understand controlled reproduction process of a human population. During the past decade, some demographers tried to extend the stable population theory as it can take into account the parity structure. Das Gupta (1976) provided a model which recognizes age, parity and marital status. However, the fertility schedule of Gupta’s model was independent of the duration variable. Next Feeney (1983) proposed, what we call, the parity progression model, in which the parity progression schedule depends on parity status and the duration variable but it is independent of age. Recently, Lamas (1985) first provided an extended stable population model which takes into account age, party and the duration, and he tried to give a proof for the strong ergodic theorem of his model. But his proof was not complete, since his model lacked the dynamical equation for the parity-structured population. In the
following, we formulate the dynamical system for an age-parity-duration-dependent population. An important feature of the parity-structured population is its stratification structure, that is, the movement between parity status is always made in the direction which makes the parity increase. As a result, we will be able to reduce the multistate system into a scalar-type renewal integral equation and to give a complete proof of the strong ergodic theorem. Since the argument could be performed in the similar way as in the previous sections, we shall only state the results without proofs below.

Let \( p_i(a,s,t)dsda, \ i \geq 1 \) be the number of female population of parity \( i \) at time \( t \) in the age interval \((a,a+da)\) who have spent time between \( s \) and \( s+ds \) since the \( i \)-th birth and let \( p_0(a,t)da \) be the zero parity population at time \( t \) in the age interval \((a,a+da)\). Let \( \lambda_i(a,s), \ i \geq 1 \) be the force of transition from parity \( i \) to parity \((i+1)\) at age \( a \) and the duration \( s \) and let \( \lambda_0(a) \) be the force of transition from parity zero to parity one at age \( a \). That is, \( \lambda_i(a,s) \) is the instantaneous birth rate of \( i \)-th parity population at age \( a \) and the duration \( s \). If we assume that a female produce at most one child in the unit of time interval, then we can formulate the parity-structured population model as follows:

\[
\begin{align*}
\frac{\partial}{\partial a} + \frac{\partial}{\partial s} + \frac{\partial}{\partial t} p_i(a,s,t) &= (-\mu(a) - \lambda_i(a,s))p_i(a,s,t), \ i \geq 1, \quad \text{(5.1a)} \\
p_i(a,0,t) &= \int_0^a p_{i-1}(a,s)p_i(a,s,t)ds, \ i \geq 2, \quad \text{(5.1b)} \\
p_1(a,0,t) &= \lambda_0(a)p_0(a,t), \quad \text{(5.1c)} \\
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) p_0(a,t) &= (-\mu(a) - \lambda_0(a))p_0(a,t), \quad \text{(5.1d)} \\
p_0(0,t) &= \gamma \sum_{i \geq 1} \int_0^\infty p_i(a,0,t)da, \quad \text{(5.1e)} \\
p_i(a,s,0) &= f_i(a,s), \ p_0(a,0) = f_0(a), \quad \text{(5.1f)}
\end{align*}
\]

where \( \gamma \) is the ratio of female new born children to the total new born children and \( \mu(a) \) is the force of mortality. As well as the case of (2.1), we can define the cohort functions for the system (5.1) as follows:

\[
\begin{align*}
\begin{aligned}
u_0(a+h;a) &= \exp(-\int_0^h [\mu(a+\rho) + \lambda_0(a+\rho)]d\rho), \\
u_i(a+h,s+h;a,s) &= \exp(-\int_0^h [\mu(a+\rho) + \lambda_i(a+\rho,s+\rho)]d\rho), \ i \geq 1.
\end{aligned} \quad \text{(5.2)}
\end{align*}
\]

