Threshold and stability results for an age-structured epidemic model

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Abstract. We study a mathematical model for an epidemic spreading in an age-structured population with age-dependent transmission coefficient. We formulate the model as an abstract Cauchy problem on a Banach space and show the existence and uniqueness of solutions. Next we derive some conditions which guarantee the existence and uniqueness for non-trivial steady states of the model. Finally the local and global stability for the steady states are examined.

Key words: Age-structure — Epidemic model — Threshold theorem — Semigroup — Cauchy problem

1. Introduction

In this paper, we consider a mathematical model for an epidemic spreading in an age-structured population where the transmission coefficient depends on age. The model is derived for an SIR disease in a constant-sized population, that is, a susceptible individual who contracts the disease will become infective but will eventually recover with permanent immunity and the total population is assumed to be in a demographically stationary state. The SIR-type age-independent epidemic model has already been investigated satisfactory, and its threshold theorem is well known (Hethcote 1974). On the other hand, since the work of McKendrick (1926), it has been recognized that the age-structure of a population is an important factor which affects the dynamics of disease transmission. Therefore several authors introduced age-structure into their epidemic models
Recently, Greenhalgh (1988b) investigated the age-structured SIR-type epidemic model in case that the transmission coefficient depends on the age of both susceptibles and infectious, and he conjectured that

1. The threshold phenomenon can be formulated in terms of the spectral radius \( r(T) \) of a certain integral operator \( T \).
2. An endemic steady state is possible if and only if \( r(T) > 1 \) and if this state exists, it is unique.
3. The equilibrium with no disease present always exists, and it is locally (in fact globally) stable if \( r(T) < 1 \) and locally unstable if \( r(T) > 1 \).
4. For realistic values of parameters, the endemic equilibrium state is asymptotically stable.

Our main purpose in this paper is to prove Greenhalgh’s conjecture. First, we shall formulate the model by using the McKendrick-type partial differential equation system. Then we rewrite it into an abstract Cauchy problem on a Banach space, and show the existence and uniqueness of its solutions. Next, under appropriate conditions, we shall prove the existence and uniqueness results for non-trivial steady states of this model. Finally, we investigate the local and global stability for the steady states.

2. The basic model

We subdivide a closed population into three compartments containing susceptibles, infectives and immunes. There is no incubating class, so a person who catches the disease becomes infectious instantaneously. Moreover, we assume that the population is in a stationary demographic state. Let \( N(a) \), \( 0 \leq a \leq \omega \) (the number \( \omega \) denotes the life span of the population) be the density with respect to age of the total number of individuals. Under our assumption, \( N(a) \) satisfies

\[
N(a) = \mu^* N \exp \left( - \int_0^a \mu(\sigma) \, d\sigma \right) ,
\]

where \( \mu(a) \) denotes the instantaneous death rate at age \( a \) of the population, the constant \( N \) is the total size of the population and \( \mu^* \) is the crude death rate. We assume that \( \mu(a) \) is nonnegative, locally integrable on \([0, \omega]\) and satisfies

\[
\int_0^\omega \mu(\sigma) \, d\sigma = +\infty .
\]

The crude death rate is determined such that

\[
\mu^* \int_0^\omega \ell(a) \, da = 1 ,
\]
where

$$\ell(a) := \exp \left( - \int_0^a \mu(\sigma) \, d\sigma \right)$$

is the survival function which is the proportion of individuals who survive to age $a$. Then we have the relation

$$N(a) = \mu N \ell(a). \quad (2.2)$$

Next let $X(a, t)$, $Y(a, t)$ and $Z(a, t)$ be the age-densities of respectively the susceptible, infected and immune population at time $t$, so that

$$N(a) = X(a, t) + Y(a, t) + Z(a, t). \quad (2.3)$$

Let $\gamma^{-1}$ be the average infectious period, i.e. the probability of still being infected at the duration $s$ elapsed since initial infection is $\exp(-s\gamma)$. Let $\beta(a, b)$ be the age-dependent transmission coefficient, that is, the probability that a susceptible person of age $a$ meets an infectious person of age $b$ and becomes infected, per unit of time. Define the force of infection $\lambda(a, t)$ by

$$\lambda(a, t) = \int_0^a \beta(a, \sigma) Y(\sigma, t) \, d\sigma. \quad (2.4)$$

Then the transmission from the susceptible to the infectious state is a Poisson process, i.e. the probability that a susceptible individual becomes infected during the small interval $(a, a + da)$ at time $t$ is $\lambda(a, t) \, da$. Moreover we assume that the death rate of the population is not affected by the presence of the disease. Under the above assumptions, the spread of the disease can be described by the system of partial differential equations

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) X(a, t) = -\lambda(a, t) X(a, t) - \mu(a) X(a, t), \quad (2.5a)$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) Y(a, t) = \lambda(a, t) X(a, t) - (\mu(a) + \gamma) Y(a, t), \quad (2.5b)$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) Z(a, t) = \gamma Y(a, t) - \mu(a) Z(a, t), \quad (2.5c)$$

with boundary conditions

$$X(0, t) = \mu N, \quad Y(0, t) = 0, \quad Z(0, t) = 0. \quad (2.6)$$

Consider the fractions of susceptible, infectious and immune population at age $a$ and time $t$:

$$x(a, t) := \frac{X(a, t)}{N(a)}, \quad y(a, t) := \frac{Y(a, t)}{N(a)}, \quad z(a, t) := \frac{Z(a, t)}{N(a)}.$$
Then the system (2.5a)–(2.5c) can be rewritten to a simpler form

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) x(a, t) &= -\lambda(a, t)x(a, t), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) y(a, t) &= \lambda(a, t)x(a, t) - \gamma y(a, t), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) z(a, t) &= \gamma y(a, t), \\
x(0, t) &= 1, \quad y(0, t) = 0, \quad z(0, t) = 0,
\end{align*}
\] (2.7)

where

\[
\lambda(a, t) = \int_0^\sigma \beta(a, \sigma)N(\sigma) y(\sigma, t) \, d\sigma, \quad N(a) = \mu N'(a). \quad \tag{2.9}
\]

In the following, we mainly consider the system (2.7)–(2.9) with initial conditions

\[
x(a, 0) = x_0(a), \quad y(a, 0) = y_0(a), \quad z(a, 0) = z_0(a),
\] (2.10)

where \(x_0(a) + y_0(a) + z_0(a) = 1\). Then it follows that for all \(t \geq 0\)

\[
x(a, t) + y(a, t) + z(a, t) = 1. \quad \tag{2.11}
\]

**Remark.** For the system (2.5), threshold and stability results have been investigated for special forms of the transmission coefficient \(\beta\); \(\beta = \text{constant}\) (Dietz 1975; Greenhalgh 1987); \(\beta(a, b) = f(a)\) (Gripenberg 1983; Webb 1985); \(\beta(a, b) = f(a)g(b)\) (Dietz and Schenzle 1985; Greenhalgh 1988b; Castillo-Chavez et al. 1989). In particular, the models formulated by Gripenberg, Dietz and Schenzle treat a more general situation than the present model in the sense that the infectivity depends on the duration of an infection. However it should be noted that the existence problem of steady states for the duration-dependent model is reduced to the same kind of a nonlinear integral equation as discussed in Sect. 4 of this paper. Tudor (1985) reduced the system (2.5) into an ordinary differential equation system by discretizing the age variable. It seems that his theoretical and numerical results support Greenhalgh’s conjectures.

### 3. Existence and uniqueness of solutions

In this section we shall show that the initial-boundary value problem (2.7a)–(2.7c) has a unique solution. First we note that it suffices to consider the system in terms of only \(x(a, t)\) and \(y(a, t)\) since, once these functions are known, \(z(a, t)\) can be obtained by \(z(a, t) = 1 - x(a, t) - y(a, t)\).