Next we define the probability densities \( m_i(b,a), \ b \geq a, \ i \geq 0 \) successively as

\[
\begin{align*}
m_0(b,a) &= \lambda_0(b)u_0(b;a), \\
m_i(b,a) &= \int_0^{b-a} \lambda_i(b,s)u_i(b,s;b-s,0)m_{i-1}(b-s,a)ds, \ i \geq 1. \quad \text{(5.3)}
\end{align*}
\]
Then \( m_i(b,a) \) denotes the probability density of an individual at age \( a \) and parity zero having the \((i+1)\)-th birth at age \( b \). Hence \( A_{i+1} \equiv \int_0^\infty m_i(a,0)da \) is the probability that a woman has at least \((i+1)\) births. Therefore \( u_0 = A_1, a_i = A_{i+1}/A_i \) \((i \geq 1)\) gives the parity progression ratio (LERIDON, 1977, p.19) and

\[
R_0 \equiv \gamma \sum_{n \geq 1} n (A_n - A_{n+1}) = \gamma \sum_{n \geq 0} \int_0^\infty m_n(a,0)da = \gamma \sum_{n=0}^\infty \prod_{k=0}^n a_k, \quad (5.4)
\]
gives the expected number of female births that a woman reproduces, which is the net reproduction rate (KEYFITZ, 1985, p.346). Next let

\[
\xi_0(a) \equiv u_0(a;0),
\]

\[
\xi_i(a) = \int_0^a u_i(a,s;a-s,0)m_{i-1}(a-s,0)ds, \quad i \geq 1. \quad (5.5)
\]

Then \( \xi_i(a) \) is the probability that an individual will survive and be at age \( a \) and parity \( i \). By using the survival rate \( \xi_i(a) \), it is easily seen that the expected sojourn time \( T_i \) of a woman at parity \( i \) is given as \( T_i = \int_0^\infty \xi_i(a)da \).

As well as Proposition 3.1, we know that the following holds, although we omit the proof.

\[
p_i(a,0,t) = \begin{cases} 
  m_{i-1}(a,0)p_0(0,t-a), & t-a > 0, \\
  \sum_{n=0}^{i-1} g_n^{(i-n-1)}(a,t), & a-t \geq 0, 
\end{cases} \quad (5.6)
\]

where \( g_j^{(0)}(a,t) \), \( j \geq 0 \), \( i \geq 0 \), \( a-t \geq 0 \) are the density of the populations who are at parity \( j \) and time zero and will have an \((i+j+1)\)-th birth at age \( a \) defined as

\[
g_0^{(i)}(a,t) = m_i(a,a-t)f_0(a-t),
\]

\[
g_j^{(0)}(a,t) = \int_0^a \lambda_j(a,s)u_j(a,s;a-t,s-t)f_j(a-t,s-t)ds, \quad j \geq 1,
\]

\[
g_j^{(i)}(a,t) = \int_0^t \lambda_{i+j}(a,s)u_{i+j}(a,s;a-s,0)g_j^{(i-1)}(a-s,t-s)ds, \quad i \geq 1, \quad j \geq 0.
\]

Let \( B_i(t) \), \( i \geq 1 \) be the number of female newborn children per unit time interval produced by the \((i-1)\)-th parity subpopulation at time \( t \). Then we obtain

\[
B_i(t) = \gamma \int_0^\infty p_i(a,0,t)da
\]
\[
= \gamma \int_0^t m_{i-1}(a,0)p_0(0,t-a)da + \gamma \int_0^\infty \sum_{k=0}^{t-1} g_k^{(i-k-1)}(a,t)da.
\]

In addition, if we let \( B_0(t) \) be the total number of female new born children per unit time at time \( t \); \( B_0(t) = p_0(0,t) = \sum_{i \geq 1} B_i(t) \), then we have the renewal integral equation for \( B_0(t) \),

\[
B_0(t) = G(t) + \int_0^t \Psi(a)B_0(t-a)da, \quad (5.7)
\]

where \( \Psi(a) \) and \( G(t) \) are defined as

\[
\Psi(a) \stackrel{\text{def}}{=} \gamma \sum_{i \geq 0} m_i(a,0), \quad G(t) \stackrel{\text{def}}{=} \gamma \int_0^\infty \sum_{i \geq 0} \sum_{k=0}^{i} g_k^{(i-k-1)}(a,t)da. \quad (5.8)
\]

Using the solution \( B_0(t) \) of (5.7), we can immediately construct the solution of system (5.1) as follows:

\[
p_0(a,t) = \begin{cases} 
    u_0(a)B_0(t-a), & t-a > 0, \\
    u_0(a;a-t)f_0(a-t), & a-t \geq 0,
\end{cases} \quad (5.9a)
\]

\[
p_i(a,s,t) = \begin{cases} 
    u_i(a,s;a-s,0)m_{i-1}(a-s,0)B_0(t-a), & t > a > s, \\
    u_i(a,s;a-s,0)\sum_{k=0}^{i-1} g_k^{(i-k-1)}(a-s,t-s), & a \geq t > s, \\
    u_i(a,s;a-t,s-t)f_i(a-t,s-t), & a \geq s \geq t.
\end{cases} \quad (5.9b)
\]

If we assume that \( \lambda_i(a,s) = 0, \lambda_0(a) = 0 \) for \( a \geq \beta \), that is, the population lose its ability of reproduction after age \( \beta \), then we have \( \Psi(a) = 0, a \geq \beta \) and \( G(t) = 0, t \geq \beta \). Under the assumption, it is well known that there exists a strictly dominant real characteristic root \( r_0 \) of (5.7). That is, \( r_0 \) is only real root of the characteristic equation

\[
\int_0^\beta \Psi(a)\exp(-ra)da = 1, \quad (5.10)
\]

such that it is greater than the real part of any other characteristic root (Keyfitz, 1977). By virtue of classical renewal theorem, we can state that

\[
\lim_{t \to \infty} \exp(-r_0t)B_0(t) = Q, \quad (5.11)
\]

where the constant \( Q \) is given by

\[
Q = \frac{\int_0^\beta \exp(-r_0t)G(t)dt}{\int_0^\beta a\cdot\exp(-r_0a)\Psi(a)da}. \quad (5.12)
\]
Therefore we can arrive at the conclusion that the following holds uniformly on any finite age interval:

\[
\lim_{t \to \infty} \exp(-r_0 t)p_0(a,t) = \exp(-r_0 a)u_0(a;0)Q,
\]

\[
\lim_{t \to \infty} \exp(-r_0 t)p_i(a,s,t) = \exp(-r_0 a)u_i(a,s;a-s,0)m_{i-1}(a-s,0)Q, \quad i \geq 1.
\]

(5.13)

This is the strong ergodic theorem for the parity-structured population, which shows that the asymptotic distribution is independent of the initial distribution.

6. Relation to the semi-Markov theory

In this section we consider the model (2.2) by using the terminologies of the semi-Markov model mainly developed by C.J. Mode (1977, 1982, 1985). From the practical point of view, the semi-Markov approach has been found to be very useful and some techniques of computation for the probabilities have been developed (Hennessey, 1980; Rajulton, 1985). It will be shown that the stable distribution of the model (2.2) can be computed by the way analogous to the algorithm developed in the semi-Markov model.

Since the force of transition \( \lambda_{ij}(a,s) \) can be interpreted stochastically as the latent risk function of the semi-Markov model, the first passage probability \( A_{ij}(a,t) \) of our model is given as

\[
A_{ij}(a,t) = \int_0^t \lambda_{ij}(a + \rho, \rho)u_j(a + \rho, \rho; a, 0)d\rho, \quad 1 \leq i, j \leq N,
\]

(6.1)

Then \( A_{ij}(a,t), i \geq 1 \) denotes the conditional probability that an individual aged \( a \) enters state \( j \) and makes a one-step transition to state \( i \) during the age interval \( (a, a + t] \), \( t > 0 \) and \( A_{0j}(a,t) \) denotes the conditional probability that an individual aged \( a \) enters state \( j \) and will die during the age interval \( (a, a + t] \), \( t > 0 \). Then we obtain a relation

\[
u_j(a + t, t; a, 0) = 1 - \sum_{i>0} A_{ij}(a,t).
\]

(6.2)

The one-step transition density \( F_{ij}(a,t) \) is given as

\[
F_{ij}(a,t) = \frac{\partial}{\partial t} A_{ij}(a,t) = \lambda_{ij}(a + t, t)u_j(a + t, t; a, 0), \quad 0 \leq i \leq N,
\]