First we introduce a new variable \(\hat{x}\) by \(\hat{x}(a, t) := x(a, t) - 1\). Then we obtain the new system for \(\hat{x}\) and \(y\).
\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \dot{x}(a, t) = -\lambda(a, t)(1 + \dot{x}(a, t)), \quad (3.1a)
\]
\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} y(a, t) = \lambda(a, t)(1 + \dot{x}(a, t)) - \gamma y(a, t), \quad (3.1b)
\]
\[
\dot{x}(0, t) = 0, \quad y(0, t) = 0.
\]
Consider the initial-boundary value problem of the system composed of (3.1a) and (3.1b) as an abstract Cauchy problem on the Banach space \( X := L^1(0, \omega; \mathbb{C}^2) \) that is the set of equivalence classes of Lebesgue integrable functions from \( [0, \omega] \) to \( \mathbb{C}^2 \) equipped with the \( L^1 \)-norm. Let \( A \) be a linear operator defined by
\[
(A\phi)(a) := \left( -\frac{d}{da} \phi_1(a), -\frac{d}{da} \phi_2(a) - \gamma \phi_2(a) \right)^t, \quad (3.2)
\]
where \( p^t \) is the transpose of the vector \( p \) and the domain \( D(A) \) is given as
\[
D(A) = \{ \phi \in X: \phi_i \in AC[0, \omega], \phi(0) = (0, 0)^t \},
\]
where \( AC[0, \omega] \) denotes the set of absolutely continuous functions. Suppose that \( \beta(a, b) \in L^\infty((0, \omega) \times (0, \omega)) \). We define a nonlinear operator \( F: X \to X \) by
\[
F(\phi)(a) = (-(P\phi_2)(a)(1 + \phi_1(a)), (P\phi_2)(a)(1 + \phi_1(a)))^t, \quad \phi \in X, \quad (3.3)
\]
where \( P \) is a bounded linear operator on \( L^1(0, \omega; \mathbb{C}) \) given by
\[
(P\psi)(a) = \int_0^\omega \beta(a, \sigma) N(\sigma) \psi(\sigma) d\sigma, \quad \psi \in L^1(0, \omega). \quad (3.4)
\]
Note that \( P\psi \in L^\infty(0, \omega) \) for \( \psi \in L^1(0, \omega) \) and hence the nonlinear operator \( F \) is defined on the whole space \( X \). Let \( u(t) := (\dot{x}(\cdot, t), y(\cdot, t))^t \in X \). Then we can rewrite the initial-boundary problem (3.1a)-(3.1b) as the abstract semilinear Cauchy problem in \( X \)
\[
\frac{d}{dt} u(t) = Au(t) + F(u(t)), \quad u(0) = u_0 \in X, \quad (3.5)
\]
where \( u_0(a) := (\dot{x}_0(a), y_0(a))^t, \dot{x}_0(a) := x_0(a) - 1 \). It is easily seen that the operator \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( T(t), \ t \geq 0 \) and \( F \) is continuously Frechet differentiable on \( X \). Then for each \( u_0 \in X \), there exists a maximal interval of existence \( [0, t_0) \), and a unique continuous (mild) solution \( t \to u(t; u_0) \) from \( [0, t_0) \) to \( X \) such that
\[
u(t; u_0) = T(t)u_0 + \int_0^t T(t - s)F(u(s; u_0)) \, ds, \quad (3.6)
\]
for all \( t \in [0, t_0) \) and either \( t_0 = \infty \) or \( \lim_{t \to t_0} \| u(t; u_0) \| = \infty \). Moreover, if \( u_0 \in D(A) \), then \( u(t; u_0) \in D(A) \) for \( 0 \leq t < t_0 \) and the function \( t \to u(t; u_0) \) is continuously differentiable and satisfies (3.5) on \( [0, t_0) \) (Webb, Proposition 4.16).
Lemma 3.1. Let

\[ \Omega := \{ (\dot{x}, y) \in X: \dot{x} \geq -1, y \geq 0 \} \]

and let

\[ \Omega_0 = \{ (\dot{x}, y) \in X: -1 \leq \dot{x} \leq 0, 0 \leq y \leq 1 \}. \]

Then the mild solution \( u(t; u_0) \), \( u_0 \in \Omega \) of (3.5) enters into \( \Omega_0 \) after finite time and the set \( \Omega_0 \) is positively invariant.

Proof. From (2.7a), we have the representation

\[
x(a, t) = \begin{cases} 
\exp \left( - \int_0^a \lambda(t, t-a+\varrho) \, d\varrho \right), & t-a > 0, \\
x_0(a-t) \exp \left( - \int_{a-t}^t \lambda(t-a+\varrho, \varrho) \, d\varrho \right), & a-t > 0,
\end{cases}
\]

which shows that \( \dot{x}(a, t) \geq -1 \) when \( x_0(a) \geq 0 \). If we write (3.1b) as an abstract Cauchy problem

\[
\frac{d}{dt} y(t) = By(t) + (Py(t))(1 + \dot{x}(t)), \quad y(0) = y_0 \in L^1(0, \omega),
\]

where the operator \( B \) is defined by

\[
B = -\frac{d}{da} - \gamma, \quad D(B) = \{ \psi \in L^1(0, \omega): \psi \in AC[0, \omega], \psi(0) = 0 \},
\]

then we obtain

\[
y(t) = S(t)y(0) + \int_0^t S(t-s)(Py(s))(1 + \dot{x}(s)) \, ds,
\]

where \( S(t) := \exp(tB) \) is the \( C_0 \)-semigroup generated by the closed operator \( B \). If we assume that \( \dot{x}(t) \geq -1, y_0 \geq 0 \), (3.9) shows that \( y(t) \) is positive because \( S(t), t \geq 0 \) is a positive semigroup and \( y(t) \) can be obtained by monotone iteration

\[
y(t) = S(t)y(0) \sum_0^t S(t-s)(PS(s)y(0))(1 + \dot{x}(s)) \, ds \cdot \cdot \cdot
\]

Hence we know that \( u(t; u_0) \in \Omega \) for all \( t \geq 0 \) when \( u_0 \in \Omega \). Next let \( w(t) := \dot{x}(t) + y(t) \). Then we have

\[
\frac{d}{dt} w(t) = Cw(t) - \gamma y(t), \quad w(0) = \dot{x}_0 + y_0 \in L^1(0, \omega),
\]

where the operator \( C \) is given by

\[
C = -\frac{d}{da}, \quad D(C) = \{ \psi \in L^1(0, \omega): \psi \in AC[0, \omega], \psi(0) = 0 \}.
\]
From (3.10), it follows that for positive \( y(t) \)
\[
w(t) = U(t)w(0) - \int_0^t U(t - s)w(s) \, ds \leq U(t)w(0),
\]
(3.11)
where \( U(t) \), \( t \geq 0 \) is the positive \( C_0 \)-semigroup generated by the operator \( C \). Since \( U(t) \) is a nilpotent translation semigroup, we have \( w(t)(a) \leq \hat{x}_0(a - t) + y_0(a - t) \), \( a > t \) and \( w(t) \leq 0 \) for \( t \geq \omega \). Then it follows that the mild solution \( u(t; u_0), u_0 \in \Omega \) enters into \( \Omega_0 \) for \( t \geq \omega \), and if \( u_0 \in \Omega_0 \), then \( u(t; u_0) \in \Omega_0 \) for all \( t \geq 0 \). This, completes the proof.

By Lemma 3.1, we know that the norm of the local solution \( u(t; u_0), u_0 \in \Omega \), of (3.5) is finite as far as it is defined. Thus we arrive at the following result:

**Proposition 3.2.** The abstract Cauchy problem (3.5) has a unique global classical solution on \( X \) with respect to the initial data \( u_0 \in \Omega \cap D(A) \).

Therefore it follows immediately that the initial-boundary value problem (2.7) – (2.9) has a unique positive global (mild) solution with respect to the positive initial data.

### 4. Existence of steady states

Let \((x^*(a), y^*(a))\) be the steady state solution for the equation (2.7a)–(2.7b). Then it is easy to verify the following:

\[
x^*(a) = \exp \left( - \int_0^a \hat{x}^*(\sigma) \, d\sigma \right),
\]
(4.1a)

\[
y^*(a) = \int_0^a \exp(-\gamma(a - \sigma)) \hat{y}^*(\sigma) \exp \left( - \int_0^\sigma \hat{x}^*(\eta) \, d\eta \right) \, d\sigma,
\]
(4.1b)

\[
\hat{x}^*(a) = \int_0^a \beta(a, \sigma) N(\sigma) y^*(\sigma) \, d\sigma.
\]
(4.1c)

Substituting (4.1b) into (4.1c) and integrating by parts, we obtain an equation for \( \hat{x}^*(a) \):

\[
\hat{x}^*(a) = \int_0^a \phi(a, \sigma) \hat{x}^*(\sigma) \exp \left( - \int_0^\sigma \hat{x}^*(\eta) \, d\eta \right) \, d\sigma,
\]
(4.2)

\[
\phi(a, \sigma) = \int_\sigma^\omega \beta(a, \zeta) N(\zeta) \exp(-\gamma(\zeta - \sigma)) \, d\zeta.
\]

From (4.1c), it follows that \( \| \hat{x}^*(a) \| \leq \mu^* N \| \beta \|_\infty \| y^* \|_1 \), where \( \| \cdot \|_\infty \) and \( \| \cdot \|_1 \) denote \( L^\infty \)-norm and \( L^1 \)-norm respectively. Then it follows from \( y^* \in L^1(0, \omega) \) that \( \hat{x}^* \in L^\infty(0, \omega) \). It is clear that one solution of (4.2) is \( \hat{x}^*(a) \equiv 0 \), which corresponds to the equilibrium state with no disease. In order to investigate non-trivial positive solutions for (4.2), we define a nonlinear operator \( \Phi(\psi) \) in the
Banach space $E := L^1(0, \omega)$ with the positive cone $E_+ := \{ \psi \in E: \psi \geq 0 \text{ a.e.} \}$ by

$$
\Phi(\psi)(a) := \int_0^\infty \phi(a, \sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \, d\sigma, \quad \psi \in E. \tag{4.3}
$$

Since the range of $\Phi$ is included in $L^\infty(0, \omega)$, the solutions of (4.2) correspond to fixed points of the operator $\Phi$. Observe that the operator $\Phi$ has a positive linear majorant $T$ defined by

$$
(T\psi)(a) := \int_0^\infty \phi(a, \sigma)\psi(\sigma) \, d\sigma, \quad \psi \in E. \tag{4.4}
$$

Here we summarize the Perron–Frobenius theory for positive operators on the ordered Banach space as far as it is needed for our purpose. Let $E$ be a real or complex Banach space and let $E^*$ be its dual, i.e. the space of all linear functionals on $E$. The value of $F \in E^*$ at $\psi \in E$ is denoted by $\langle F, \psi \rangle$. A closed subset $E_+$ is called a cone if the following holds: (1) $E_+ + E_+ \subseteq E_+$, (2) $\lambda E_+ \subseteq E_+$ for $\lambda \geq 0$, (3) $E_+ \cap (-E_+) = \{0\}$, (4) $E_+ \neq \{0\}$. We write $x \leq y$ if and only if $y - x \in E_+$ and write $x < y$ if $y - x \in E_+ \setminus \{0\}$. The cone $E_+$ is called total if the set $\{\psi - \phi: \psi, \phi \in E_+\}$ is dense in $E$. The dual cone $E_+^*$ is the subset of $E_+^*$ consisting of all positive linear functionals on $E$, i.e. $F \in E_+^*$ if and only if $F \in E_+^*$ and $\langle F, \psi \rangle > 0$ for all $\psi \in E_+ \setminus \{0\}$. A positive linear functional $F \in E_+^*$ is called strictly positive if $\langle F, \psi \rangle > 0$ for all $\psi \in E_+ \setminus \{0\}$. Let $B(E)$ be the set of bounded linear operators on $E$ into $E$. $T \in B(E)$ is called positive with respect to the cone $E_+$ if $T(E_+) \subseteq E_+$. We say $T \geq S$ if $(T - S)(E_+) \subseteq E_+$ for $T, S \in B(E)$. We denote the spectral radius of $T \in B(E)$ by $r(T)$.

Although several formally different concepts about positivity of operators have been introduced to extend the Perron–Frobenius theory since the work of Krein and Rutman (1948), it seems that Sawashima's concepts are most natural and convenient for our purpose (see Sawashima 1964; Marek 1970; Heijmans 1986):

**Definition 4.1** (Sawashima 1964). A positive operator $T \in B(E)$ is called semi-nonsupporting if and only if for every pair $\psi \in E_+ \setminus \{0\}, F \in E_+^* \setminus \{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $\langle F, T^n\psi \rangle > 0$. A positive operator $T \in B(E)$ is called nonsupporting if and only if for every pair $\psi \in E_+ \setminus \{0\}, F \in E_+^* \setminus \{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $\langle F, T^n\psi \rangle > 0$ for all $n \geq p$.

The reader may refer to Sawashima (1964), Niiro and Sawashima (1966) for the proof of the following theorem:

**Proposition 4.2.** Let the cone $E_+$ be total, $T \in B(E)$ be semi-nonsupporting with respect to $E_+$ and let $r(T)$ be a pole of the resolvent $R(\lambda, T)$. Then the following holds:

1. $r(T) \in P_\sigma(T) \setminus \{0\}$ ($P_\sigma(T)$ denotes the point spectrum of $T$) and $r(T)$ is a simple pole of the resolvent.

2. The eigenspace corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\psi \in E_+$ is a nonsupporting point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\psi$ for some constant $c$. 


The eigenspace of $T^*$ corresponding to $r(T)$ is also a one-dimensional subspace of $E^*$ spanned by a strictly positive functional $F \in E_*^+$.

The following comparison theorem is due to Marek (1970):

**Proposition 4.3.** Suppose that $E$ is a Banach lattice. Let $S$ and $T$ be positive operators in $B(E)$.

1. If $S \leq T$, then $r(S) \leq r(T)$.
2. If $S$ and $T$ are semi-nonsupporting and compact operators, then $S \leq T$, $S \not\equiv T$ and $r(T) \neq 0$ imply that $r(S) < r(T)$.

After the above preparations, we first consider the nature of the majorant operator $T$ defined by (4.4). In the following, we shall make an assumption:

**Assumption 4.4.**

1. $\beta(a, \zeta) \in L^\infty_{\infty}((0, \omega) \times (0, \omega))$.
2. $\lim_{h \to 0} \int_0^\omega \beta(a + h, \zeta) - \beta(a, \zeta) \, da = 0$ uniformly for $\zeta \in \mathbb{R}$, (4.5)

where $\beta$ is extended by $\beta(a, \zeta) = 0$ for $a, \zeta \in (-\infty, 0) \cup (\omega, \infty)$.

3. There exist numbers $\alpha$ with $\omega > \alpha > 0$ and $\epsilon > 0$ such that

$$
\beta(a, \zeta) \geq \epsilon \text{ for almost all } (a, \zeta) \in (0, \omega) \times (\omega - \alpha, \omega). \quad (4.6)
$$

Then we can prove that

**Lemma 4.5.** Under Assumption 4.4, the operator $T : E \to E$ is nonsupporting and compact.

**Proof.** Define the positive linear functional $f_0 \in E_*^+$ by

$$
\langle f_0, \psi \rangle := \int_0^\omega \left[ \int_0^\omega s(\zeta) N(\zeta) \exp(-\gamma(\zeta - \sigma)) \, d\zeta \right] \psi(\sigma) \, d\sigma, \quad \psi \in E, \quad (4.7)
$$

where the function $s(\zeta)$ is defined as $s(\zeta) = 0, \zeta \in [0, \omega - \alpha)$; $s(\zeta) = \epsilon, \zeta \in (\omega - \alpha, \omega)$. Hence $\beta(a, \zeta) \geq s(\zeta)$ for almost all $(a, \zeta) \in [0, \omega] \times [0, \omega]$.

It is easily seen that the functional $f_0$ is strictly positive and

$$
T\psi \geq \langle f_0, \psi \rangle e, \quad e = 1 \in E_+, \quad \psi \in E_+. \quad (4.8)
$$

Then for any integer $n$, we have

$$
T^{n+1}\psi \geq \langle f_0, \psi \rangle \langle f_0, e \rangle^n e.
$$

Therefore we obtain $\langle F, T^n\psi \rangle > 0, \, n \geq 1$ for every pair $\psi \in E_+ \setminus \{0\}$, $F \in E_*^+ \setminus \{0\}$, that is, $T$ is nonsupporting. Next observe that

$$
\int_0^\omega \left( \phi(a + h, \sigma) - \phi(a, \sigma) \right) \, da \leq \mu N \int_0^\omega \left( \beta(a + h, \zeta) - \beta(a, \zeta) \right) da \, d\zeta.
$$

Then it follows from Assumption 4.4 (1), (2) and a well-known compactness criterium in $L^1$ that the operator $T$ is compact.

From Proposition 4.2, it follows that the spectral radius $r(T)$ of the operator $T$ is the only positive eigenvalue with a positive eigenvector and also an
eigenvalue of the dual operator \( T^* \) with a strictly positive eigenfunctional. Now we can prove the following:

**Proposition 4.6** (threshold results). Let \( r(T) \) be the spectral radius of the operator \( T \) defined by (4.4). Then the following holds:

1. If \( r(T) \leq 1 \), the only nonnegative solution \( \psi \) of the equation \( \psi = \Phi(\psi) \) is the trivial solution \( \psi = 0 \).
2. If \( r(T) > 1 \), the equation \( \psi = \Phi(\psi) \) has at least one non-zero positive solution.

**Proof.** Suppose that \( r(T) \leq 1 \). It is easily checked that \( T\psi - \Phi(\psi) \in E_+ \setminus \{0\} \) for \( \psi \in E_+ \setminus \{0\} \). If there exists a \( \psi_0 \in E_+ \setminus \{0\} \) being a solution of \( \psi = \Phi(\psi) \), then \( \psi_0 = \Phi(\psi_0) \leq T\psi_0 \). Let \( F_0^* \in E_+^* \setminus \{0\} \) be the adjoint eigenvector of \( T \) corresponding to \( r(T) \). Taking duality pairing, we find \( \langle F_0^*, T(\psi_0) - \psi_0 \rangle = (r(T) - 1)\langle F_0^*, \psi_0 \rangle > 0 \) because \( T(\psi_0) - \psi_0 \in E_+ \setminus \{0\} \) and \( F_0^* \) is strictly positive. Then we have \( r(T) > 1 \), which is a contradiction. Next we assume that \( r(T) > 1 \).