(6.3)

where \( \lambda_{ij}(a + t, t) = \mu_j(a + t), \quad 1 \leq j \leq N. \) One of technical problems in the semi-Markov model is to determine the state probability \( s_j(a,t) \), which denotes the probability that an individual who entered state \( j \) at age \( a \) will be in state \( i \) at age \( a + t \). Let \( F(a,t) \) be a \( N \times N \) matrix whose \( (i,j) \)-th element is \( F_{ij}(a,t) \), \( 1 \leq i, j \leq N \), and let \( S(a,t) \) be a \( N \times N \) matrix whose \( (i,j) \)-th element is
s_j(a,t). Then we immediately obtain the renewal integral equation as

$$S(a,t) = U(a + t,t;a,0) + \int_0^t S(a + \rho,t - \rho)F(a,\rho)d\rho,$$

which is an integral version of the backward equation (MODE, 1982). Let $R(a,t)$ be the solution matrix of the resolvent equation

$$R(a,t) = F(a,t) + \int_0^t F(a + \rho,t - \rho)R(a,\rho)d\rho,$$

Then the $(i,j)$-th element $R_{ij}(a,t)$ of $R(a,t)$ denotes the probability density that an individual who entered state $j$ at age $a$ moves to state $i$ at age $a + t$. We can find the solution $S(a,t)$ of (6.4) as

$$S(a,t) = U(a + t,t;a,0) + \int_0^t U(a + t,t - \rho;a + \rho,0)R(a,\rho)d\rho.$$  

(6.6)

This is the continuous version of Littman's algorithm to seek the state probability (MODE, 1985). From (6.2) and (6.3), we know that $S(a,t)$ can be computed by the knowledge of the first passage probability. Subsequently we show that by the similar method we can compute the stable distribution in our model (2.2). From (3.14), we can formulate the following integral equation for our model (2.2);

$$\Omega(a;0) = \Omega^0(a;0) + \int_0^a F(a - s,s)\Omega(a - s;0)ds,$$

(6.7)

where the $(i,j)$-th element $\omega_{ij}(a;0)$ of $\Omega(a;0)$ is the probability density that an individual born in state $j$ enters state $i$ at age $a$ and the $(i,j)$-th element $\omega^0_{ij}(a;0)$ of $\Omega^0(a;0)$ the probability density that an individual born in state $j$ enters state $i$ at age $a$ by the first movement. The first passage probability matrix $A^0(a)$ is given as

$$A^0(a) = \int_0^a \Omega^0(a;0)da = \int_0^a \Lambda^0(a)U^0(a;0)da.$$  

(6.8)

Solving the integral equation (6.7), it is easily seen that the probability density matrix $\Omega(a;0)$ is given as

$$\Omega(a;0) = \Omega^0(a;0) + \int_0^a R(s,a - s)\Omega^0(s;0)ds.$$  

(6.9)

Using the probability density $\Omega(a;0)$, the survival rate matrix $L(a)$ is given as

$$L(a) = U^0(a;0) + \int_0^a U(a,s;a - s,0)\Omega(a - s;0)ds.$$  

(6.10)

As is seen in the section 4, we can compute the intrinsic growth rate $r_0$ and the Frobenius eigenvector $Q$ by using $L(a)$ and $M(a)$. Let $p(a,s,t)$ (or $p^0(a,t)$)
be $N$-vector whose $i$-th element is $p_i(a,s,t)$ (or $p_i^0(a,t)$). Then the stable distribution vectors are given by

$$ p^0(a,t) = \exp(r_0(t-a))U^0(a;0)Q, $$

$$ p(a,s,t) = \exp(r_0(t-a))U(a,s;a-s,0)\Omega(a-s;0)Q. \tag{6.11} $$

Therefore we know that the stable distribution of the duration-dependent model (2.2) can be computed in principle by the knowledge of the first passage probabilities $A^0_{ij}(a)$, $A_{ij}(a,t)$. Furthermore, using the survival rate matrix, we obtain the stable distribution $n(a,t)$ aggregated over the duration as

$$ n(a,t) \stackrel{\text{def}}{=} p^0(a,t) + \int_0^a p(a,s,t)ds $$

$$ = \exp(r_0(t-a))(U^0(a;0) + \int_0^a U(a,s;a-s,0)\Omega(a-s;0)ds)Q $$

$$ = \exp(r(t-a))L(a)Q, $$

which has the same form as the stable distribution of the duration-independent multistate model.

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