Under Assumption 4.4, in the same manner as the proof of Lemma 4.5, we can see that the operator \( \Phi \) is a completely continuous operator in the Banach space \( E \). Moreover, if we define the number \( M_0 \) by

\[
M_0 := \sup_{0 \leq \sigma \leq \omega} \int_0^\omega \phi(a, \sigma) \, da,
\]

the set \( \Omega := \{ \psi \in E: 0 \leq \psi, \|\psi\| \leq M_0 \} \) is invariant (in fact \( \Phi(E_+) \subset \Omega \)) under the operator \( \Phi \). We define an operator \( \Phi_r \) by

\[
\Phi_r(\psi) = \begin{cases} 
\Phi(\psi), & \text{if } \|\psi\| \geq r, \psi \in E_+ \\
\Phi(\psi) + (r - \|\psi\|)\psi_0, & \text{if } \|\psi\| \leq r, \psi \in E_+ 
\end{cases}
\]

where \( \psi_0 \) is the positive eigenvector of \( T \) corresponding to \( r(T) > 1 \). Then \( \Phi_r \) is also completely continuous and transforms the set

\[
\Omega_r := \{ \psi \in E: 0 \leq \psi, \|\psi\| \leq M_0 + r\|\psi_0\| \}
\]

into itself. Since \( \Omega_r \) is bounded, convex and closed in \( E, \Phi_r \) has a fixed point \( \psi_r \in \Omega_r \) (Schauder's principle). Observe that the Frechet derivative of \( \Phi(\psi) \) at \( \psi = 0 \) is the operator \( T \) and \( T \) does not have eigenvectors in \( E_+ \) corresponding to the eigenvalue one. Then we can apply the method of Krasnoselskii (1964a), Theorem 4.11, and it can be shown that the norms of these fixed points are greater than \( r \) if \( r \) is sufficiently small. That is, \( \Phi \) has a positive fixed point.

Subsequently, in order to investigate the uniqueness problem for non-trivial positive fixed points of the operator \( \Phi \), we introduce the concept of concave operators (see Krasnoselskii 1964a,b).

**Definition 4.7.** Let \( E_+ \) be a cone in a real Banach space \( E \) and \( \leq \) be the partial ordering defined by \( E_+ \). A positive operator \( A : E_+ \to E_+ \) is called a concave operator if there exists a \( \psi_0 \in E_+ \setminus \{0\} \) which satisfies the following:

1. For any \( \psi \in E_+ \setminus \{0\} \) there exist \( \alpha = \alpha(\psi) > 0 \) and \( \beta = \beta(\psi) > 0 \) such that \( \alpha \psi \leq A \psi \leq \beta \psi_0 \), that is, \( A \psi \) is comparable with \( \psi_0 \).
2. \( A(t\psi) \geq tA\psi \) for \( 0 \leq t \leq 1 \) and for every \( \psi \in E_+ \) such that \( \alpha(\psi)\psi_0 \leq \psi \leq \beta(\psi)\psi_0 \) with \( \alpha(\psi) > 0, \beta(\psi) > 0 \).
Here we introduce a new class of concave operators which has at most one positive fixed point. This type of operator is closely related to the \textit{e-sublinear operator} introduced by Amann (1972).

\textbf{Lemma 4.8.} Suppose that the operator $A : E_+ \to E_+$ is monotone and concave. If for any $\psi \in E_+$ satisfying $\alpha_1 \psi_0 \leq \psi \leq \beta_1 \psi_0$ ($\alpha_1 = \alpha_1(\psi) > 0$, $\beta_1 = \beta_1(\psi) > 0$) and any $0 < t < 1$, there exists $\eta = \eta(\psi, t) > 0$ such that

$$A(t\psi) \triangleright \eta \psi_0,$$

(4.9)

then $A$ has at most one positive fixed point.

\textbf{Proof.} Suppose that $\psi_1 \in E_+ \setminus \{0\}$ and $\psi_2 \in E_+ \setminus \{0\}$ are two positive fixed points of $A$. From the concavity of $A$, we can choose positive constants $\alpha_1 = \alpha_1(\psi_1) > 0$ and $\beta_2 = \beta_2(\psi_2) > 0$ such that

$$\psi_1 = A\psi_1 \geq \alpha_1 \beta_2^{-1} \beta_2 \psi_0 \geq \alpha_1 \beta_2^{-1} A\psi_2 = \alpha_1 \beta_2^{-1} \psi_2.$$

If we define $k = \sup\{\mu : \psi_1 \geq \mu \psi_2\}$, then we see that $k > 0$ from the above inequality. If we assume that $0 < k < 1$, then there exists $\eta = \eta(\psi_2, k) > 0$ such that

$$\psi_1 = A\psi_1 \geq A(k\psi_2) \geq k A\psi_2 + \eta \psi_2 \geq k A\psi_2 + \eta \beta_2^{-1} A\psi_2 = (k + \eta \beta_2^{-1}) \psi_2,$$

which contradicts the definition of $k$. Hence we know that $k \geq 1$ and $\psi_1 \geq \psi_2$. In the same way, we can prove $\psi_2 \geq \psi_1$. Thus $\psi_1 = \psi_2$.

Here we introduce another assumption:

\textbf{Assumption 4.9.} For all $(a, \sigma) \in [0, \omega] \times [0, \omega]$, the inequality,

$$\beta(a, \sigma)N(\sigma) - \gamma \int_{\sigma}^{\omega} \beta(a, \zeta) N(\zeta) \exp(-\gamma(\zeta - \sigma)) d\zeta > 0,$$

(4.10)

holds.

Then we can prove the following:

\textbf{Proposition 4.10.} Suppose that Assumption 4.9 holds. If $r(T) > 1$, $\Phi$ has only one positive fixed point.

\textbf{Proof.} From Lemma 4.8 and Proposition 4.6, it is sufficient to show that under Assumption 4.9, the operator $\Phi$ is a monotonic concave operator satisfying the condition (4.9). From (4.3), it follows that

$$\Phi(\psi)(a) = \int_0^\omega \phi(a, \sigma) \left( -\frac{d}{d\sigma} \right) \exp \left( -\int_0^\sigma \psi(\eta) d\eta \right) d\sigma$$

$$= \phi(a, 0) - \int_0^\omega [\beta(a, \sigma)N(\sigma) - \gamma \phi(a, \sigma)] \exp \left( -\int_0^\sigma \psi(\eta) d\eta \right) d\sigma.$$

Then the operator $\Phi$ is monotonic under Assumption 4.9. Next observe that

$$\alpha(\psi) \psi_0 \leq \Phi(\psi) \leq \beta(\psi) \psi_0,$$
where $\psi_0 \equiv 1$ and

$$
\alpha(\psi) := \int_0^\infty f(\sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \, d\sigma,
$$

$$
\beta(\psi) := M \int_0^\infty g(\sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \, d\sigma,
$$

where $M := \text{ess sup} \beta(a, b) < \infty$, $f(\sigma)$ and $g(\sigma)$ are defined by

$$
f(\sigma) := \int_\sigma^\infty s(\zeta)N(\zeta) \exp(-\gamma(\zeta - \sigma)) \, d\zeta,
$$

$$
g(\sigma) := \int_\sigma^\infty N(\zeta) \exp(-\gamma(\zeta - \sigma)) \, d\zeta.
$$

Then it follows that $\alpha(\psi) > 0$ and $\beta(\psi) > 0$ for $\psi \in E_+ \setminus \{0\}$. Moreover we obtain

$$
\begin{align*}
\Phi(t\psi)(a) - t\Phi(\psi)(a) &= \int_0^\infty f(\sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \left[ \exp \left( (1-t) \int_0^\sigma \psi(\eta) \, d\eta \right) - 1 \right] \, d\sigma \\
&\geq t \int_0^\infty f(\sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \left[ \exp \left( (1-t) \int_0^\sigma \psi(\eta) \, d\eta \right) - 1 \right] \, d\sigma,
\end{align*}
$$

from which we conclude that $\Phi$ is a concave operator and the condition (4.9) is satisfied by letting $\psi_0 = 1$ and

$$
\eta(t, \sigma) := \int_0^\infty f(\sigma)\psi(\sigma) \exp \left( -\int_0^\sigma \psi(\eta) \, d\eta \right) \left[ \exp \left( (1-t) \int_0^\sigma \psi(\eta) \, d\eta \right) - 1 \right] \, d\sigma.
$$

This completes the proof.

Note that Assumption 4.9 holds if $\beta(a, \sigma)N(\sigma)$ is non-increasing as a function of $\sigma$. In fact, we have

$$
\beta(a, \sigma)N(\sigma) - \gamma \int_\sigma^\infty \beta(a, \zeta)N(\zeta) \exp(-\gamma(\zeta - \sigma)) \, d\zeta = \gamma \int_\sigma^\infty [\beta(a, \sigma)N(\sigma) - \beta(a, \zeta)N(\zeta)] \exp(-\gamma(\zeta - \sigma)) \, d\zeta
$$

$$
+ \exp(-\gamma(\omega - \sigma)\beta(a, \sigma)N(\sigma),
$$

which is nonnegative for all $(a, \sigma) \in [0, \omega] \times [0, \omega]$ if $\beta(a, \sigma)N(\sigma) - \beta(a, \zeta)N(\zeta) \geq 0$ for $\zeta \geq \sigma$. In particular, Assumption 4.9 holds if $\beta$ is independent of the age of infectives $\sigma$, because $N(\sigma)$ is a decreasing function. Another type of condition which guarantees Assumption 4.9 is given as follows:

$$
\ell(a) \geq k(1 - \exp(-\gamma(\omega - a)), \quad (4.11)
$$

if the constant $k$ defined by

$$
k := \frac{\sup \beta(a, b)}{\inf \beta(a, b)}, \quad (4.12)
$$
is finite. Since $N(a) = \mu \cdot N' \leq \mu \cdot N$, the sufficiency of condition (4.11) follows from the inequality

$$\beta(a, \sigma)N(\sigma) - \gamma \phi(a, \sigma) \geq \inf \beta(a, b)\mu \cdot N[\ell(\sigma) - k(1 - \exp(-\gamma(\omega - \sigma)))].$$

**Remark 4.11.** No matter whether Assumption 4.9 holds, if $\beta(a, b)$ can be factorized as $u(a)v(b)$ (which is called the proportionate mixing assumption, see Dietz and Schenzle 1985), it is easily seen that there always exists a unique non-trivial steady state under the condition

$$r(T) = \int_0^\infty \phi(\sigma, \sigma) \, d\sigma > 1. \quad (4.13)$$

In this case, $u(a)$ is the eigenvector of the operator $T$ corresponding to the spectral radius $r(T)$ (see Greenhalgh 1988b).

### 5. Stability analysis for equilibrium solutions

In order to investigate the local stability of the equilibrium solutions $(x^*(a), y^*(a))^T$ of (2.7a)–(2.7b), we rewrite (2.7a)–(2.7b) into the equation for small perturbations: Let

$$x(a, t) = x^*(a) + \zeta(a, t), \quad y(a, t) = y^*(a) + \eta(a, t).$$

Then we have

$$\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \zeta(a, t) &= -\lambda(a, t)(\zeta(a, t) + x^*(a)) - \lambda^*(a)\zeta(a, t), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \eta(a, t) &= \lambda(a, t)(\zeta(a, t) + x^*(a)) + \lambda^*(a)\zeta(a, t) - \gamma \eta(a, t),
\end{align*} \quad (5.1a)$$

where

$$\begin{align*}
\lambda(a, t) &= \int_0^\infty \beta(a, \sigma)N(\sigma)\eta(\sigma, t) \, d\sigma, \\
\lambda^*(a) &= \int_0^\infty \beta(a, \sigma)N(\sigma)y^*(\sigma) \, d\sigma.
\end{align*}$$

Therefore we can formulate (5.1) as an abstract semilinear problem on the Banach space $X$:

$$\frac{d}{dt} u(t) = Au(t) + G(u(t)), \quad u(t) = (\zeta(t), \eta(t))^T \in X, \quad (5.2)$$

where the generator $A$ is defined by (3.2). The nonlinear term $G$ is defined as

$$G(u) = -(Pu_2)(u_1 + x^*) - \lambda^* u_1, \quad (Pu_2)(u_1 + x^*) + \lambda^* u_1)^T,$$

for $u = (u_1, u_2)^T \in X$, where the operator $P$ is defined by (3.4). The linearized equation of (5.2) around $u = 0$ is given by

$$\frac{d}{dt} u(t) = (A + C)u(t), \quad (5.3)$$
where the bounded linear operator $C$ is the Frechet derivative of $G(u)$ at $u = 0$
and is given by
\[
Cu = (-(Pu_2)x^* - \lambda^*u_1, (Pu_2)x^* + \lambda^*u_1)^*.
\]
Now let us consider the resolvent equation for $A + C$:
\[
(\lambda - (A + C))\phi = \psi, \quad \phi \in D(A), \psi \in X, \quad \lambda \in \mathbb{C}.
\]
Then we have
\[
\phi_1' + (\lambda + \lambda^*)\phi_1 = \psi_1 - (P\phi_2)x^*,
\]
\[
\phi_2' + (\lambda + \gamma)\phi_2 = \psi_2 + (P\phi_2)x^* + \lambda^*\phi_1.
\]
From (5.5a), we obtain
\[
\phi_1(a) = \exp(-\lambda a)\Pi(a) \int_0^a [\psi_1(\sigma) - (P\phi_2)(\sigma)\Pi(\sigma)] \exp(\lambda \sigma)\Pi^{-1}(\sigma) \, d\sigma, \quad (5.6)
\]
where $\Pi(a)$ is defined by
\[
\Pi(a) := \exp \left( - \int_0^a \lambda^*(\sigma) \, d\sigma \right) = x^*(a).
\]
From (5.5b), we have
\[
\phi_2(a) = \int_0^a \exp(-\gamma(a - \sigma)) \psi_2(\sigma) + (P\phi_2)(\sigma)\Pi(\sigma) + \lambda^*(\sigma)\phi_1(\sigma) \, d\sigma. \quad (5.7)
\]
On the other hand, from (5.6), we can write
\[
\lambda^*(\sigma)\phi_1(\sigma) = \exp(-\lambda \sigma)\Pi(\sigma)\lambda^*(\sigma)
\]
\[
\times \int_0^\sigma [\psi_1(\eta) - (P\phi_2)(\eta)\Pi(\eta)] \exp(\lambda \eta)\Pi^{-1}(\eta) \, d\eta. \quad (5.8)
\]
Using (5.7) and (5.8), we obtain
\[
(P\phi_2)(a) = I(a) + J(a) + K(a) + L(a), \quad (5.9)
\]
where
\[
I(a) := \int_0^a \beta(a, \sigma)N(\sigma) \int_0^\sigma \exp(-\gamma + \lambda) (\sigma - \eta) \psi_2(\eta) \, d\eta \, d\sigma,
\]
\[
J(a) := \int_0^a \beta(a, \sigma)N(\sigma) \int_0^\sigma \exp(-\gamma + \lambda) (\sigma - \eta) (P\phi_2)(\eta) \Pi(\eta) \, d\eta \, d\sigma,
\]
\[
K(a) := \int_0^a \beta(a, \sigma)N(\sigma) \int_0^\sigma \exp(-\gamma + \lambda) (\sigma - \eta) \exp(-\lambda \eta)\Pi(\eta)\lambda^*(\eta)
\]
\[
\times \int_0^\sigma [\psi_1(\zeta) \exp(\lambda \zeta)\Pi^{-1}(\zeta) \, d\zeta \, d\sigma,
\]
\[
L(a) := -\int_0^a \beta(a, \sigma)N(\sigma) \int_0^\sigma \exp(-\gamma + \lambda) (\sigma - \eta) \exp(-\lambda \eta)\Pi(\eta)\lambda^*(\eta)
\]
\[
\times \int_0^\eta (P\phi_2)(\zeta) \exp(\lambda \zeta) \, d\zeta \, d\sigma.
\]
Define
\[
\phi_\lambda(a, \sigma) := \int_{-\sigma}^{\infty} \beta(a, \eta) N(\eta) \exp(-\gamma - \lambda)(\eta - \sigma) \, d\eta,
\]  
then we can rewrite the above representations for \(I, J, K, L\) as
\[
I(a) = \int_{0}^{\infty} \phi_\lambda(a, \sigma) \psi_\lambda(\sigma) \, d\sigma,
\]
\[
J(a) = \int_{0}^{\infty} \phi_\lambda(a, \sigma) \Pi(\sigma)(P\phi_\lambda)(\sigma) \, d\sigma,
\]
\[
K(a) = \int_{0}^{\infty} \phi_\lambda(a, \sigma) \exp(-\lambda\sigma) \Pi(\sigma)\lambda^*(\sigma) \int_{0}^{\sigma} \psi_1(\eta) \Pi^{-1}(\eta) \exp(\lambda\eta) \, d\eta \, d\sigma,
\]
\[
L(a) = -\int_{0}^{\infty} \phi_\lambda(a, \sigma) \exp(-\lambda\sigma) \Pi(\sigma)\lambda^*(\sigma) \int_{0}^{\sigma} (P\phi_\lambda)(\eta) \exp(\lambda\eta) \, d\eta \, d\sigma,
\]
If we define linear operators on the Banach space \(L^1(0, \infty)\) by
\[
(T_\lambda\psi)(a) := \int_{0}^{\infty} \phi_\lambda(a, \sigma) \Pi(\sigma)\psi(\sigma) \, d\sigma,
\]
\[
(U_\lambda\psi)(a) := \int_{0}^{\infty} \phi_\lambda(a, \sigma) \exp(-\lambda\sigma) \Pi(\sigma)\lambda^*(\sigma) \int_{0}^{\sigma} \psi(\eta) \exp(\lambda\eta) \, d\eta \, d\sigma,
\]
\[
(V_\lambda\psi)(a) := (T_\lambda\psi)(a) - (U_\lambda\psi)(a),
\]
then the following expression holds:
\[
(V_\lambda\psi)(a) = \int_{0}^{\infty} \chi_\lambda(a, \sigma) \psi(\sigma) \, d\sigma,
\]
\[
\chi_\lambda(a, \sigma) := \int_{0}^{\sigma} \Pi(\zeta)[\beta(a, \zeta) N(\zeta) - \gamma\phi_\lambda(a, \zeta)] \exp(-\lambda(\zeta - \sigma)) \, d\zeta.
\]
It is easy to verify the above expression if we note that
\[
(U_\lambda\psi)(a) = \int_{0}^{\infty} \left( -\frac{d\Pi(\sigma)}{d\sigma} \right) \phi_\lambda(a, \sigma) \exp(-\lambda\sigma) \int_{0}^{\sigma} \psi(\eta) \exp(\lambda\eta) \, d\eta \, d\sigma
\]
\[
= (T_\lambda\psi)(a) - \int_{0}^{\infty} \Pi(\sigma)[\beta(a, \sigma) N(\sigma) - \gamma\phi_\lambda(a, \sigma)]
\]
\[
\times \int_{0}^{\sigma} \psi(\eta) \exp(-\lambda(\sigma - \eta)) \, d\eta \, d\sigma.
\]
From the above definitions and (5.9), it follows that
\[
(P\phi_\lambda)(a) = (T_\lambda\psi_\lambda\Pi^{-1})(a) + (T_\lambda P\phi_\lambda)(a) + (U_\lambda\psi_\lambda\Pi^{-1})(a) - (U_\lambda P\phi_\lambda)(a).
\]
Hence we have
\[
(P\phi_\lambda)(a) = (I - V_\lambda)^{-1}[(T_\lambda\psi_\lambda\Pi^{-1})(a) + (U_\lambda\psi_\lambda\Pi^{-1})(a)].
\]
Lemma 5.1. The perturbed operator $A + C$ has a compact resolvent and
\[ \sigma(A + C) = P_\sigma(A + C) = \{ \lambda \in \mathbb{C} : 1 \in P_\sigma(V_{\lambda}) \}, \]  \hspace{1cm} (5.15)
where $\sigma(A)$ and $P_\sigma(A)$ denote the spectrum of $A$ and the point spectrum of $A$ respectively.

Proof. From (5.6) and (5.14), we obtain the expression for $\phi_1$
\[ \phi_1(a) = J(\psi_1)(a) - K(\psi_1, \psi_2)(a), \]
where the operators $J$ and $K$ are defined by
\[ J(\psi_1)(a) = \int_0^a G(a, \sigma) \psi_1(\sigma) \, d\sigma, \]
\[ K(\psi_1, \psi_2)(a) = \int_0^a G(a, \sigma) \Pi(\sigma)(I - V_{\lambda})^{-1}[(T_{\lambda} \psi_2 \Pi^{-1})(\sigma) + (U_{\lambda} \psi_1 \Pi^{-1})(\sigma)] \, d\sigma, \]
where $G(a, \sigma) := \exp[-\lambda(a - \sigma)] \Pi(a) \Pi^{-1}(\sigma)$. Since $J$ is a Volterra operator with a continuous kernel, it is a compact operator on $L^1(0, \omega)$. On the other hand, in the same manner as the proof of Lemma 4.5, we can prove that $T_{\lambda}$ and $U_{\lambda}$ are compact for all $\lambda \in \mathbb{C}$. Let $\Lambda := \{ \lambda \in \mathbb{C} : 1 \in \sigma(V_{\lambda}) \}$. Then it follows that when $\lambda \in \mathbb{C} \setminus \Lambda$ the operator $K$ is a compact operator from $X$ to $L^1(0, \omega)$. In the same way, we can prove that $\phi_2(a)$ can be represented by a compact operator from $X$ to $L^1(0, \omega)$. Then we know that $A + C$ has a compact resolvent, so it follows that $\sigma(A + C) = P_\sigma(A + C)$ (see Kato, p. 187). From the above argument, it follows that $\mathbb{C} \setminus \Lambda \subset \rho(A + C)$ ($\rho(A + C)$ denotes the resolvent set of $A + C$), that is, $\Lambda \supset \sigma(A + C) = P_\sigma(A + C)$. Since $V_{\lambda}$ is a compact operator, we know that $\sigma(V_{\lambda}) \setminus \{0\} = P_\sigma(V_{\lambda}) \setminus \{0\}$ and if $\lambda \in \Lambda$, there exists an eigenfunction $\psi_{\lambda}$ such that $V_{\lambda} \psi_{\lambda} = \psi_{\lambda}$. Then it is easily seen that if we define the following functions:
\[ \phi_1(a) = -\exp(-\lambda a) \Pi(a) \int_0^a \exp(\lambda \sigma) \psi_{\lambda}(\sigma) \, d\sigma, \]
\[ \phi_2(a) = \int_0^a \exp[-\lambda(a - \sigma)] [\psi_{\lambda}(\sigma) \Pi(\sigma) - \lambda^*(\sigma) \phi_1(\sigma)] \, d\sigma, \]
$(\phi_1, \phi_2)^\ast$ gives an eigenvector of $A + C$ corresponding to the eigenvalue $\lambda$. Then $\Lambda \subset P_\sigma(A + C)$ and we conclude that (5.15) holds.

Lemma 5.2. Let $T(t), t \geq 0$ be the $C_0$-semigroup generated by the perturbed operator $A + C$. Then $T(t), t \geq 0$ is eventually norm continuous and
\[ \omega_0(A + C) = s(A + C) := \sup\{ \text{Re} \, \mu : \mu \in \sigma(A + C) \}, \]  \hspace{1cm} (5.16)
where $\omega_0(A + C)$ denotes the growth bound of the semigroup $T(t), t \geq 0$ and $s(A + C)$ is the spectral bound of the generator $A + C$.

Proof. We define bounded operators $C_1$ and $C_2$ by
\[ C_1 \phi = (-\lambda^* \phi_1, \lambda^* \phi_1)^\ast, \quad C_2 \phi = (-x^*(P\phi_2), x^*(P\phi_2))^\ast, \quad \phi \in X. \]
Then \( C = C_1 + C_2 \) and \( A + C_1 \) generates a \( C_0 \)-semigroup \( S(t) \), \( t \geq 0 \). Since \( S(t) \) is a nilpotent semigroup, so it is eventually norm continuous. Using Assumption 4.4, we can prove that \( C_2 \) is a compact operator in \( X \). Therefore, from Theorem 1.30 in Nagel (1986), p. 44, \( T(t) \) is also eventually norm continuous. Since the spectral mapping theorem holds for the eventually norm continuous semigroup (Nagel, p. 87), we obtain (5.16).

If \( \omega_0(A + C) < 0 \), the equilibrium \( u = 0 \) of system (5.2) is locally exponentially asymptotically stable (the principle of linearized stability, see Webb 1985; Desch and Schappacher 1986). Therefore, in order to study the stability of equilibrium states, we have to know the structure of the set of singular points \( \Lambda = \{ \lambda \in \mathbb{C} : 1 \in P_\sigma(V_\lambda) \} \). Since \( \|V_\lambda\| \to 0 \) if \( \Re \lambda \to \infty \), \( I - V_\lambda \) is invertible for large values of \( \Re \lambda \). By the theorem of Steinberg (1968), the function \( \lambda \to (I - V_\lambda)^{-1} \) is meromorphic in the complex domain, and hence the set \( \Lambda \) is a discrete set whose elements are poles of \( (I - V_\lambda)^{-1} \) of finite order.

Now we shall make use of positive operator theory once more. Our main purpose here is to determine the dominant singular point, i.e. the element of the set \( \Lambda \) with the largest real part. From (5.15) and (5.16), the dominant singular point gives the growth bound of the semigroup \( T(t) \) generated by \( A + C \). First we show that

**Lemma 5.3.** Suppose that the following assumption holds:

**Assumption 5.4.**

\[
p^* (\omega) < e^{-\gamma_0}.
\]

Then the operator \( V_\lambda, \lambda \in \mathbb{R} \) is nonsupporting with respect to \( E_+ \) and the following holds:

\[
\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} r(V_\lambda) = 0.
\]

**Proof.** By changing order of integration in the expression (5.13), it can be shown that

\[
\chi_\lambda(a, \sigma) = \int_{\sigma}^{\infty} \left[ \Pi(\zeta) \gamma \int_{\sigma}^{\zeta} \Pi(\eta) \exp(-\gamma(\zeta - \eta)) d\eta \right] \beta(a, \zeta) N(\zeta) d\zeta.
\]

If we define

\[
G_\sigma(\zeta) := \Pi(\zeta) \gamma \int_{\sigma}^{\zeta} \Pi(\eta) \exp(-\gamma(\zeta - \eta)) d\eta,
\]

then the operator \( V_\lambda, \lambda \in \mathbb{R} \) is positive if \( G_\sigma(\zeta) > 0 \) for almost all \( \zeta \in [\sigma, \omega] \), \( 0 \leq \sigma \leq \omega \). Since \( G_\sigma(\zeta) \exp(\gamma \zeta) \) is monotone decreasing for the variable \( \zeta, G_\sigma(\zeta) \geq \exp(\gamma(\omega - \zeta)) G_\sigma(\omega) \) for all \( \zeta \in [\sigma, \omega] \). From \( G_\sigma(\omega) \geq G_\sigma(\omega) \), we know that \( G_\sigma(\omega) > 0 \) is sufficient to guarantee \( G_\sigma(\zeta) > 0 \) for all \( 0 \leq \sigma \leq \zeta \leq \omega \). Integrating (5.20) by parts, we have

\[
G_\sigma(\zeta) = \Pi(\sigma) \exp(-\gamma(\zeta - \sigma)) - \int_{\sigma}^{\zeta} \lambda^* (\eta) \Pi(\eta) \exp(-\gamma(\zeta - \eta)) d\eta.
\]
Then we know that $G_0(\omega) = \exp(-\gamma \omega) - y^*(\omega)$ and the operator $V_\lambda, \lambda \in \mathbb{R}$ is positive under Assumption 5.4. From the expression (5.19), we have for $\lambda \in \mathbb{R}$,

$$\chi_\lambda(a, \sigma) \geq G_0(\omega)\phi_\lambda(a, \sigma).$$

(5.21)

Therefore, in order to show the nonsupporting property of $V_\lambda, \lambda \in \mathbb{R}$, it suffices to prove that the integral operator $2P_\lambda$ defined by

$$(2P_\lambda \phi)(a) := \int_0^\infty \phi_\lambda(a, \sigma)\psi(\sigma)\,d\sigma, \quad \psi \in E,$$

(5.22)

is nonsupporting. It is easy to verify the inequality

$$\hat{T}_\lambda \psi \geq \langle f_\lambda, \psi \rangle e, \quad e = 1 \in E_+, \quad \psi \in E_+,$$

(5.23)

where the positive linear functional $f_\lambda, \lambda \in \mathbb{R}$ is defined by

$$\langle f_\lambda, \psi \rangle := \int_0^\infty \left[ \int_0^\infty s(x)N(x) \exp(-(\lambda + \gamma)(x - \sigma))\,dx \right] \psi(\sigma)\,d\sigma.$$

Then it follows that for all integers $n = 0, 1, 2, \ldots$, 

$$\hat{T}_\lambda^{n+1} \psi \geq \langle f_\lambda, \psi \rangle^n e.$$

Since $f_\lambda$ is strictly positive and the constant function $e = 1$ is a quasi-interior point of $L^1(0, \infty)$, it follows that $\langle F, \hat{T}_\lambda \psi \rangle > 0, n \geq 1$ for every pair $\psi \in E_+\setminus\{0\}, F \in E_+^* \setminus\{0\}$. Then $\hat{T}_\lambda, \lambda \in \mathbb{R}$ is nonsupporting. Next we show (5.18). From (5.21) and (5.23), we obtain

$$V_\lambda \psi \geq G_0(\omega)\hat{T}_\lambda \psi \geq G_0(\omega)\langle f_\lambda, \psi \rangle e, \quad \lambda \in \mathbb{R}, \quad \psi \in E_+.$$

Taking duality pairing with the eigenfunctional $F_\lambda$ of $V_\lambda$ that corresponds to $r(V_\lambda)$, we have

$$r(V_\lambda)\langle F_\lambda, \psi \rangle \geq G_0(\omega)\langle F_\lambda, e \rangle \langle f_\lambda, \psi \rangle.$$

If we let $\psi = e$, we arrive at the inequality

$$r(V_\lambda) \geq G_0(\omega)\langle f_\lambda, e \rangle,$$

where

$$\langle f_\lambda, e \rangle = \int_0^\infty \int_0^\infty s(x)N(x) \exp[-(\lambda + \gamma)(x - \sigma)]\,dx\,d\sigma$$

$$= \int_0^\infty s(x)N(x) \left[ \frac{1 - e^{-(\lambda + \gamma)x}}{\lambda + \gamma} \right] dx \geq e \int_{\omega - \alpha}^\infty N(x) \left[ \frac{1 - e^{-(\lambda + \gamma)x}}{\lambda + \gamma} \right] dx.$$

Since $N(x) > 0$ for $x \in [\omega - \alpha, \omega)$, we know that $\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty$. On the other hand, we obtain

$$V_\lambda \psi \leq T_\lambda \psi \leq \hat{T}_\lambda \psi \leq \langle g_\lambda, \psi \rangle e, \quad \lambda \in \mathbb{R}, \quad \psi \in E_+,$$

where the positive functional $g_\lambda$ is defined by

$$\langle g_\lambda, \psi \rangle := M \int_0^\infty \left[ \int_0^\infty N(x) \exp[-(\lambda + \gamma)(x - \sigma)]\,dx \right] \psi(\sigma)\,d\sigma,$$
where $M := \text{ess sup } \beta(a, \zeta)$. Then we obtain the estimate

$$r(V_\lambda) \leq \langle g_\lambda, \epsilon \rangle = M \int_0^\infty N(x) \left[ \frac{1 - e^{-(\lambda + \gamma)x}}{\lambda + \gamma} \right] dx,$$

from which we know that $\lim_{\lambda \to +\infty} r(V_\lambda) = 0$. This completes the proof.

From Assumption 5.4 and the expression (5.19), the kernel $\chi_\lambda(a, \sigma)$ is strictly decreasing as a function of $\lambda \in \mathbb{R}$. Using Proposition 4.3, we know that the function $\lambda \to r(V_\lambda)$ is strictly decreasing for $\lambda \in \mathbb{R}$. Moreover if there exists $\lambda \in \mathbb{R}$ such that $r(V_\lambda) = 1$, then $\lambda \in A$, because $r(V_\lambda) \in P_\sigma(V_\lambda)$. From the monotonicity of $r(V_\lambda)$ and (5.18), it is easy to see that the following holds:

**Lemma 5.5.** Under Assumption 5.4, there exists a unique $\lambda_0 \in \mathbb{R} \cap A$ such that $r(V_{\lambda_0}) = 1$, and $\lambda_0 > 0$ if $r(V_0) > 1; \lambda_0 = 0$ if $r(V_0) = 1; \lambda_0 < 0$ if $r(V_0) < 1$.

Using the similar argument as Theorem 6.13 of Heijmans (1986), we can prove that $\lambda_0$ is the dominant singular point:

**Lemma 5.6.** Suppose that Assumption 5.4 holds. If there exists a $\lambda \in A, \lambda \neq \lambda_0$, then $\text{Re } \lambda < \lambda_0$.

**Proof.** Suppose that $\lambda \in A$ and $V_\lambda \psi = \psi$. Then $\langle V_\lambda \psi, \psi \rangle = \|\psi\|$, where $\|\psi\|(a) := |\psi(a)|$. From the expression (5.19), it follows that $V_{\text{Re } \lambda} \|\psi\| \geq \|\psi\|$. Taking duality pairing with $F_{\text{Re } \lambda} \in E_+^*$ on both sides, we have $r(V_{\text{Re } \lambda}) \langle F_{\text{Re } \lambda}, \|\psi\| \rangle \geq \langle F_{\text{Re } \lambda}, \|\psi\| \rangle$, from which we conclude that $r(V_{\text{Re } \lambda}) \geq 1$, because $F_{\text{Re } \lambda}$ is strictly positive. Since $r(V_\lambda), \lambda \in \mathbb{R}$ is a decreasing function, we obtain that $\text{Re } \lambda < \lambda_0$. If $\text{Re } \lambda = \lambda_0$, then $V_{\lambda_0} \|\psi\| = \|\psi\|$. In fact, if we suppose that $V_{\lambda_0} \|\psi\| > \|\psi\|$, taking duality pairing with the eigenfunctional $F_0$ corresponding to $r(V_{\lambda_0}) = 1$ on both sides yields $\langle F_0, \|\psi\| \rangle > \langle F_0, \|\psi\| \rangle$ which is a contradiction. Then we can write that $\|\psi\| = \psi_\circ c$ for some constant $c$ which we may assume to be one, where $\psi_\circ$ is the eigenfunction corresponding to $r(V_{\lambda_0}) = 1$. Hence $\psi(a) = \psi_\circ(a) \exp(i\alpha(a))$ for some real-valued function $\alpha$. If we substitute this relation into $V_{\lambda_0} \psi_\circ = \|V_\lambda \psi\|$, we obtain

$$\int_0^\infty \int_0^\infty G_\sigma(\xi) \beta(a, \zeta) N(\zeta) \exp(-\lambda_0(\zeta - \sigma)) \psi_\circ(\sigma) d\zeta d\sigma.$$ 

From Lemma 6.12 of Heijmans (1986), it follows that $-\text{Im } \lambda(\zeta - \sigma) + \alpha(\sigma) = \beta$ for some constant $\beta$. Using the relation $V_\lambda \psi = \psi$, we obtain that $\exp(i\beta)V_{\lambda_0} \psi_\circ = \psi_\circ \exp(i\alpha(a))$, so $\beta = \alpha(a)$, which implies that $\text{Im } \lambda = 0$. This completes the proof.

**Proposition 5.7.** Under Assumption 5.4, the equilibrium state $(x^*, y^*)$ is locally asymptotically stable if $r(V_0) < 1$ and locally unstable if $r(V_0) > 1$.

**Proof.** From Lemmas 5.5 and 5.6, we conclude that $\sup \{\text{Re } \lambda : 1 \in P_\sigma(V_{\lambda})\} = \lambda_0$. Hence it follows that $s(A + C) = \sup \{\text{Re } \lambda : 1 \in P_\sigma(V_{\lambda})\} < 0$ if $r(V_0) < 1$, and $s(A + C) > 0$ if $r(V_0) > 1$. This completes the proof.
Now we can state the local stability results for our epidemic model:

**Proposition 5.8** (local stability results). Let $r(T)$ be the spectral radius of the operator $T$ defined by (4.4). Then the following hold:

1. If $r(T) < 1$, the trivial equilibrium point of (2.7a)-(2.7b) is locally asymptotically stable.
2. If $r(T) > 1$, the trivial equilibrium point is locally unstable.
3. If $r(T) > 1$ and Assumption 5.4 holds for an endemic state, it is locally asymptotically stable.

**Proof.** By definition (4.4) and (5.22), we have $T = T_0$. Since Assumption 5.4 is satisfied for the trivial steady state, it is sufficient to consider only the case that Assumption 5.4 holds. Under Assumption 5.4, we know that $U$, $V$ are positive operators for $\lambda \in \mathbb{R}$ and it follows that

$$V \leq T \leq \hat{T}$$

for $\lambda \in \mathbb{R}$, (5.24)

which implies that $r(V_0) \leq r(T_0) = r(T)$, where the equality holds if and only if $\lambda^*(a) \equiv 0$, which corresponds to the trivial equilibrium state. Hence, for the trivial equilibrium state, Proposition 5.7 says that if $r(T) = r(V_0) < 1$, it is locally asymptotically stable and it is locally unstable if $r(T) = r(V_0) > 1$. Next we show the result (3). By Proposition 5.7, it suffices to show that $r(V_0) < 1$ for the endemic equilibrium state. From (5.11), we obtain the inequality $r(V_\lambda) < r(T_\lambda)$, $\lambda \in \mathbb{R}$, since $T_\lambda$ is nonsupporting for $\lambda \in \mathbb{R}$ and $V_\lambda \neq T_\lambda$ when $\lambda^*(a)$ is not identically zero. In particular, the nonsupporting operator $T_0$ has an expression

$$(T_0 \psi)(a) = \int_0^\infty \phi(a, \sigma) \exp \left( - \int_0^\sigma \lambda^*(\eta) \, d\eta \right) \psi(\sigma) \, d\sigma.$$  

Since $\lambda^*(a)$ is a non-trivial positive solution of $\psi = \phi(\psi)$, it follows that $T_0$ has a positive eigenfunction $\lambda^*(a)$ corresponding to the eigenvalue one. Since a nonsupporting operator has only one positive eigenfunction which corresponds to its spectral radius, we conclude that $r(T_0) = 1$, and hence $r(V_0) < 1$. This shows that the endemic equilibrium state satisfying Assumption 5.4 is locally asymptotically stable.

**Remark 5.9.** Instead of Assumption 5.4, if we adopt Assumption 4.9, we can say that there is no nonnegative element in the set $A$ of singular points for the endemic steady states. In fact, from Assumption 4.9 and the expression (5.13), we know that $V_\lambda$ is a positive operator for $\lambda \in \mathbb{R}_+ = [0, \infty)$. Suppose that there exists a $\mu \in A \cap \mathbb{R}_+$. Then there is a $\psi \in \mathbb{E}_+ \setminus \{0\}$ such that $V_\mu \psi = \psi = T_\mu \psi - U_\mu \psi$. Let $F_\mu$ be the eigenfunctional corresponding to $r(T_\mu)$. Then $F_\mu$ is strictly positive, since $T_\mu$ is nonsupporting. Since $\langle F_\mu, U_\mu \psi \rangle > 0$ for the endemic steady state, we obtain

$$\langle F_\mu, \psi \rangle < \langle F_\mu, T_\mu \psi \rangle = r(T_\mu) \langle F_\mu, \psi \rangle,$$

which shows that $r(T_\mu) > 1$, because $\langle F_\mu, \psi \rangle > 0$. On the other hand, $r(T_\lambda)$, $\lambda \in \mathbb{R}_+$ is a strictly decreasing function, it follows that $r(T_0) = 1 \geq r(T_\lambda)$.
for $\lambda \in \mathbb{R}_+$. This is a contradiction. Therefore we conclude that the set $A \cap \mathbb{R}_+$ is empty.

Finally, in the case that $r(T) < 1$, we shall prove the global stability for the trivial equilibrium state:

**Proposition 5.10** (global stability result). *If $r(T) < 1$, the trivial equilibrium point of (2.7) is globally stable with respect to positive initial conditions.*

**Proof.** By Lemma 3.1, it suffices to show the global stability for the equation (3.5) with respect to the initial data $u_0 = (x_0, y_0) \in \Omega_0$. As was seen in (3.8), the second element $y(t)$ of $u(t; u_0)$ of (3.5) is governed by the abstract equation

$$
\frac{d}{dt} y(t) = By(t) + (Py(t))(1 + \hat{x}(t)), \quad y(0) = y_0 \in L^1(0, \omega),
$$

which can be seen as a linear equation for $y(t)$ if we consider $\hat{x}(t)$ as a known function. If we define a bounded operator $C(t) : E \rightarrow E, \ t \geq 0$ by $C(t) \phi := (P\phi)(1 + \hat{x}(t))$, then we have

$$
y(t) = S(t)y(0) + \int_0^t S(t-s)C(s)y(s) \, ds
$$

because $-1 \leq \hat{x}(t) \leq 0, 0 \leq y(t) \leq 1$ for all $t \geq 0$. Therefore we conclude that $0 \leq y(t) \leq W(t)y(0)$, where $W(t), \ t \geq 0$ is a $C_0$-semigroup generated by the perturbed operator $B + P$. By the same reason as the proof of Lemma 5.2, $W(t)$ is eventually norm continuous. Moreover the resolvent $R(\lambda, B + P)$ is given by

$$
R(\lambda, B + P)\psi = \int_0^a \exp[-(\lambda + \gamma)\sigma][(I - \hat{T}_\lambda)^{-1}\psi](\sigma) \, d\sigma,
$$

where the operator $\hat{T}_\lambda$ is defined by (5.22). Then we know that $B + P$ has a compact resolvent and

$$
\sigma(B + P) = P(\sigma(B + P)) = \{\lambda \in \mathbb{C} : 1 \in \sigma(T_{\lambda})\}.
$$

Let $\Sigma := \{\lambda \in \mathbb{C} : 1 \in \sigma(T_{\lambda})\}$. Since $\hat{T}_\lambda, \lambda \in \mathbb{R}$ is compact, we obtain that $\Sigma = \{\lambda \in \mathbb{C} : 1 \in \sigma(T_{\lambda})\}$. Using similar arguments as in the proofs of Lemma 5.5 and Lemma 5.6, we know that there exists a unique $\lambda_0 \in \mathbb{R} \cap \Sigma$ such that $r(T_{\lambda_0}) = 1$ and $s(B + P) = \lambda_0$. Hence if $r(T) = r(T_0) < 1$, then $\omega_0(B + P) = s(B + P) = \lambda_0 < 0$, because $r(T_{\lambda})$ is strictly decreasing for $\lambda \in \mathbb{R}$. That is, the semigroup $W(t)$ is exponentially stable and $\lim_{t \to \infty} y(t) = 0$. From (2.9) and (3.7), it is easily seen that $\lim_{t \to \infty} x(t) = 1$. This completes the proof.
6. Summary and discussion

In this paper we have examined Greenhalgh's conjectures for an age-structured SIR-type epidemic model. Under the appropriate conditions, we could prove his conjectures, i.e. (1) there exists a threshold value $r(T)$ given as the spectral radius of the positive linear operator $T$; (2) the equilibrium with no disease is always possible and it is locally, in fact globally, stable if $r(T) < 1$ and unstable if $r(T) > 1$; (3) the endemic equilibrium state is possible if and only if $r(T) > 1$; (4) the endemic steady state is unique if the Assumption 4.9 holds and locally stable if the Assumption 5.4 is satisfied. However it should be noted that the results (1)–(3) are robust to the variation of parameters, the conditions for uniqueness and stability for the endemic steady state are rather sensitive to the values of parameters. To seek more advantageous conditions to guarantee the uniqueness and stability for the endemic equilibrium state remains as an open problem. Another important question is whether destabilization of the endemic steady state could lead to the bifurcation of time-periodic solutions. This phenomenon would give an explanation for the fact that some SIR-type diseases tend to occur in regular periodic cycles (see Greenhalgh 1988b).

On the other hand, it should be also noted that the model investigated here is based on some restrictive assumptions as an epidemic model. We have assumed that

1. the population is in a demographically steady state;
2. the latent period is negligibly short;
3. the recovery rate is constant;
4. the infectivity is independent of the duration of an infection.

The assumption of demographic steady state is appropriate for short-time argument in developed countries with low population growth rate, but in general the fact that the population growth affects the spread of disease is important in case that we consider epidemiology in populations with high growth rate or we examine the diseases with a long latent period. If we intend to take into account the latent period, it suffices to introduce the incubation class into the model. The reader may refer to McLean (1986) for more realistic model building which takes into account the incubation class and the effect of demographic growth. Most essential improvement in the model would be attained by introducing duration-dependence in the transmission process. Several authors have already introduced SIR-type age-dependent epidemic models with duration-dependent infectivity (Hoppensteadt 1974; Gripenberg 1983; Dietz and Schenzle 1985). However they assume that the transmission coefficient has a special form, and hence the general case that the transmission coefficient depends on age of both susceptibles and infectious should be investigated in future.

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References


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